Turbulent Intermittency in the Lagrangian-Averaged Alpha Model

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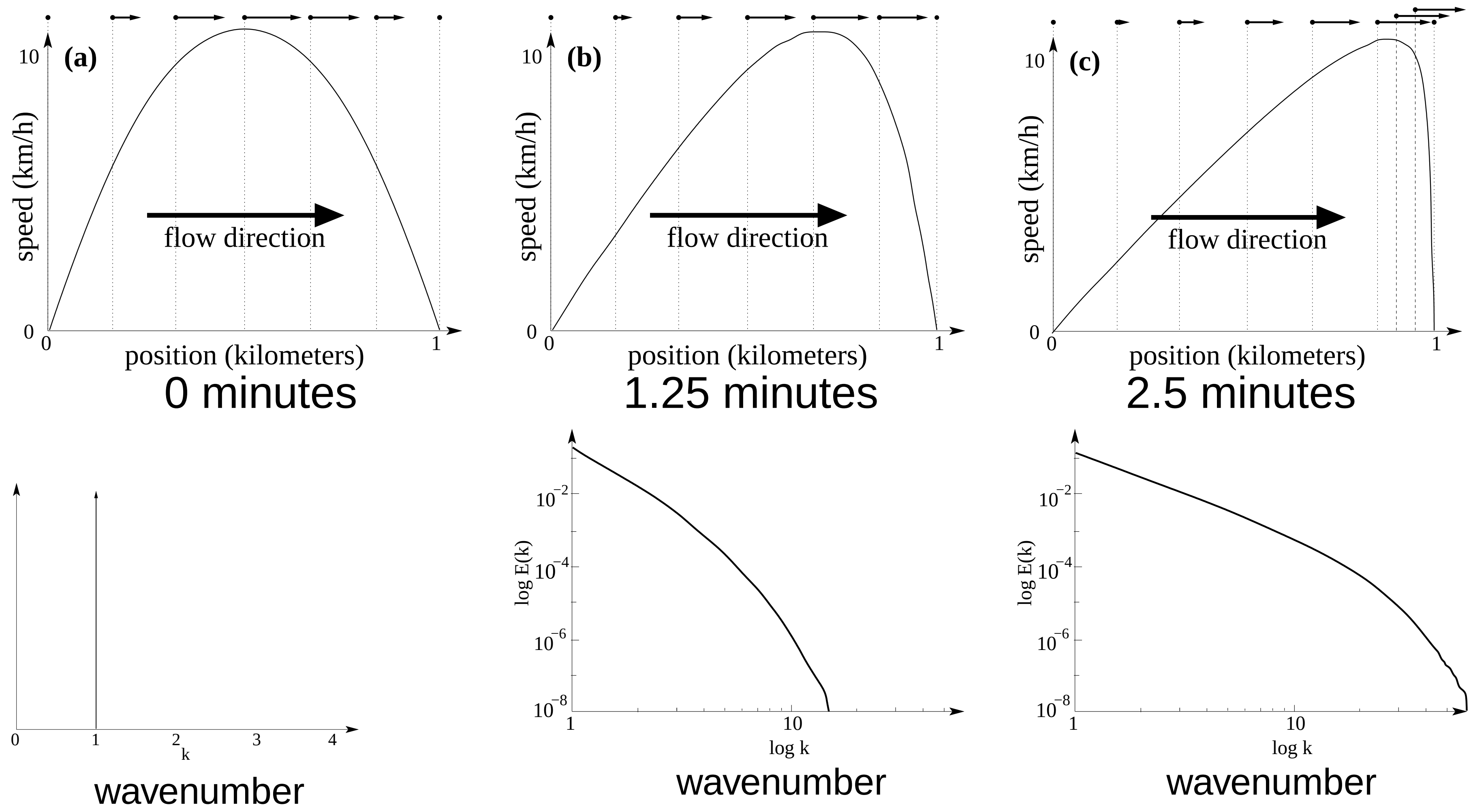
Outline

- What is Turbulence?
- What is Intermittency?
- What is the Lagrangian-Averaged Alpha Model?
- Experiments - Does it reproduce intermittency?
- Results - If so, how well?
- Summary, Conclusions, & What Next?

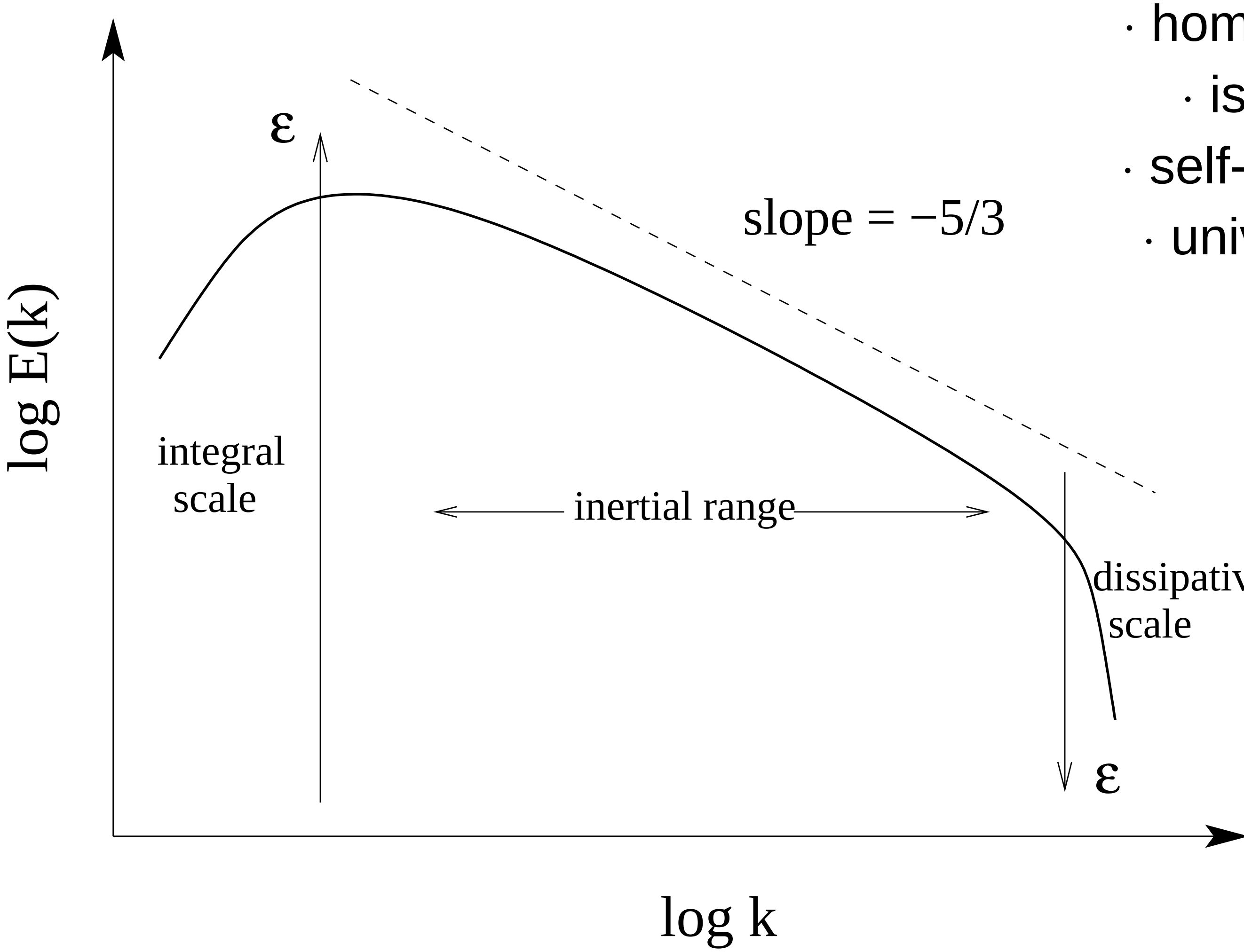
What is Turbulence?

