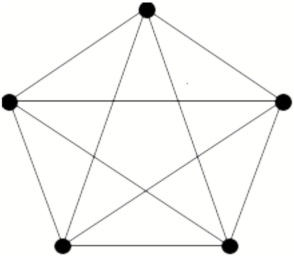
Replica Symmetry and Combinatorial Optimization

Johan Wästlund

Physics of Algorithms, Santa Fe 2009

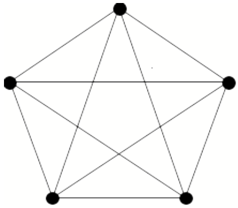
Comments (like this one) have been added in order to make sense of some of the slides. Some of those comments represent what I said, or might have said, in the talk.
The talk is based on the paper arXiv:0908.1920.

Mean Field Model of Distance



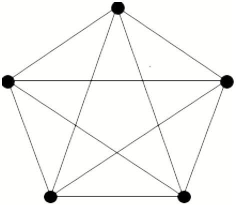
Mean Field Model of Distance

- i.i.d edge lengths, say uniform $[0, 1]$



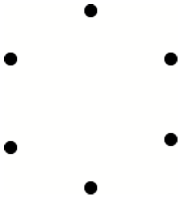
Mean Field Model of Distance

- i.i.d edge lengths, say uniform $[0, 1]$
- Quenched disorder



Optimization problems

Minimize total length:



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Minimize total length:

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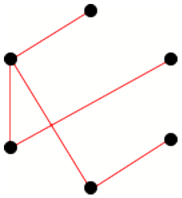

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Minimize total length:

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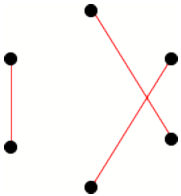
- Matching



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Minimize total length:

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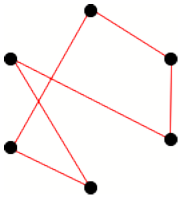
- Traveling Salesman



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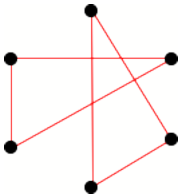
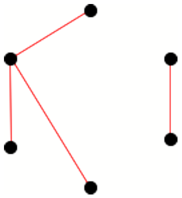
- 2-factor



Optimization problems

Minimize total length:

- Spanning Tree
- Matching
- Traveling Salesman
- 2-factor
- Edge Cover



Replica/cavity method

Replica symmetric prediction of the length of the optimum solution for large N. (Sketch of the background with a certain bias towards people in the audience...)

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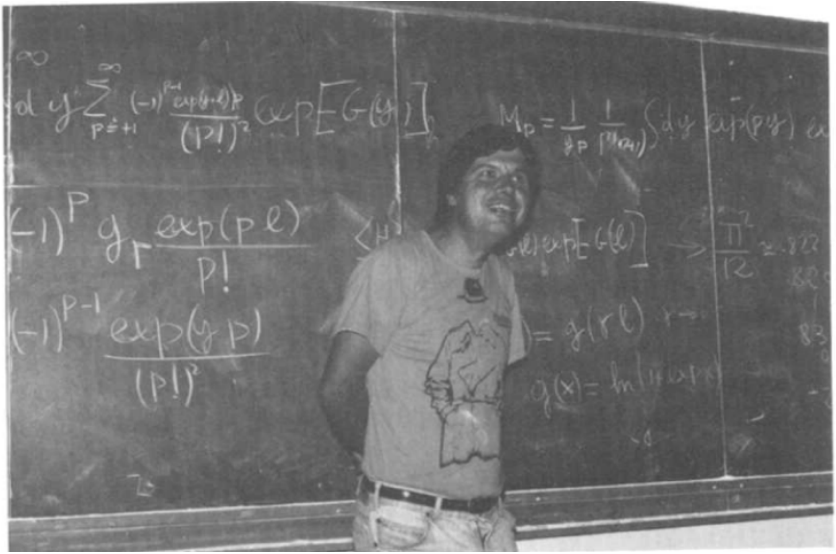

Giorgio Parisi in Les Houches 1986, having calculated the $\pi^2/12$ limit for minimum matching.

