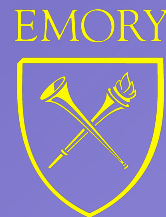


# Extremal Optimization: Dynamics and Results

Stefan  
Boettcher

[www.physics.emory.edu/faculty/boettcher/](http://www.physics.emory.edu/faculty/boettcher/)



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### Collaborator:

- ▶ Allon Percus (Los Alamos/Claremont Graduate U)

### Funding:

- ▶ NSF-DMR, Los Alamos-LDRD, Emory-URC

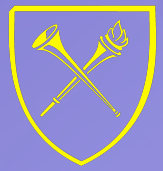
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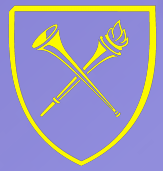




# Overview:

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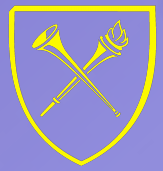




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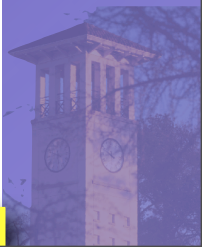
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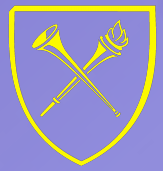




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  - Dilute Edwards-Anderson in  $d=3, \dots, 8$

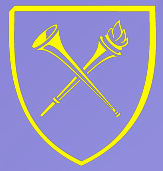




# Extremal Optimization (EO)

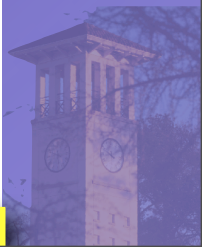
- Motivated by Self-Organized Criticality





# Extremal Optimization (EO)

- ◉ Motivated by Self-Organized Criticality
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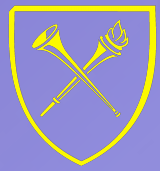




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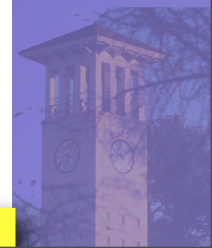
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  - Extremal Driving:
    - ★ Select and eliminate the “bad”,
    - ★ Replace it *at random*,
    - ★ Eventually, only the “good” is left!