Leonardo da Vinci’s illustration of the swirling flow of turbulence

Cascade to Small Scales

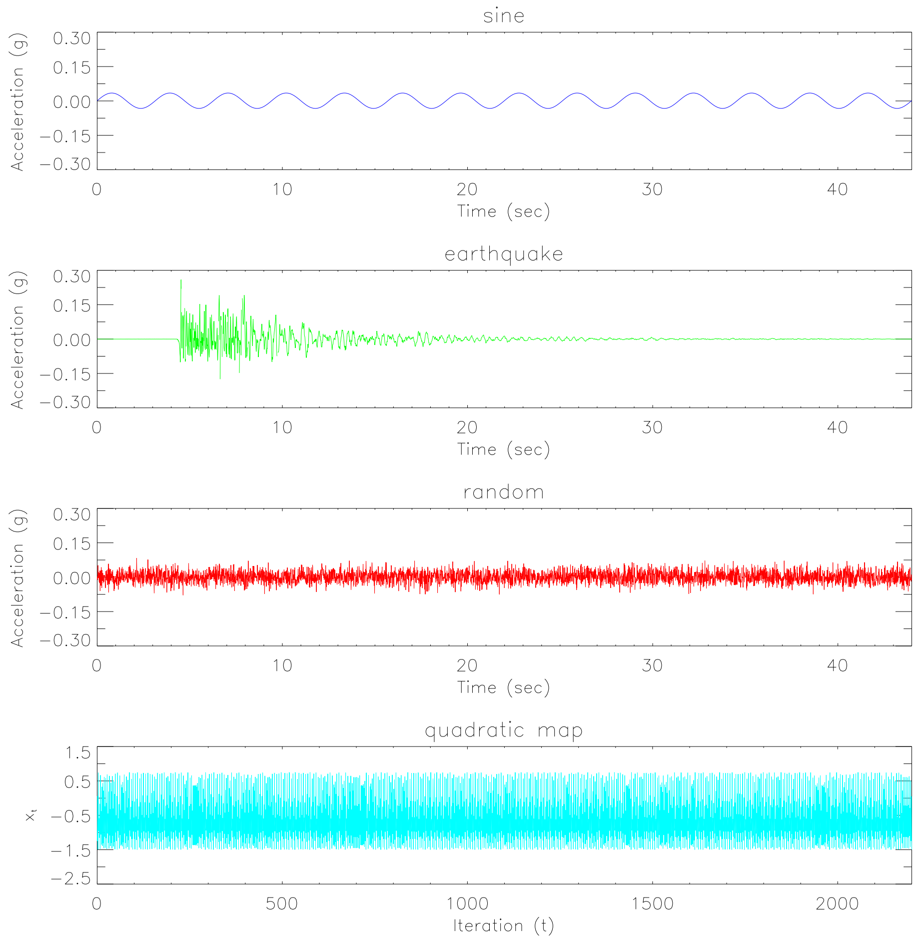


Kolmogorov and the Inertial Range



- homogeneity
- isotropy
- self-similarity
- universality

Intermittency - Examples



Structure Function Scaling - Some Math

$$\delta f(\tau) \equiv f(t+\tau) - f(t) \tag{1}$$

$$S_p^f(\tau) \equiv \langle |\delta f(\tau)|^p \rangle^1 \tag{2}$$

$$\delta f(\lambda T) = \lambda^h \delta f(T) \tag{3}$$

self-similarity

$$\tau \equiv \lambda T$$

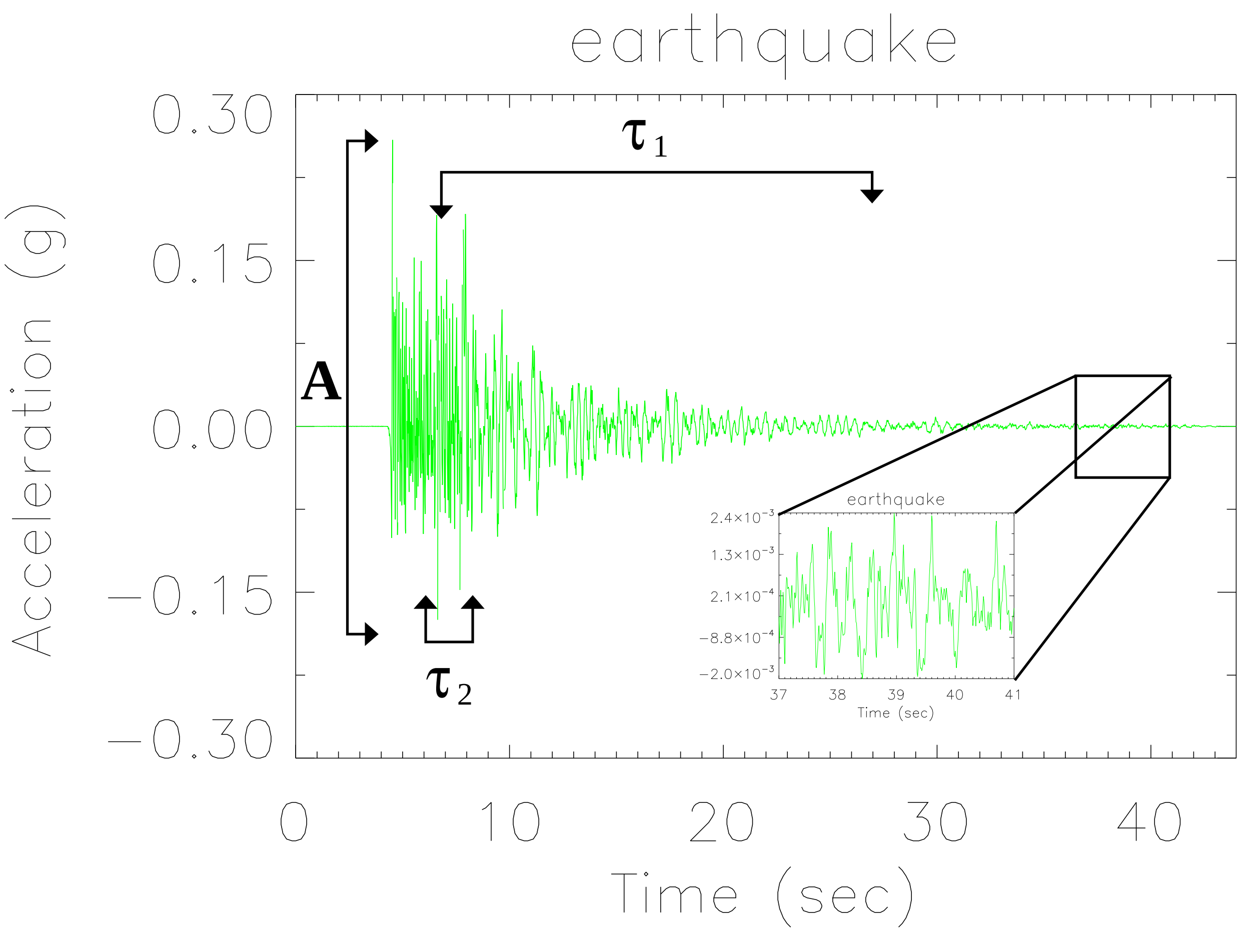
$$S_p^f(\tau) = \langle |\lambda^h \delta f(T)|^p \rangle = \lambda^{h \cdot p} \langle |\delta f(T)|^p \rangle \sim \tau^{\zeta_p^f}$$