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Limit costs for uniform $[0, 1]$ edge lengths (no normalization needed!)

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- Edge cover $\frac{1}{2}\min(x^2 + e^{-x}) \approx 0.728$ Not yet proved, but the bipartite case is done in joint work with M. Hessler

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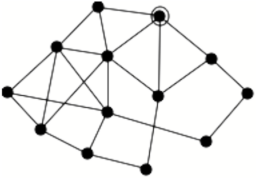
Replica symmetric predictions of $\beta_M(d)$ and $\beta_{TSP}(d)$ are correct

The right hand side is the average length of an edge in the solution, measured in the length unit at which the expected number of neighbors is 1. What is new is that the theorem holds also for $d > 1$, but I will not say so much about the parameter d in the rest of the talk.

Graph Exploration

2-person zero-sum game:



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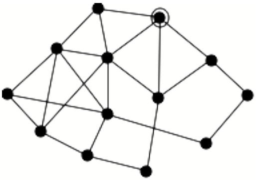

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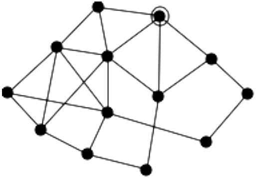

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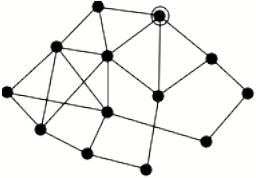

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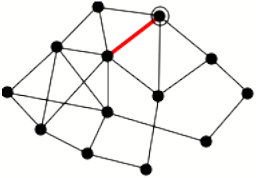

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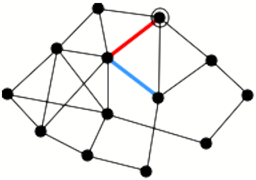

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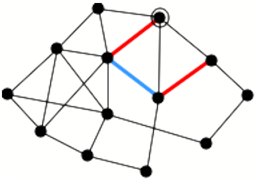

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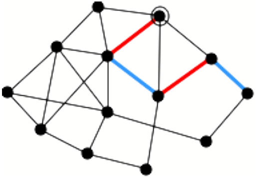

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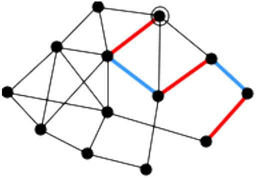

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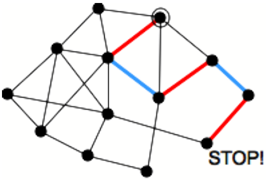

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2-person zero-sum game:

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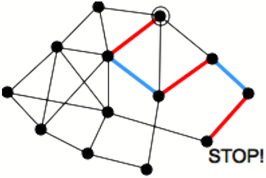
- Edges longer than θ are irrelevant!



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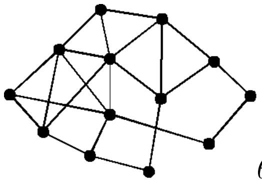
Diluted Matching Problem

Optimization:



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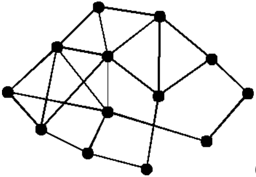
Optimization:

- Partial matching



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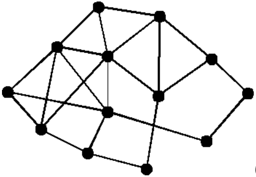
Diluted Matching Problem

Optimization:

- Partial matching
- Cost = total length of edges + $\theta/2$ for each unmatched vertex



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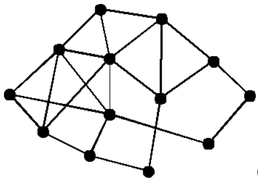
Diluted Matching Problem

Optimization:

- Partial matching
- Cost $=$ total length of edges $+ \theta/2$ for each unmatched vertex
- Feasible solutions exist also for odd N



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Solution to Graph Exploration

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Lemma

$$f(G, v) = M(G) - M(G - v)$$

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Proof.