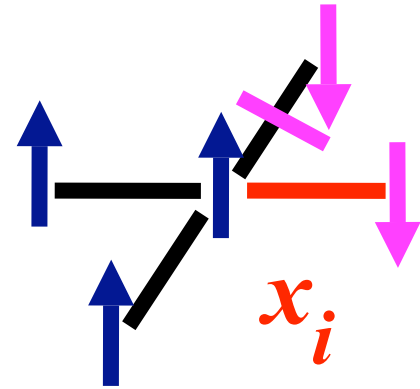
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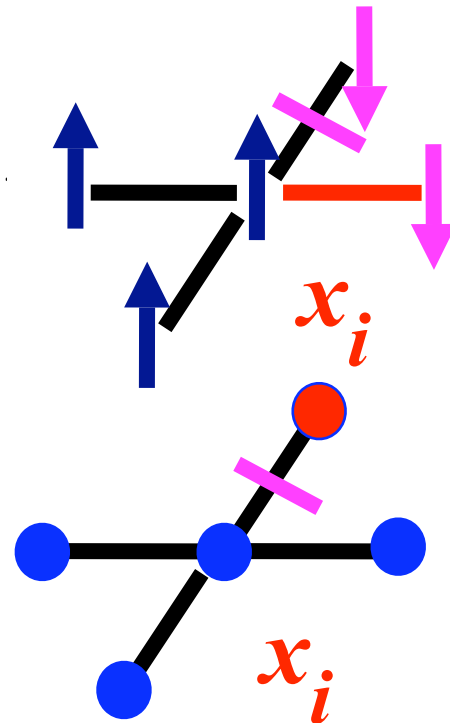
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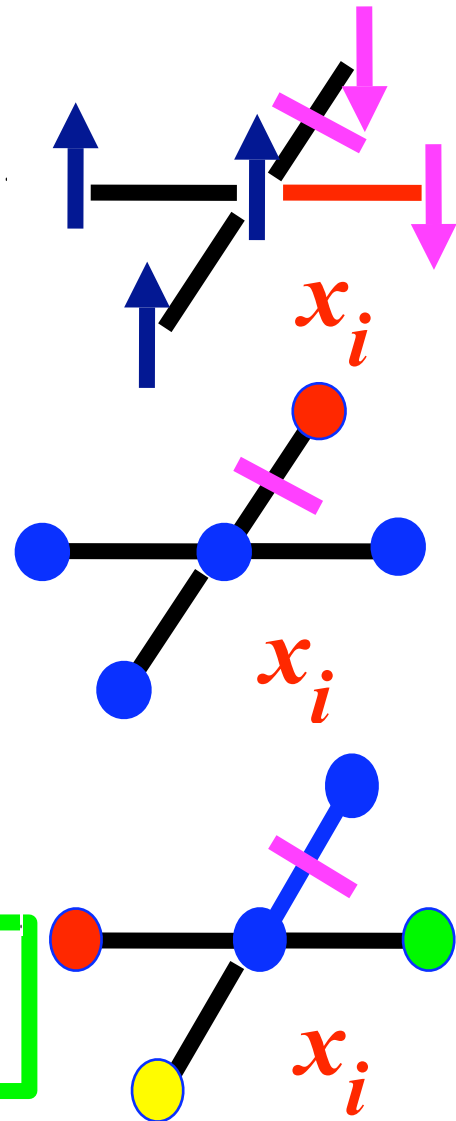
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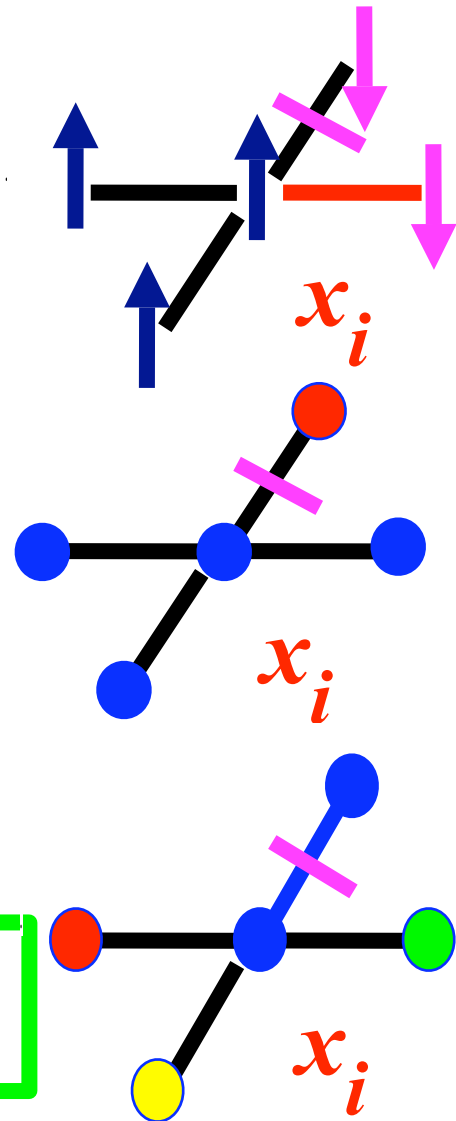
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$$Cost \propto H = -\sum_i \lambda_i$$



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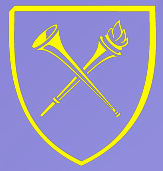
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# Typical Extremal Optimization Run:

EO-run for Partitioning ( $n=500$ ):