$$\zeta_p^f = h \cdot p \tag{4}$$

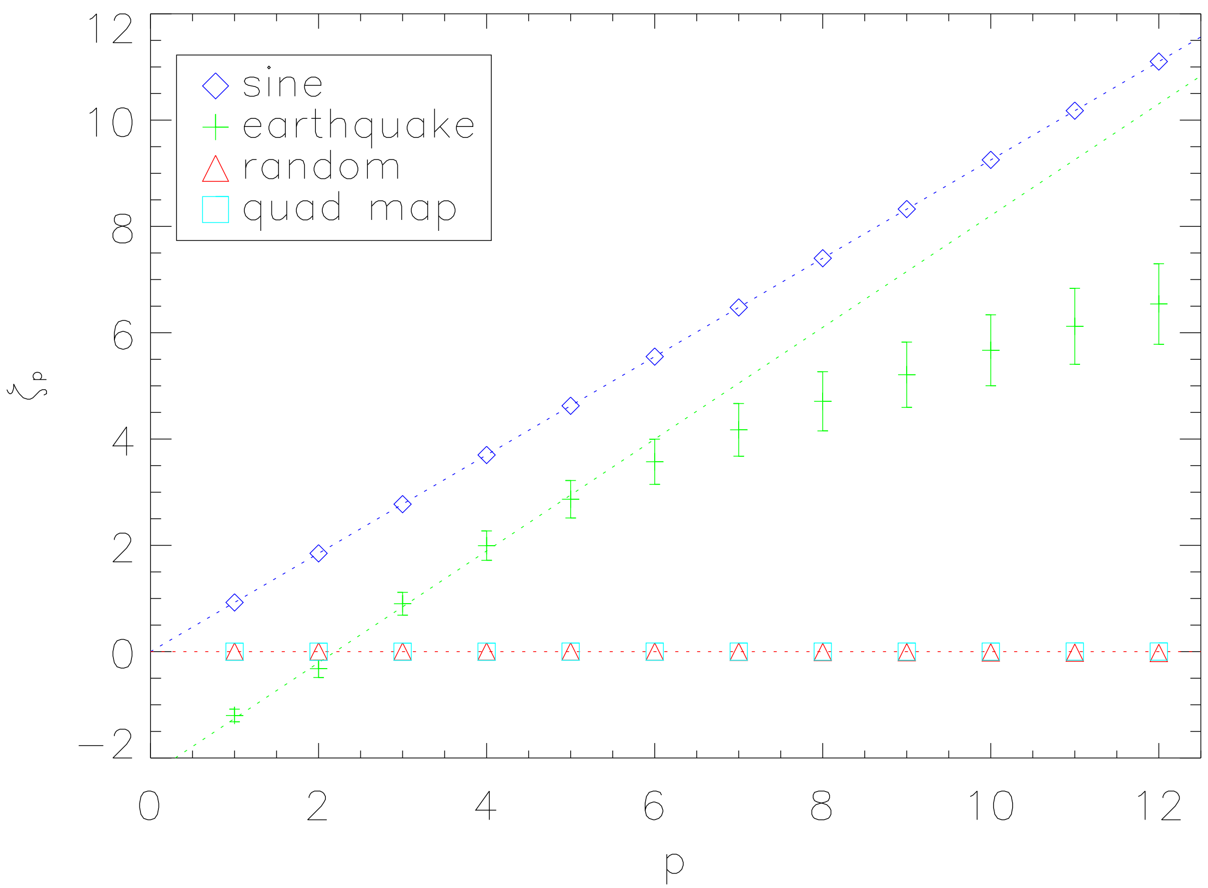
If the statistical features of the system are independent of spatial scale, it is described as self-similar and it's scaling will be linear, $\zeta_p^f \sim p$.

[1] Angle brackets, $\langle \cdot \rangle$, denote integration over the entire domain.

Structure Function Scaling for Intermittency



Structure Function Scaling - Examples

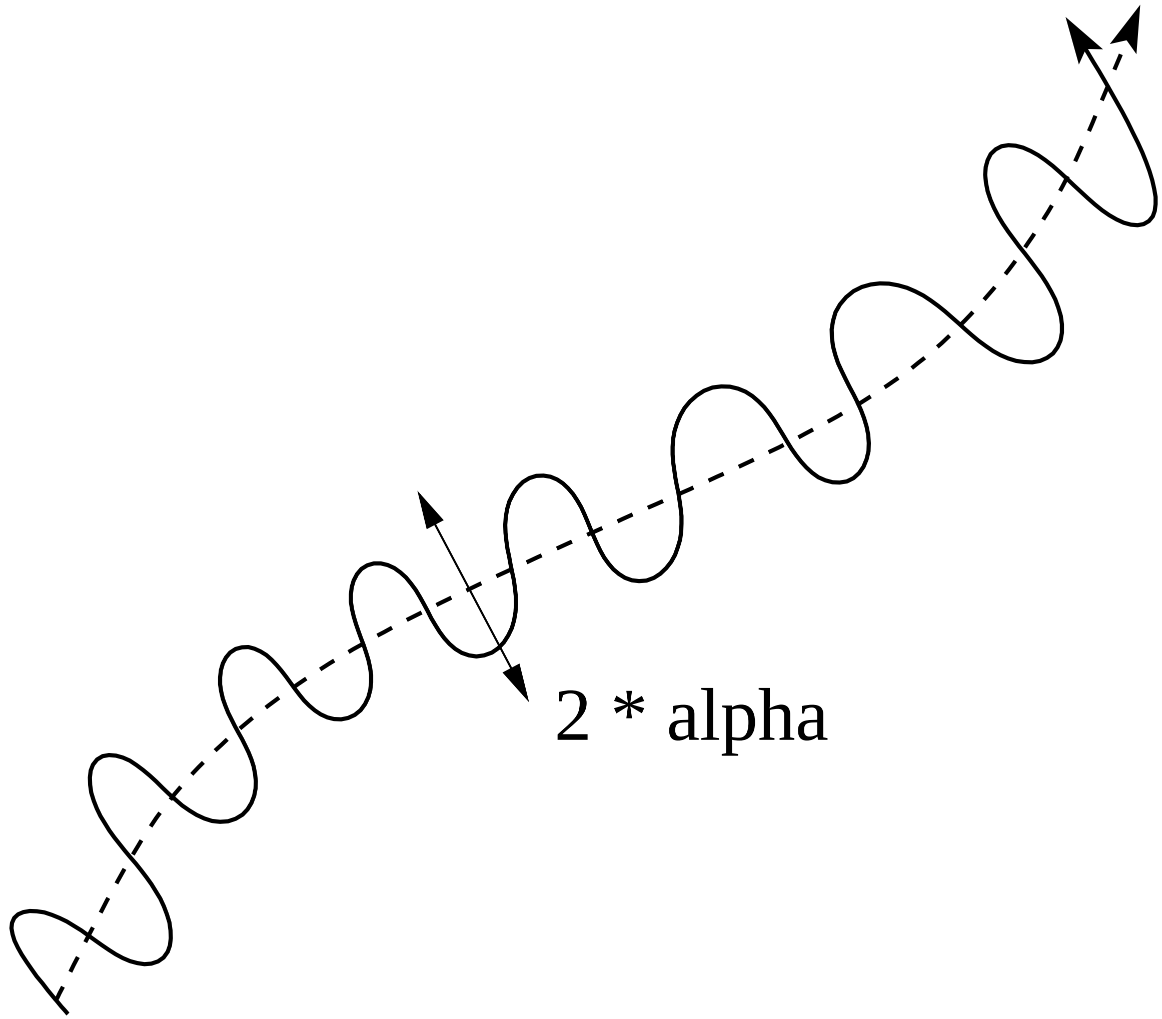


$$S_p(\tau) \sim \tau^{\zeta_p}$$

Lagrangian-Averaged Alpha Model

- this is the closure
- Lagrangian-averaging $=$ averaging along fluid trajectory

Illustration of Lagrangian Averaging



Lagrangian-Averaged Alpha Model

- this is the closure
- Lagrangian-averaging $=$ averaging along fluid trajectory
- Taylor's frozen-in turbulence hypothesis
- Hamilton's Principle
- conservation of energy *etc.* (under a new norm)
- add dissipation $\nabla^2 f$

Navier-Stokes vs. Lagrangian Averaging

(incompressible) Navier-Stokes equations,

$$\partial_t \vec{u} + \vec{u} \cdot \vec{\nabla} \vec{u} = -\vec{\nabla} p + \mathcal{F} + \nu \nabla^2 \vec{u}$$
$$\vec{\nabla} \cdot \vec{u} = 0 \tag{5}$$