$$f(G, v) = \min(\theta/2,\ l_i - f(G - v, v_i))$$

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Proof.

$$f(G, v) = \min(\theta/2,\ l_i - f(G - v, v_i))$$

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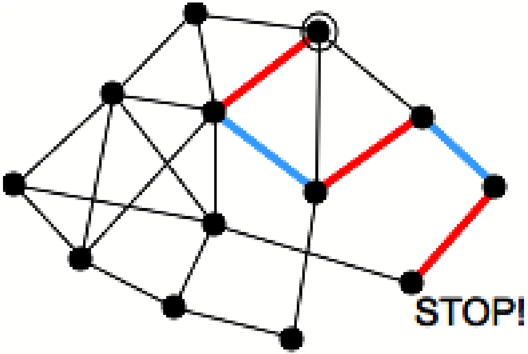
$f(G,v)$ and $M(G) - M(G-v)$ satisfy the same recursion. □

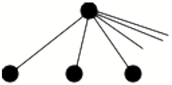
Solution to Graph Exploration

Alice's and Bob's optimal strategies are given by the optimum diluted matchings on G and $G - v$ respectively



PWIT-approximation

Poisson Weighted Infinite Tree (Aldous)



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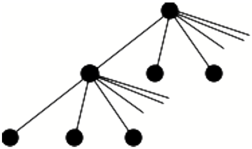

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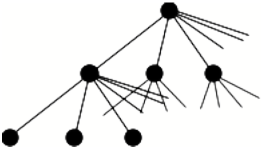

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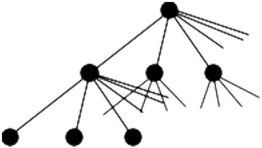

θ-cluster = component of the root after edges of cost more than θ have been deleted

PWIT-approximation

The PWIT is a local weak limit of the mean field model:

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Lemma

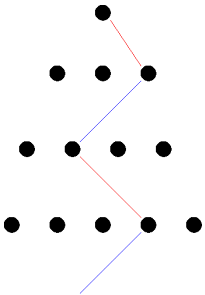
Fix positive integer k. Then there exists a coupling of the PWIT and rooted K_N such that

$$P(\textit{isomorphic } (k,\theta)\textit{-neighborhoods}) \geq 1 - \frac{(2+\theta)^k}{N^{1/3}}$$

Has to be modified slightly for $d > 1$.

Graph Exploration on the PWIT

- Study Graph Exploration on the PWIT!



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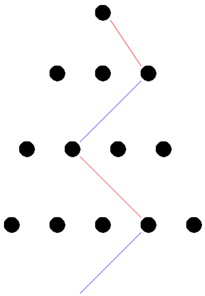

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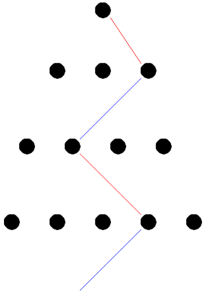

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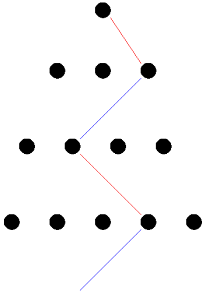
- Infinite look-ahead values f_A and f_B (when $k \to \infty$)



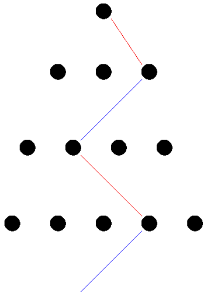
Graph Exploration on the PWIT

Theorem

Almost surely $f_A = f_B$

Sketch of proof.

□



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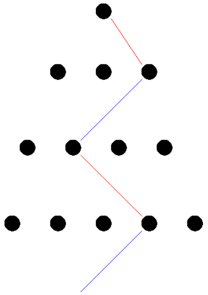

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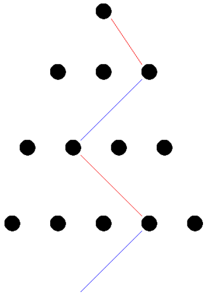

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Sketch of proof.