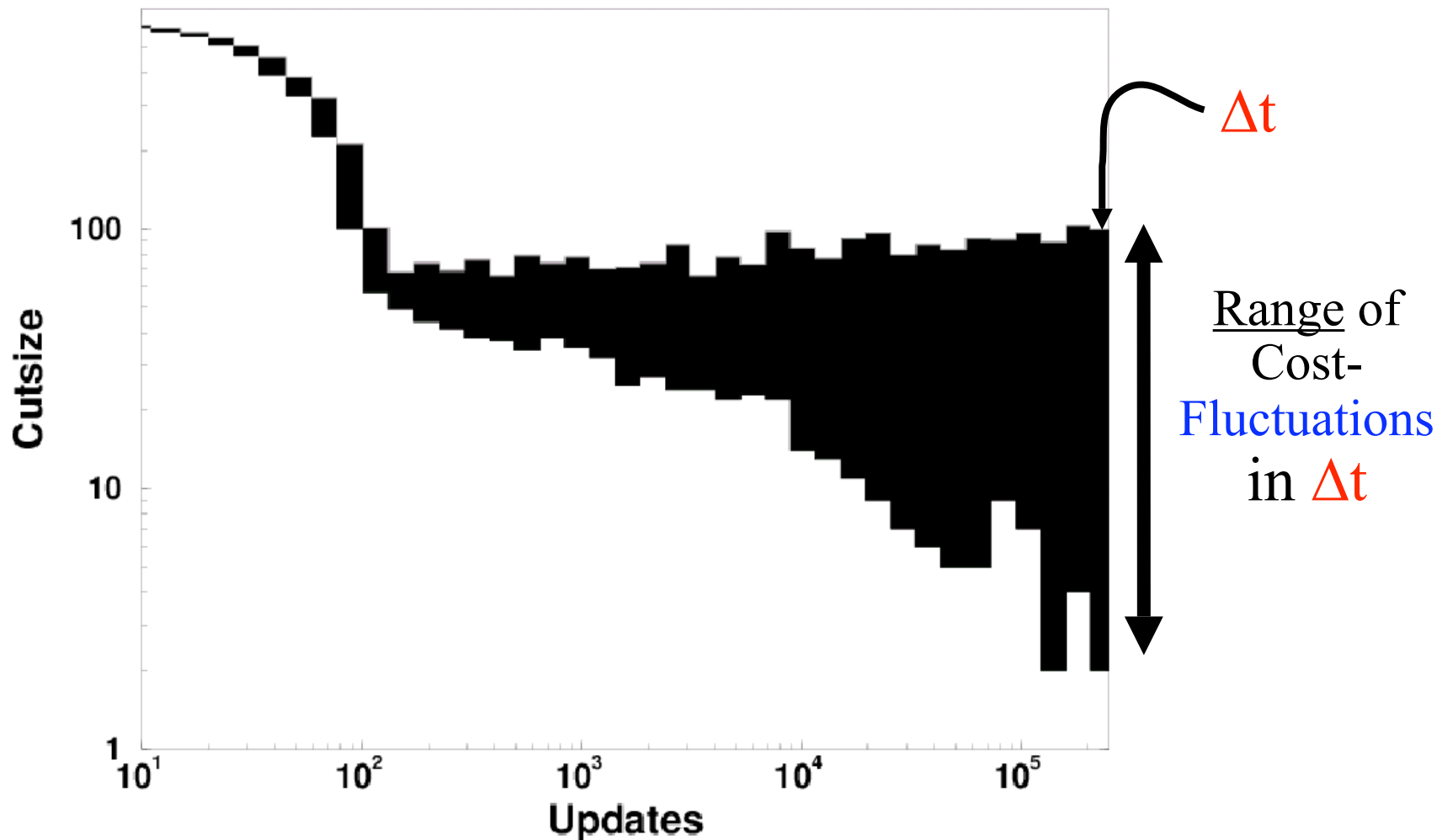


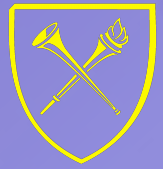