LANS-α

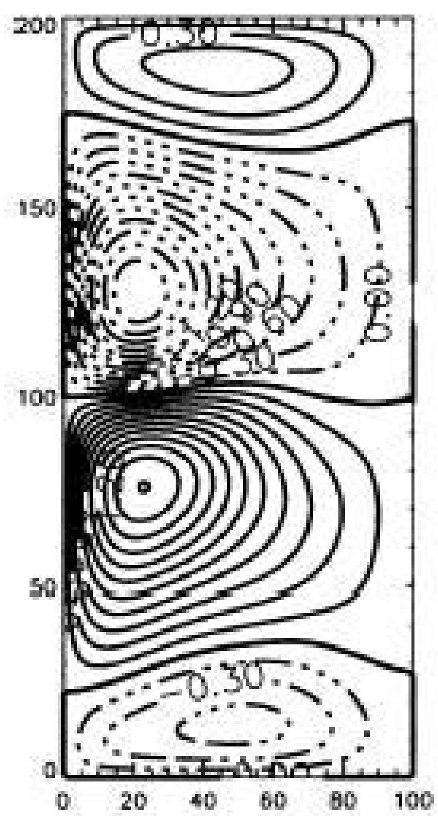
$$\hat{u}_s = f(k^2)\hat{u} \rightarrow \frac{\hat{u}}{(1 + \alpha^2 k^2)} \tag{6}$$

$$\partial_t \vec{u} + \vec{u_s} \cdot \vec{\nabla} \vec{u} + \vec{\nabla} \vec{u}_s^T \cdot \vec{u} = -\vec{\nabla} \mathcal{P} + \mathcal{F} + \nu \nabla^2 \vec{u}$$
$$\vec{\nabla} \cdot \vec{u_s} = 0 \tag{7}$$

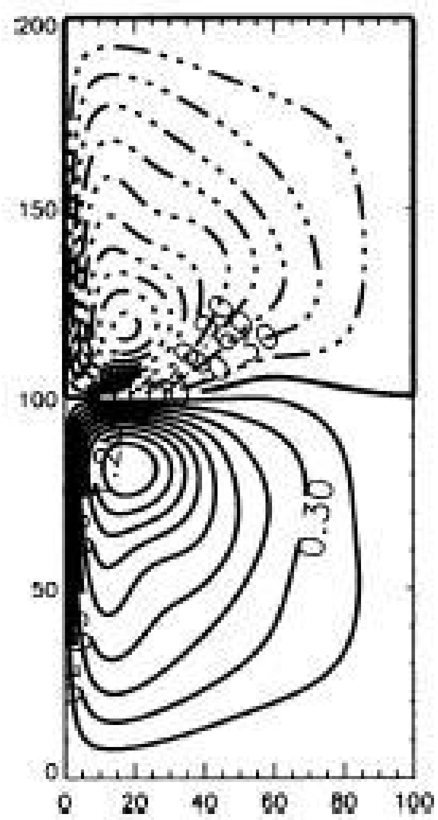
Time Savings of Lagrangian Averaging

- $\Delta t \sim \frac{1}{uN}$ (CFL)
- total cost $\sim N^{d+1}$
- $\alpha \Rightarrow \frac{N}{2} \Rightarrow$ 8 (2D), 16 (3D)
- $\alpha \Rightarrow \frac{N}{4} \Rightarrow$ 64 (2D), 256 (3D) - 12 years!
- up to $\frac{N}{4}$ gives accurate results

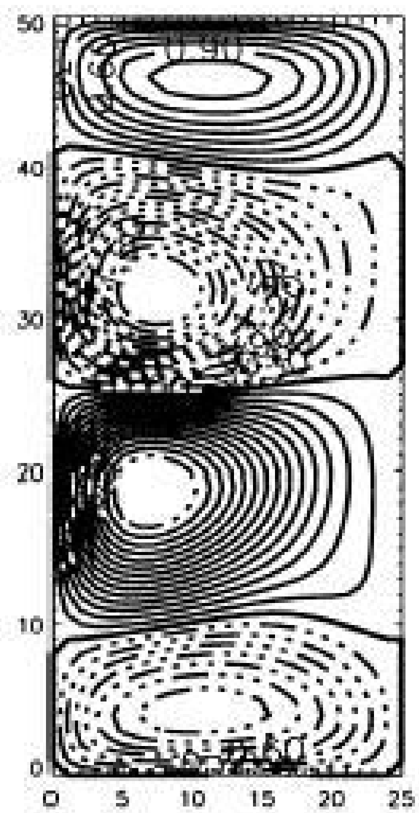
Example - Barotropic Double-Gyre Circulation[2]



physical



higher viscosity



$\alpha, N' = \frac{N}{4}$

[2]Holm, D. D. and Nadiga, B., *Modeling Mesoscale Turbulence in the Barotropic Double Gyre Circulation*, J. Phys. Oceanogr., 2355 (2003).

Experiments - What We're Looking For

- what turbulence is
- why it's hard to do
- how the alpha model can help
- intermittency versus self-similarity
- $\sim N^{d+1} \longrightarrow$ 2D vastly cheaper
- for non-conductive fluids energy $\longrightarrow$ large scales (hurricanes)
- in 2D - little transfer of energy to small scales $\longrightarrow$ no strong, localized events $\longrightarrow$ no intermittency

Magnetohydrodynamics (MHD)

- Are 2D conductive fluids intermittent?
- add Lorenz force $\vec{j} \times \vec{b}$
- no magnetic monopoles $\vec{\nabla} \cdot \vec{b} = 0$
- add an induction equation for time evolution of $\vec{b}$ (in Alfvénic units)

$$\partial_t \vec{b} = \vec{\nabla} \times (\vec{u} \times \vec{b}) + \eta \vec{\nabla}^2 \vec{b} + \mathcal{F}_M \qquad (8)$$

- make an alpha model for it $\Rightarrow$ hyperviscosity-like term

$$\partial_t \vec{b_s} = \vec{\nabla} \times (\vec{u_s} \times \vec{b_s}) + \eta \nabla^2 (1 - \alpha^2 \nabla^2) \vec{b_s} + \mathcal{F}_M \qquad (9)$$

- $\mathcal{A} = \langle a^2 \rangle \to$ large scales,[3] $E = \frac{1}{2}\langle u^2 + b^2 \rangle \to$ small scales

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[3] $\vec{b} = \vec{\nabla} \times a\vec{z}$

Numerical Method

- Pseudospectral
- exponentially convergent derivatives
- fast Poisson Eqs solution
- non-dissipative and non-dispersive
- 2-3rds rule for dealiasing
- square box with periodic boundary conditions
- initial conditions/forcing specified in Fourier space