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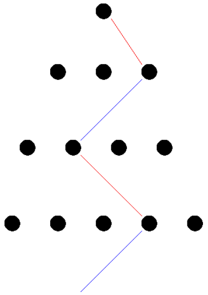
- Reasonable lines of play do not percolate

□



Graph Exploration on the PWIT

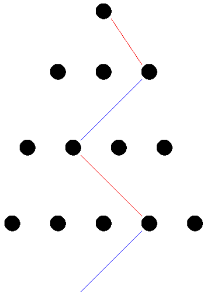
Theorem

Almost surely $f_A = f_B$

Sketch of proof.

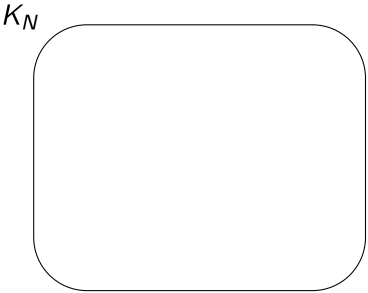
- Let both Alice and Bob be infinite look-ahead optimistic players
- If they do not agree on the value of the game, play has to go on forever
- Reasonable lines of play do not percolate
- Contradiction!

$\square$



Replica Symmetry

$f_A = f_B$ means Replica Symmetry holds: To find how to match v it suffices to look at a neighborhood of size independent of N



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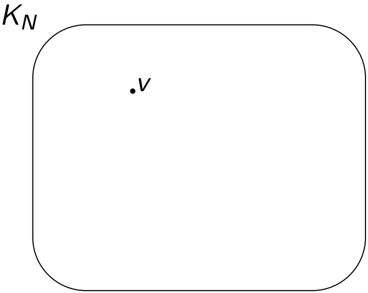

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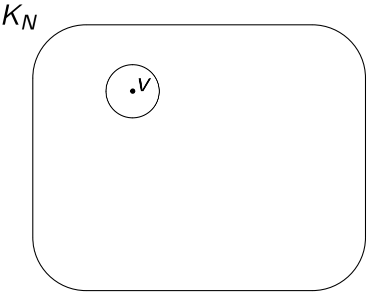

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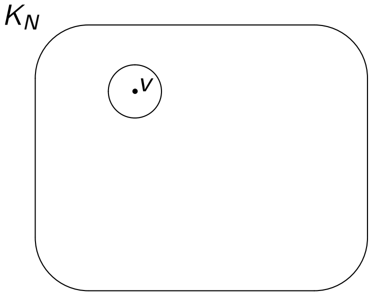

$$\frac{Cost[\text{Diluted Matching}]}{N/2} \xrightarrow{\text{p}} \beta_M(d,\theta)$$

Results

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$$\frac{Cost[\text{Matching}]}{N/2}$$

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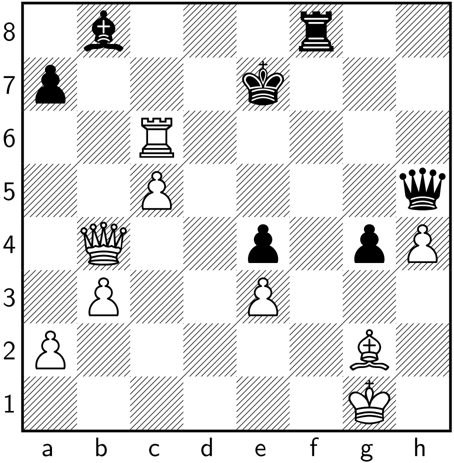
- $\beta_M(2) \approx 1.14351809919776$
- $\beta_{TSP}(2) \approx 1.285153753372032$ But how do we get results for the TSP? Let me explain by analogy to...