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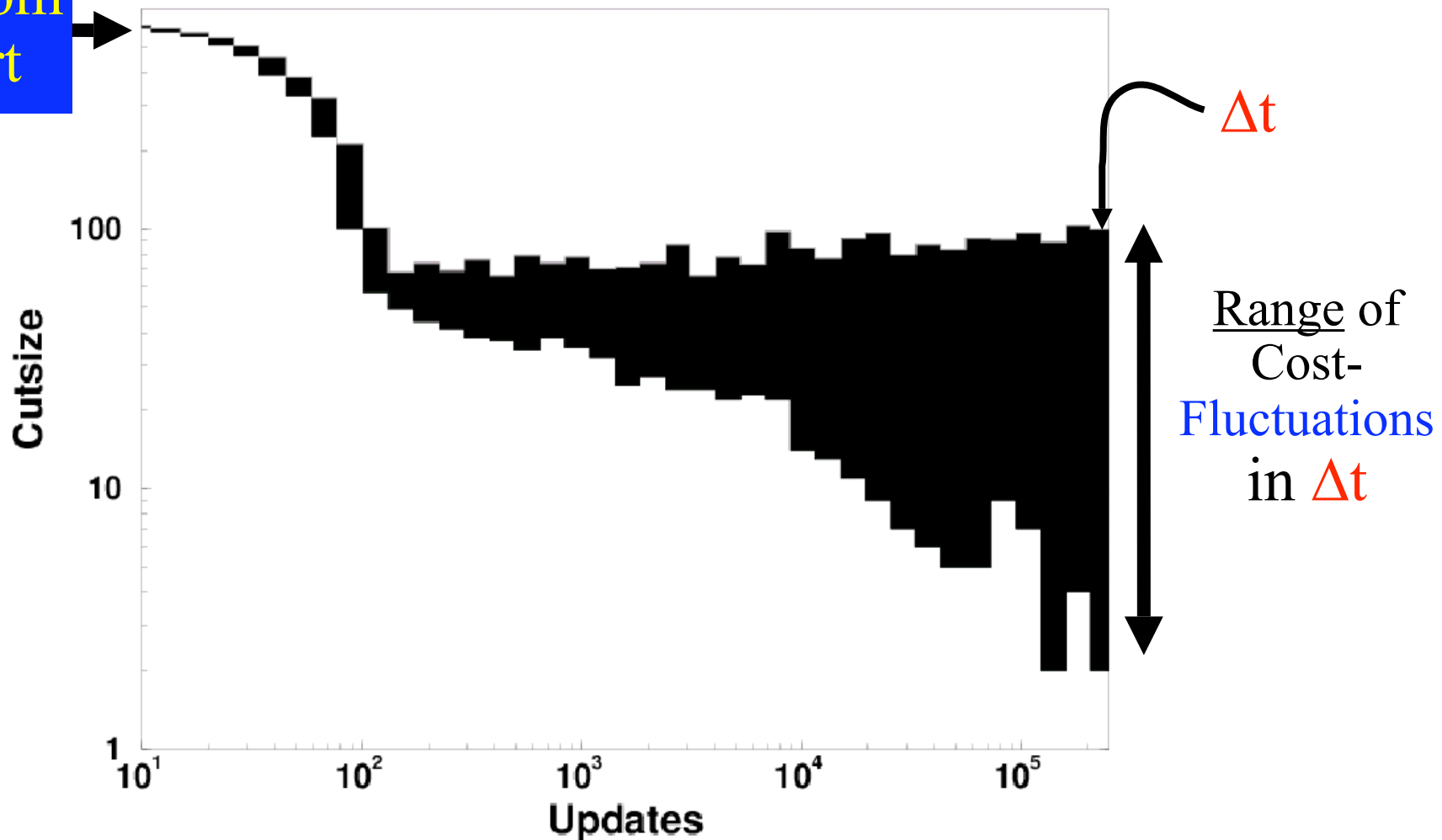