Runs

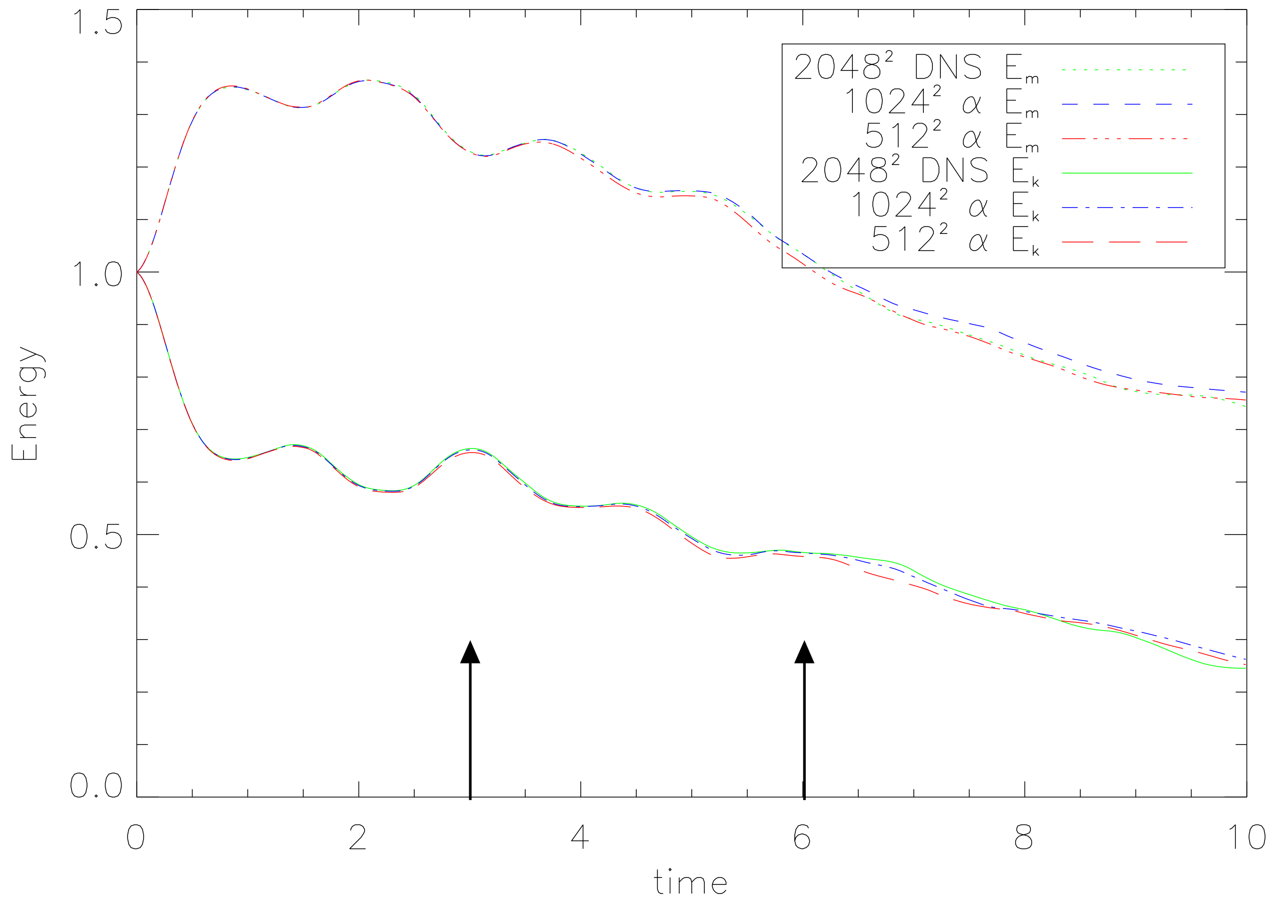
Table 1: Resolution, N, model parameter, α, viscosity, $\nu = \eta$, forcing, $\mathcal{F}_M, \mathcal{F}_K$, initial conditions, $u_o = b_0$, Taylor Reynolds number, R_λ,†

Run	N	$\alpha \cdot N$	$\nu = \eta$	$\mathcal{F}_M$	$\mathcal{F}_K$	$u_o = b_0$	R_λ
a	2048	0 (DNS)	10^{-4}	0	0	1	1500
b	1024	6	10^{-4}	0	0	1	1700
c	512	6	10^{-4}	0	0	1	1700
d	1024	0 (DNS)	$1.6 \cdot 10^{-4}$	0.2	0.45	0	1600
e	512	6	$1.6 \cdot 10^{-4}$	0.2	0.45	0	1300
f	256	6	$1.6 \cdot 10^{-4}$	0.2	0.45	0	1100

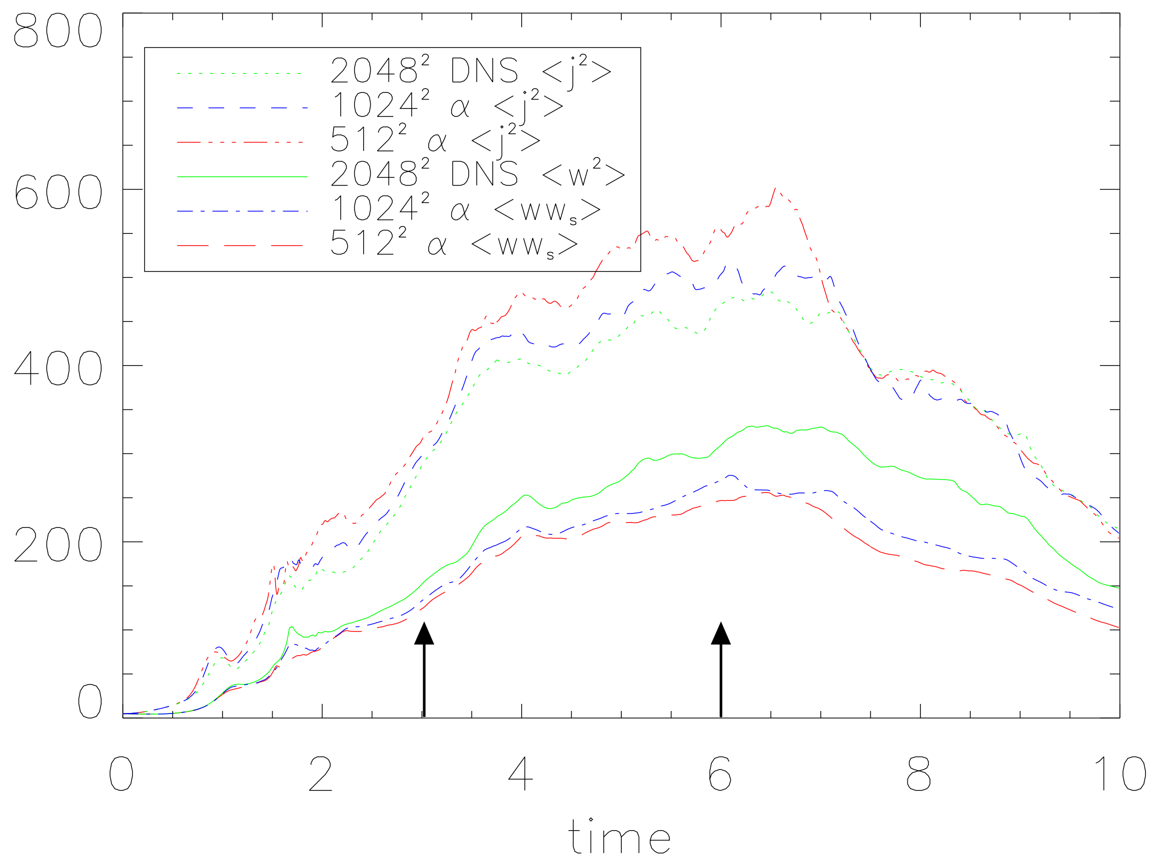
$^\dagger R_\lambda = \frac{\lambda v_{rms}}{\nu}$, computed at peak of the dissipation, $t \approx 6.5$, for freely decaying runs (a-c) and averaged over $t = [50, 150]$ for forced runs (d-f).