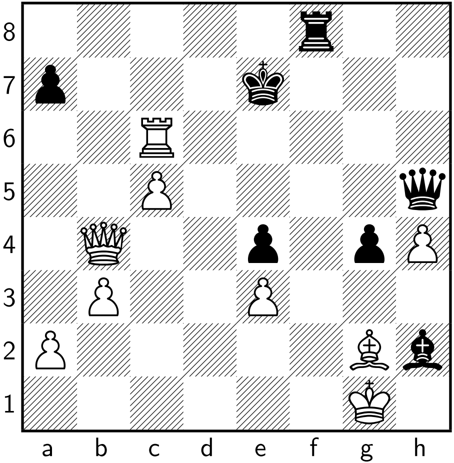
Refusal Chess



In Refusal Chess, when a player makes a move, the opponent can either accept (and play on) or refuse, in case the move is taken back and the player has to choose another move. A player has the right to refuse once per move.

For the chess players: A player is in check or checkmate if they would be in ordinary chess (so the rules are kind of illogical; for instance you cannot leave your king threatened just because you can refuse your opponent to capture it in the next move). If a player has only one legal move, the opponent cannot refuse it.

Refusal Chess



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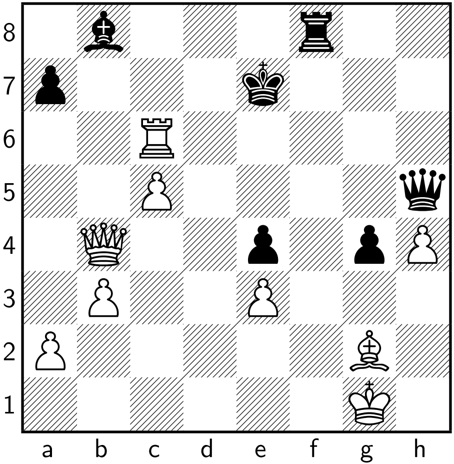
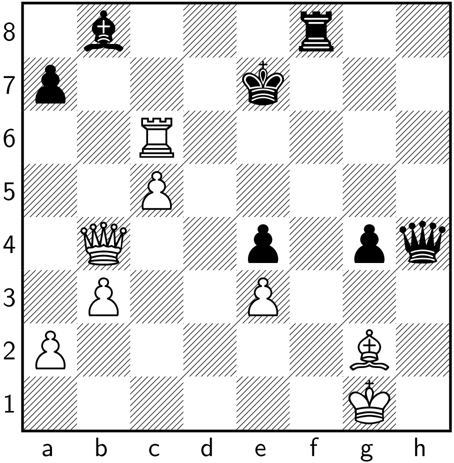


Refusal Chess



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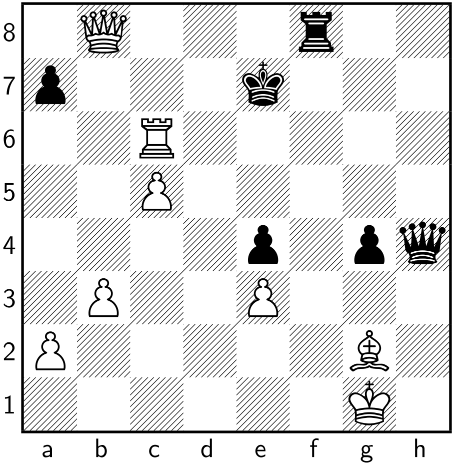
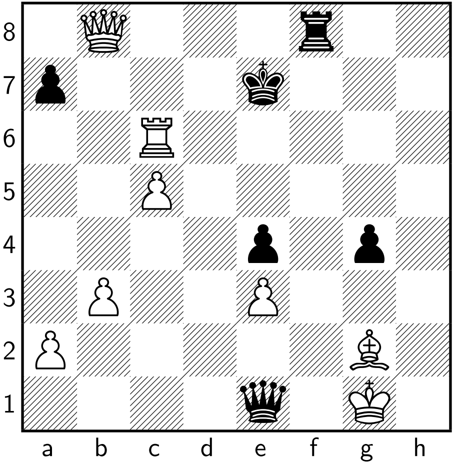


This would be a bad move in ordinary chess since Black can simply take back with the rook. But in refusal chess it is not so clear...