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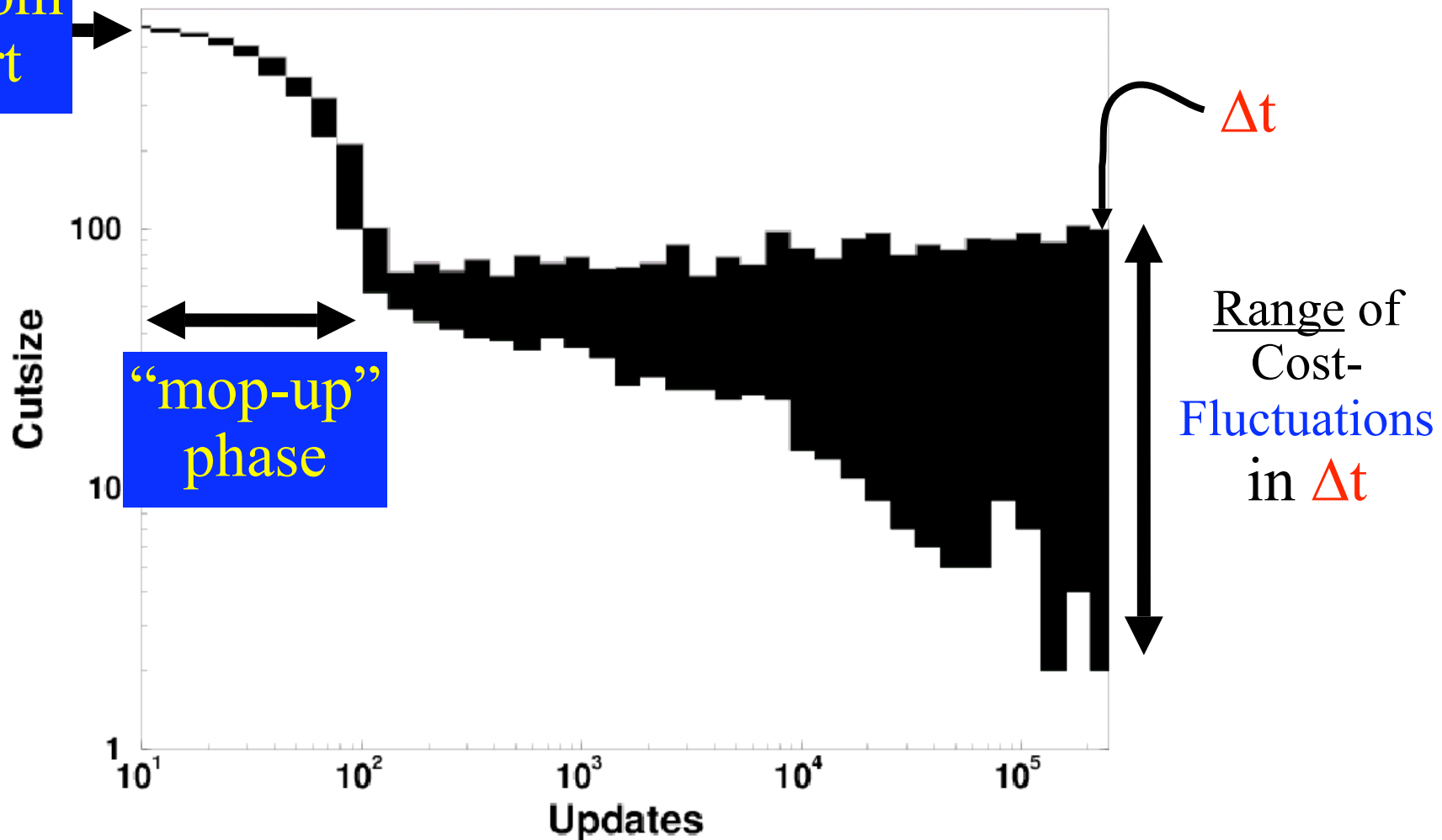
random  
start