$$\alpha^{-1} = k_{max}/2 \tag{10}$$

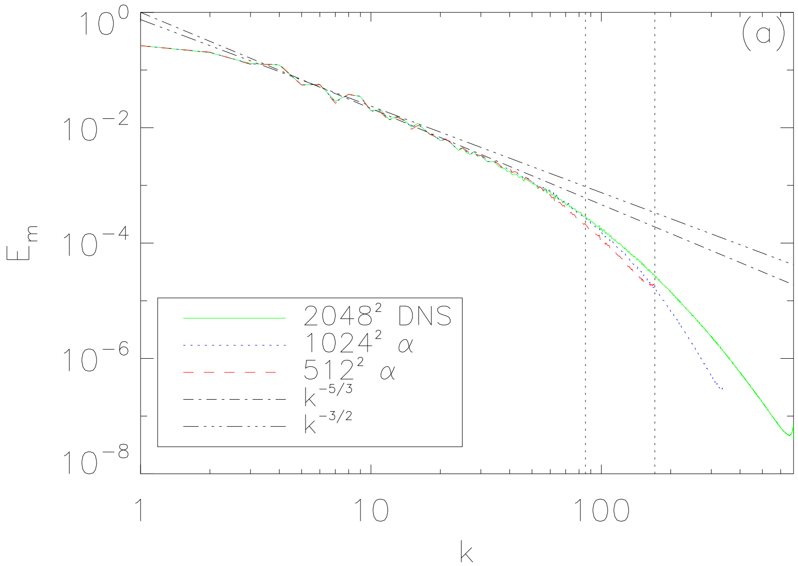
Results - Freely Decaying



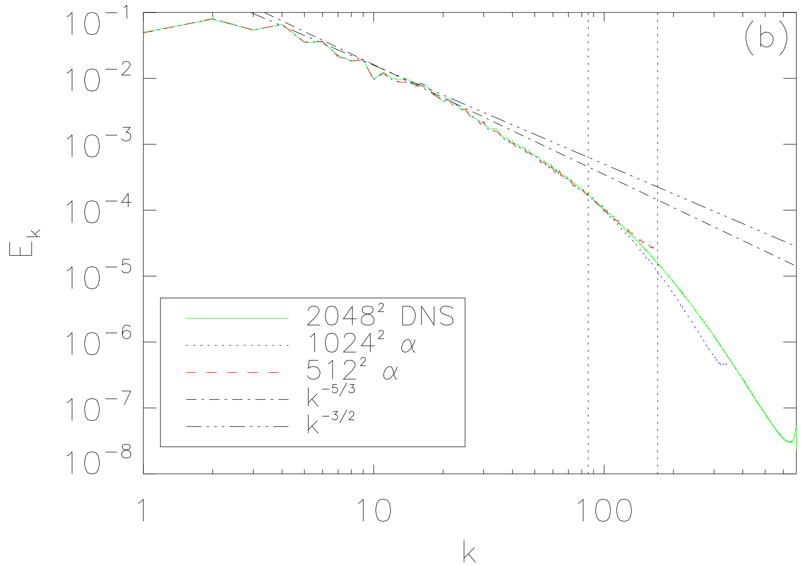
Results - Dissipation



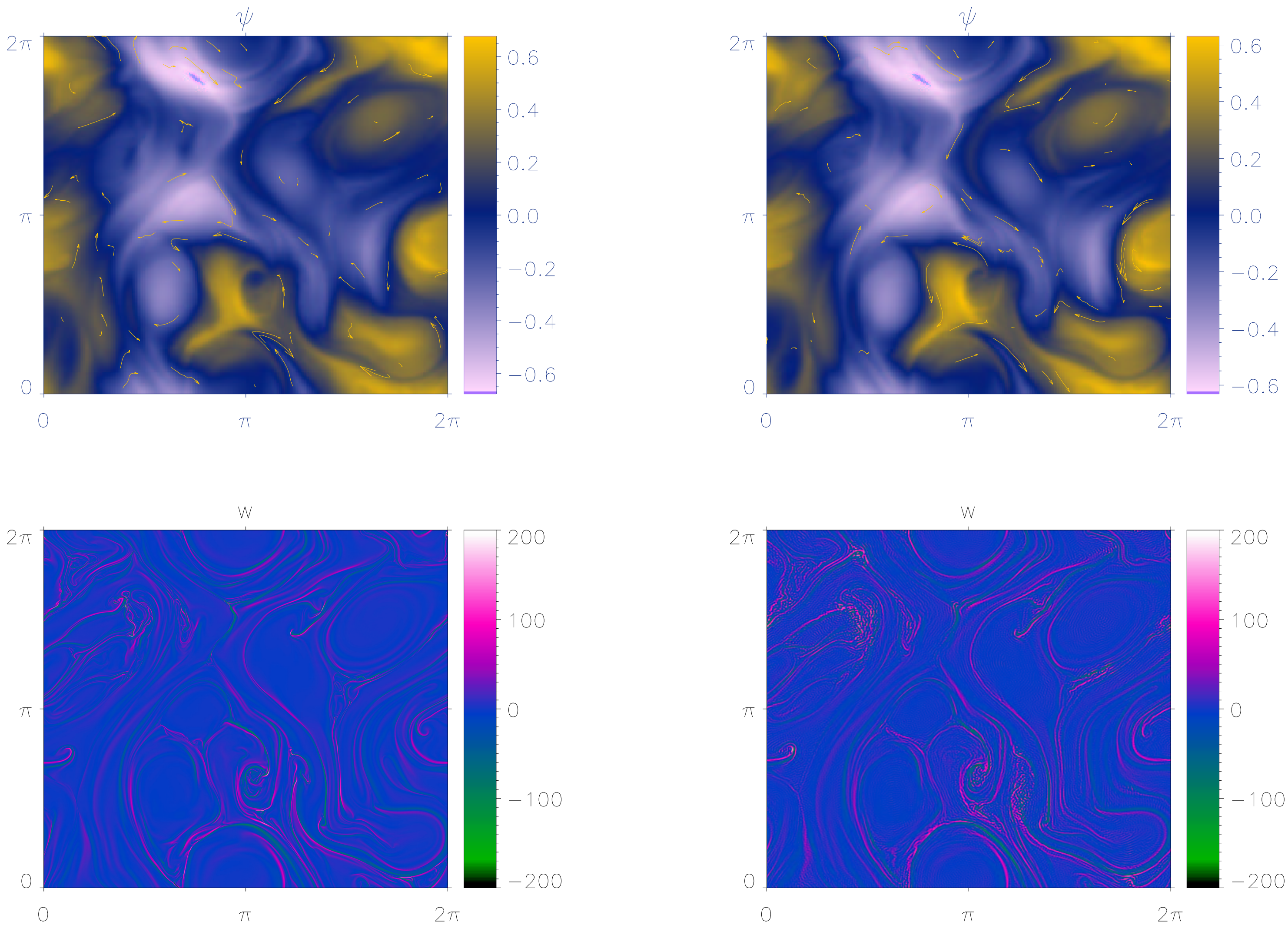
Results - Spectra



magnetic

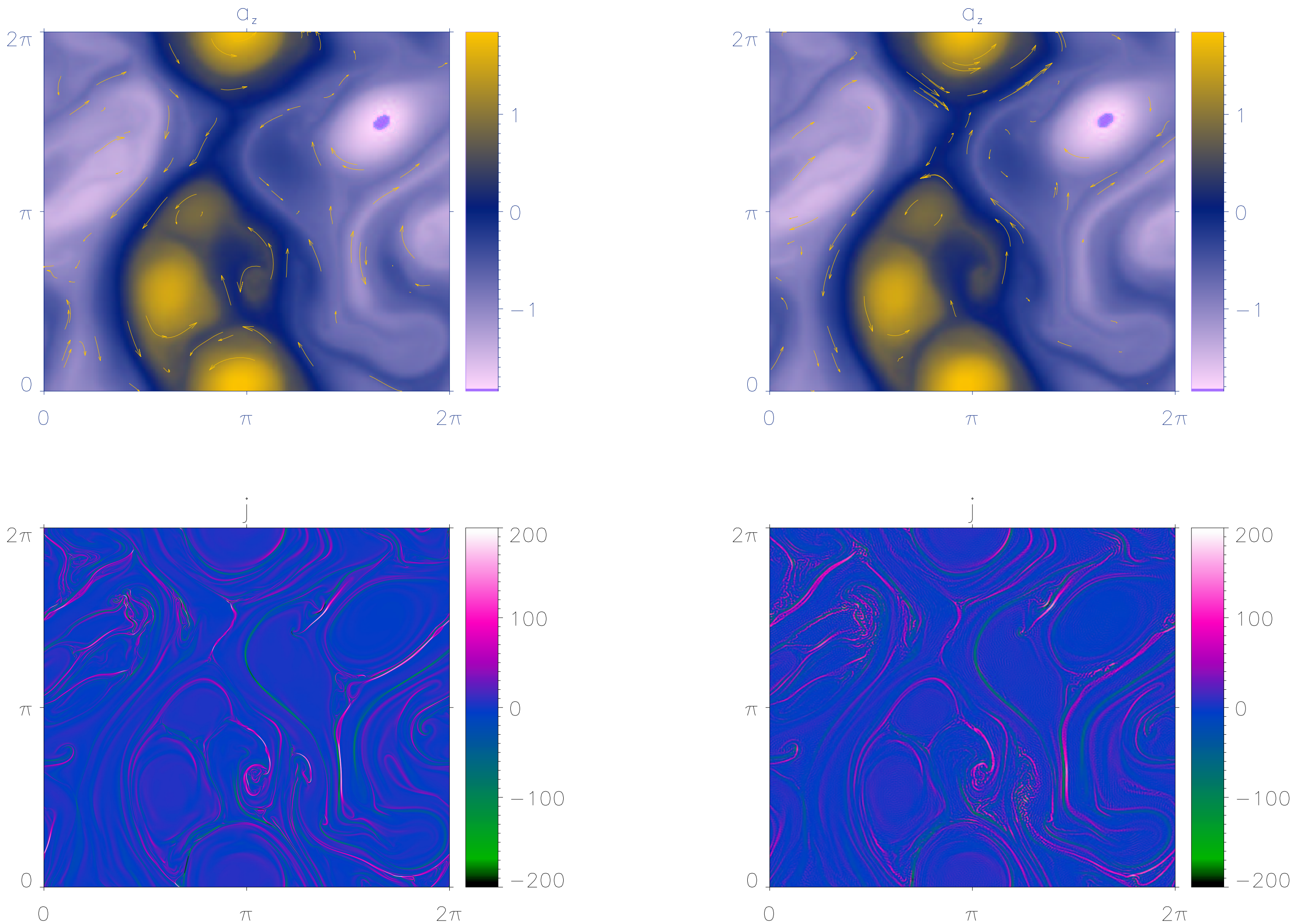


kinetic



DNS

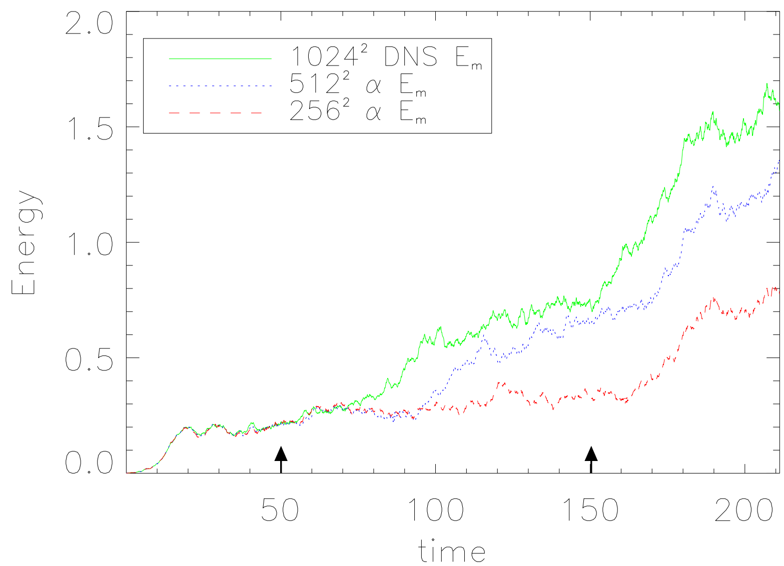