Refusal Chess

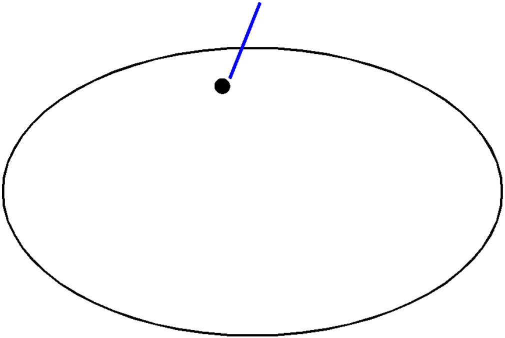


Black accepts White's move and plays on. Now White is in trouble, since refusing this move will allow Black to capture White's queen. But let us not analyze this particular position further.
The original position is taken from a spectacular finish by Jonathan Yedidia, one of the participants of the conference and former chess pro. He played 32 — Bh2+! and White resigned in view of 33. Kh1 Qxh4!, after which 34. Qb7+ (or any other move) is countered with a deadly discovered check.

My tour is better than yours!

Alice and Bob play "My tour is better than yours!"

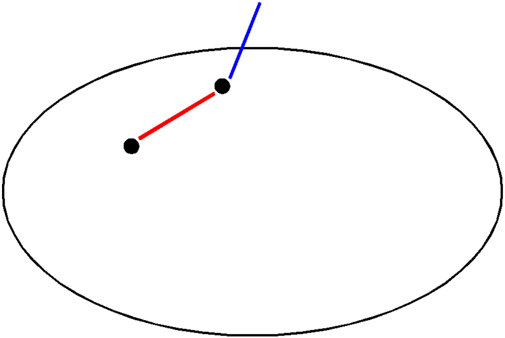
Bob has this edge in his tour



My tour is better than yours!

Alice and Bob play "My tour is better than yours!"

Alice says: "Good for you, but I have this edge in my tour!"



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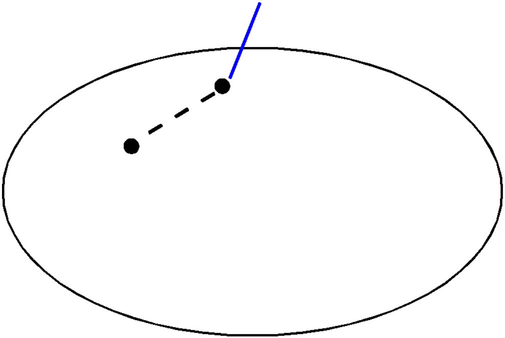
Alice and Bob play "My tour is better than yours!"

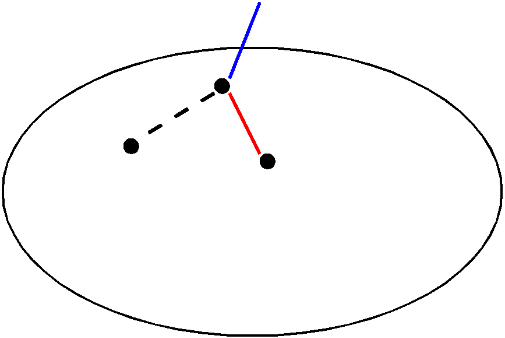
Bob says: "Well, so do I", effectively cancelling Alice's move (this is the difference from Graph Exploration)



My tour is better than yours!

Alice and Bob play "My tour is better than yours!"

"Allright", says Alice, "but I have this edge and you don't!"



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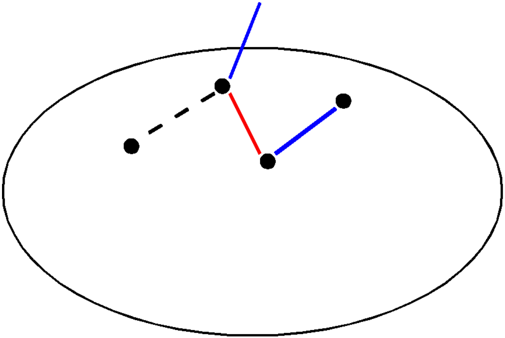
Bob has already admitted having two edges from that vertex, so he cannot cancel this move.