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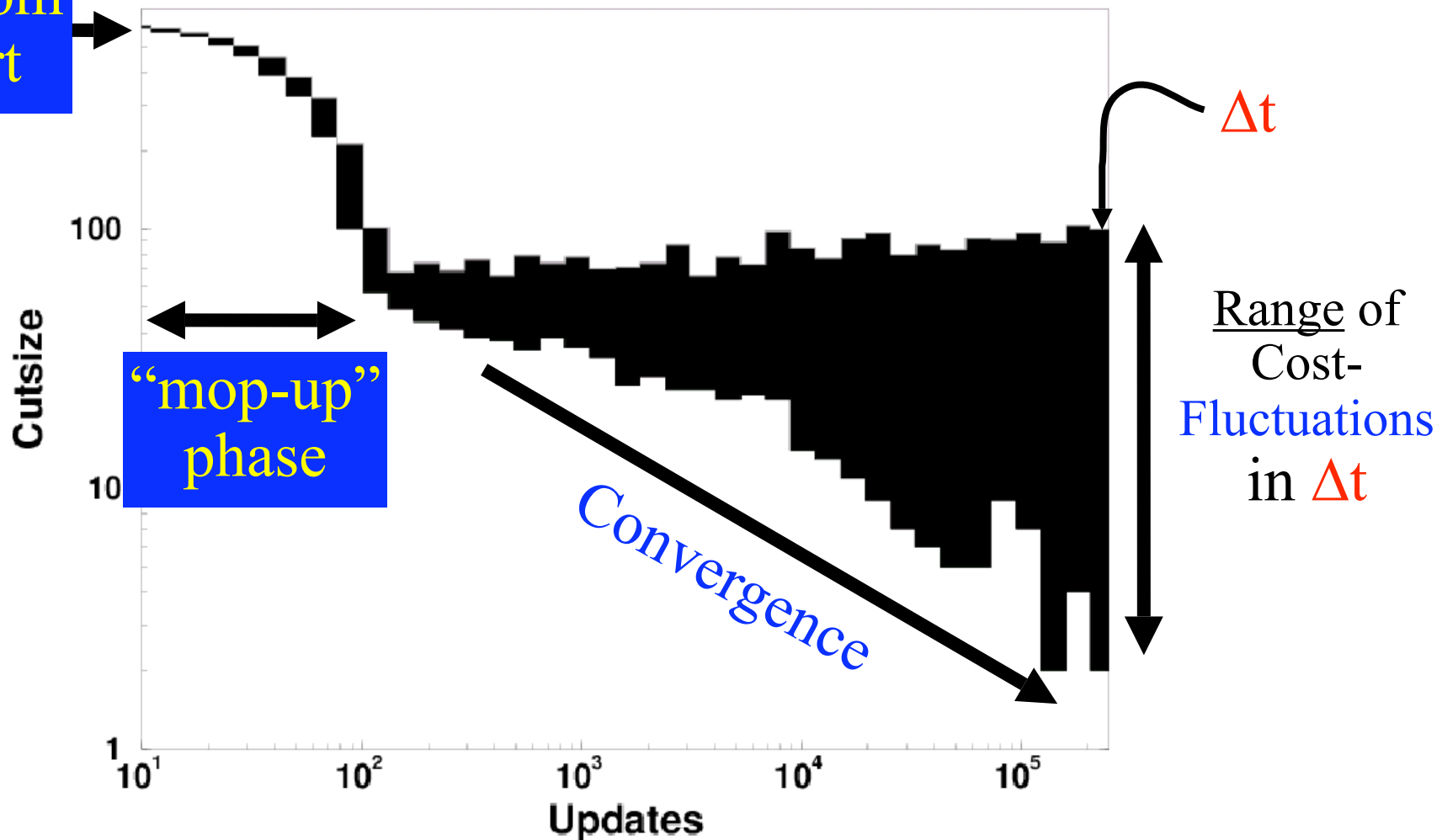
random  
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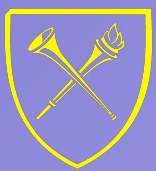