$\alpha, N' = \frac{N}{4}$



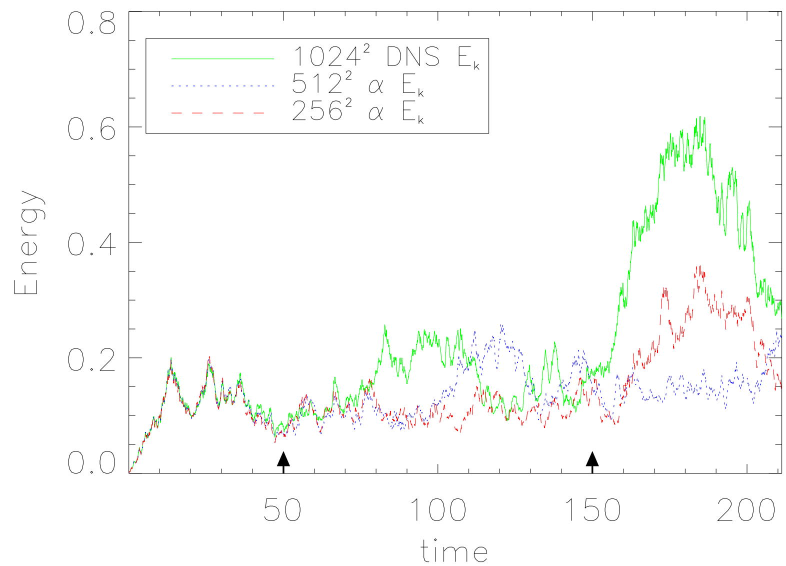
DNS

$\alpha, N' = \frac{N}{4}$

Results - Forced



magnetic



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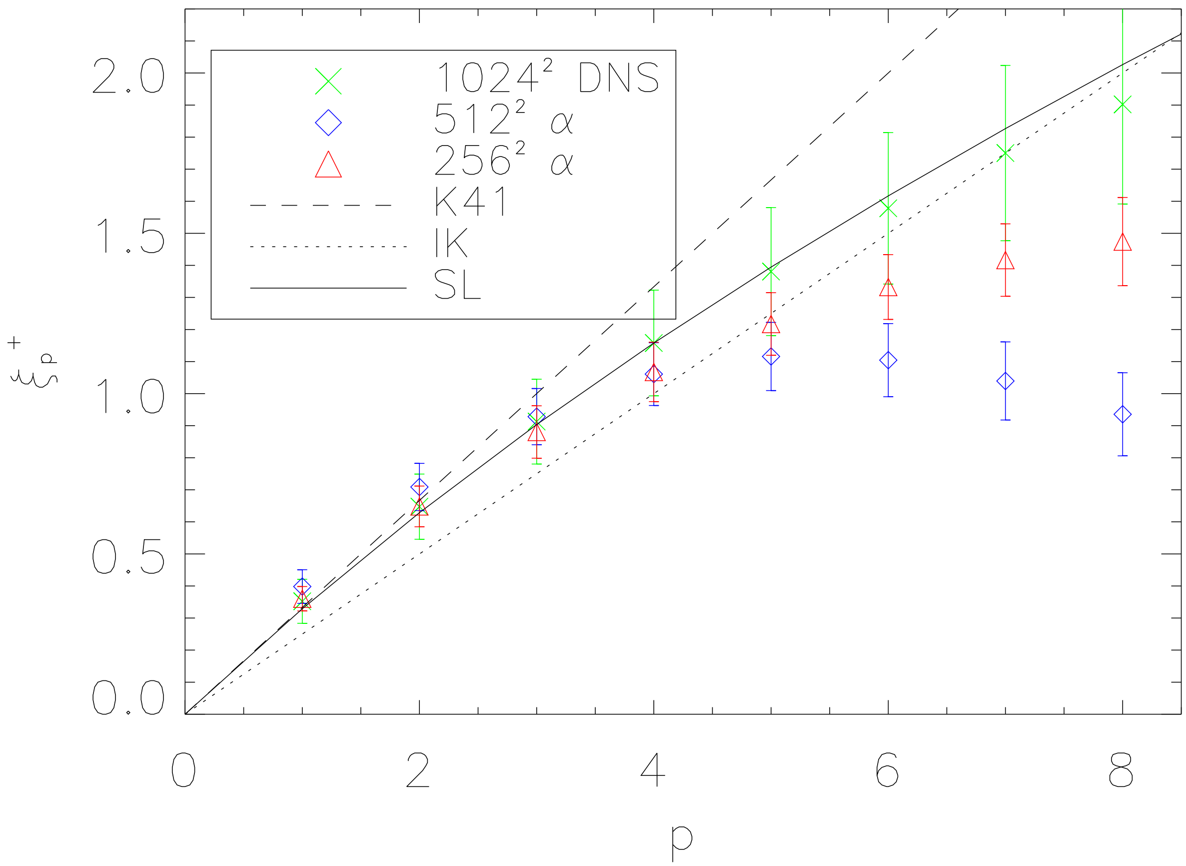
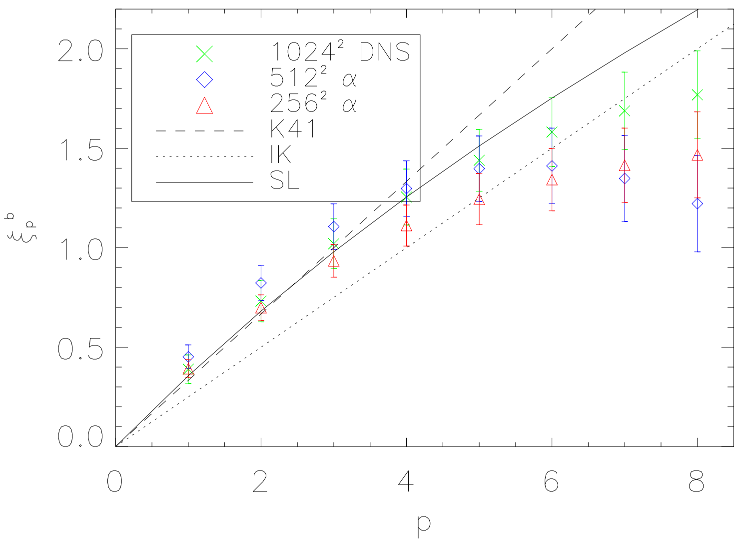
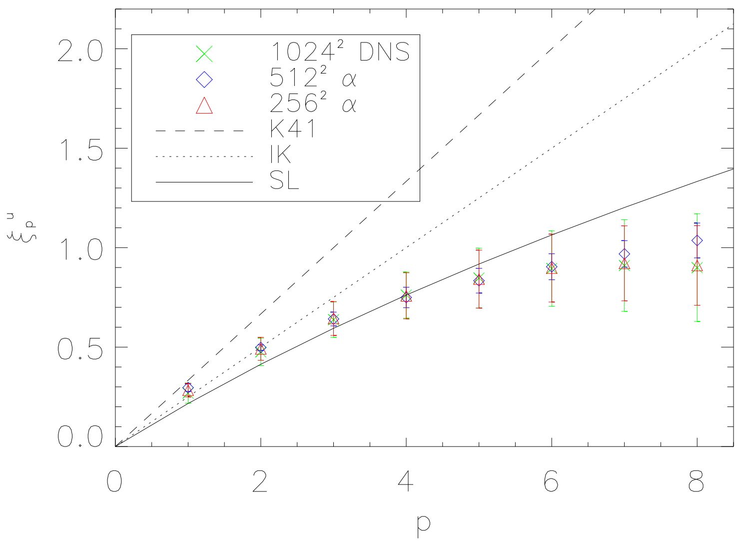
Answers - Scaling Exponents - Forced