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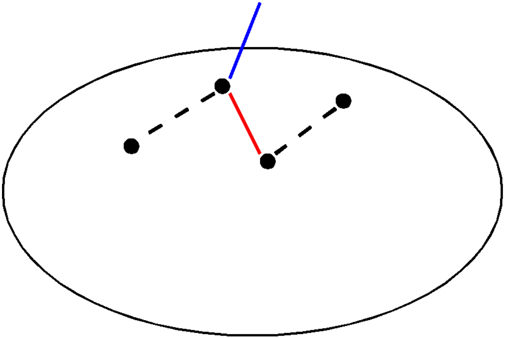
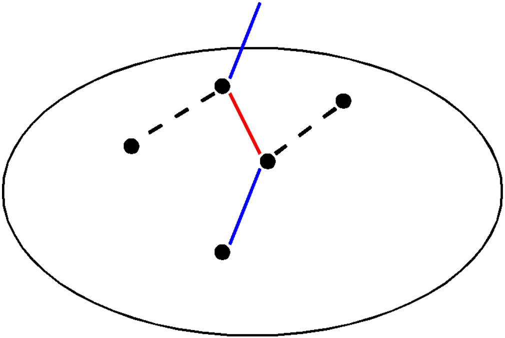
Similarly, Alice can refuse one of Bob's moves by claiming that she also has this edge in her tour.



My tour is better than yours!

Alice and Bob play “My tour is better than yours!”

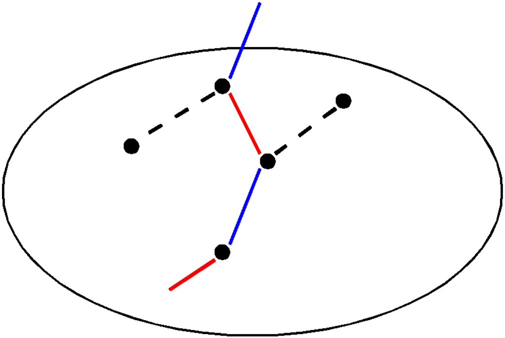
But if she does...



My tour is better than yours!

Alice and Bob play "My tour is better than yours!"

...she will have to accept Bob's second move, and so on...



Here I am sweeping a number of details under the rug, but the structure of the game is the same as in Refusal Chess (you always play your second best move; we can call the game "Refusal Exploration"). The results for this game and the conclusions for the TSP are analogous to the results for matching.

The end is near...

Future work

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- $0 < d < 1$

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- Games for other optimization problems (edge cover?)

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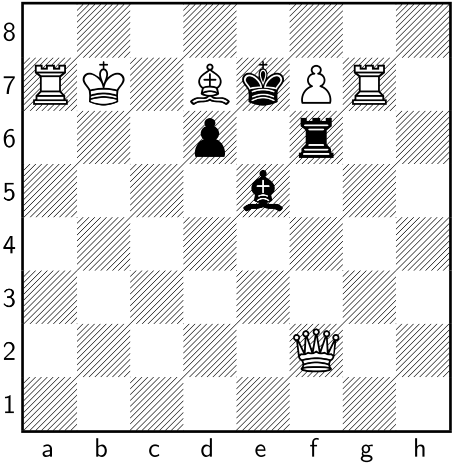
- RS holds for Chess but not Go???

Refusal Chess Problem



Mate in 2: (a) Ordinary chess (b) Refusal chess.

For the chess players: This is a two-mover with one solution for ordinary chess and a different solution for Refusal Chess. In ordinary chess, the solution is 1. Ra8, with the variations 1. — Rxf2, 2. f8Q, and 1. — Kxd7, 2. f8N.
In Refusal Chess the two keys are 1. f8R+ forbidding Kxf8, and 1. f8B+ again forbidding Kxf8. On 1. f8R+ Rf7 the mating moves are 2. Qxf7 and 2. Rgxf7, while on 1. f8B+ Kd8, the mating moves are 2. Qb6 and 2. Ra8.
So the problem is a so-called *Allumwandlung*: The white pawn promotes to queen and knight in ordinary chess, and to rook and bishop in refusal chess.
Notice that in Refusal, 1. f8Q+ doesn't work since White then cannot refuse 1. — Kxf8.