Results - Velocity vs. Magnetic



magnetic



kinetic

Summary & Conclusions

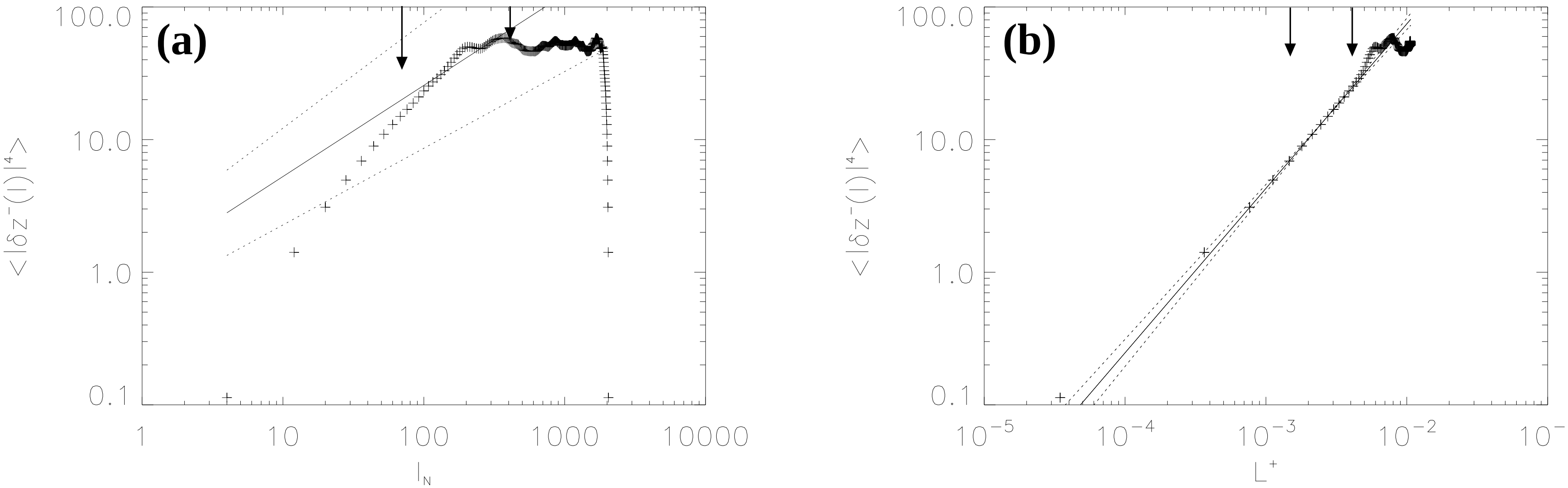
- Is the alpha model a good closure?
  - YES - it reproduces the large-wavelength component behavior
  - for velocity even better - spectra down to scale $\sim \alpha$ instead of $\sim 2\alpha$
- Does the alpha model exhibit intermittency?
  - YES
- How well?
  - up to the level of the 4th- or 5th-order statistics
  - velocity results are better - 7th-order statistics $< 3\%$ error (compared to $< 7\%$ for best LES)

What Next?

- Will modifying the magnetic dissipation make a better model?

Extended Self-Similarity - Example

4th order



length

3rd order

$$S_3 \sim l^{\zeta_3}$$

$$S_4 \sim S_3^{\xi_4}$$

$$\Rightarrow S_4 \sim l^{\xi_4 \zeta_3}$$

$$\zeta_4 = \xi_4 \zeta_3$$

von Kármán-Howarth & Extended Self-Similarity

Kolmogorov 4-5ths law:

$$\langle (\delta u_L(l))^3 \rangle = -\frac{4}{5}\varepsilon l, \tag{11}$$

for the third-order longitudinal structure function of the velocity

$$\delta u_L(l) = \Big(\vec{u}(\vec{x} + \vec{l}) - \vec{u}(\vec{x}) \Big) \cdot \vec{l}/l \tag{12}$$

$$\vec{z^+} \equiv \vec{u} + \vec{b} \tag{13}$$

$$L^+(l) \equiv \langle |\delta z_L^-||\delta z^+|^2 \rangle \propto l, \tag{14}$$

we determine the relative scaling exponents ξ_p^f from

$$S_p^f(l) \sim [L^+(l)]^{\xi_p^f}. \tag{15}$$

Conservation of Ideal Invariants

$$\frac{dE}{dt} = \frac{d}{dt}\langle\frac{1}{2}(u^2 + b^2)\rangle = -\nu\langle w^2\rangle - \eta\langle j^2\rangle \quad (16)$$

$$\frac{d\mathcal{A}}{dt} = \frac{d}{dt}\langle\frac{1}{2}a_z^2\rangle = -\nu\langle b^2\rangle^4 \quad (17)$$

$\mathcal{A} \longrightarrow$ large scales, $E \longrightarrow$ small scales

$$\frac{dE}{dt} = \frac{d}{dt}\langle\frac{1}{2}(\vec{u}\cdot\vec{u}_s + \vec{b}\cdot\vec{b}_s)\rangle = -\nu\langle w \cdot w_s\rangle - \eta\langle j^2\rangle \quad (18)$$

$$\frac{d\mathcal{A}}{dt} = \frac{d}{dt}\langle\frac{1}{2}a_{s_z}^2\rangle = -\nu\langle\vec{b}\cdot\vec{b}_s\rangle \quad (19)$$

[4] The total cross helicity, $H_C = \langle\frac{1}{2}\vec{u}\cdot\vec{b}\rangle \approx 0$, is zero by choice.

LAMHD-α

$$\partial_t \vec{u} + \vec{u}_s \cdot \vec{\nabla} u + u_j \vec{\nabla} u_s^j = -\vec{\nabla} P + j \vec{\times} b_s + \nu \nabla^2 \vec{u} + \mathcal{F}_K$$
$$\partial_t \vec{b}_s + \vec{u}_s \cdot \vec{\nabla} b_s = \vec{b}_s \cdot \vec{\nabla} u_s + \eta \nabla^2 \vec{b} + \mathcal{F}_M$$
$$\vec{\nabla} \cdot \vec{b_s} = 0$$
$$\vec{\nabla} \cdot \vec{u_s} = 0 \qquad (20)$$