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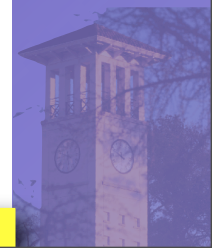
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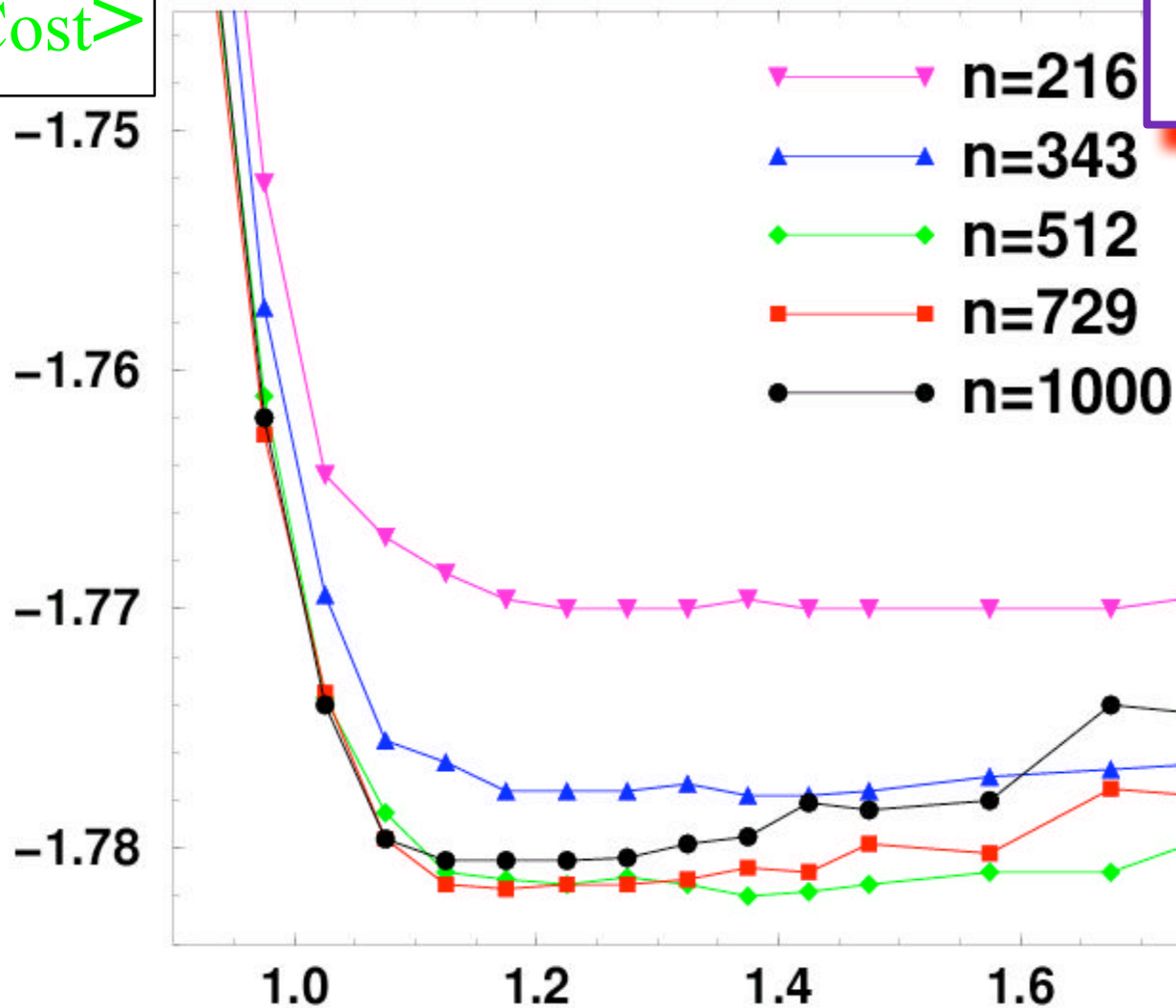
For Ranks  $\lambda_{\Pi(1)} \leq \dots \leq \lambda_{\Pi(n)}$ , update  $i = \Pi(k)$  with  
scale-free, power-law distribution

$$P(k) \propto k^{-\tau}$$

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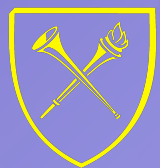
$\langle \text{Cost} \rangle$



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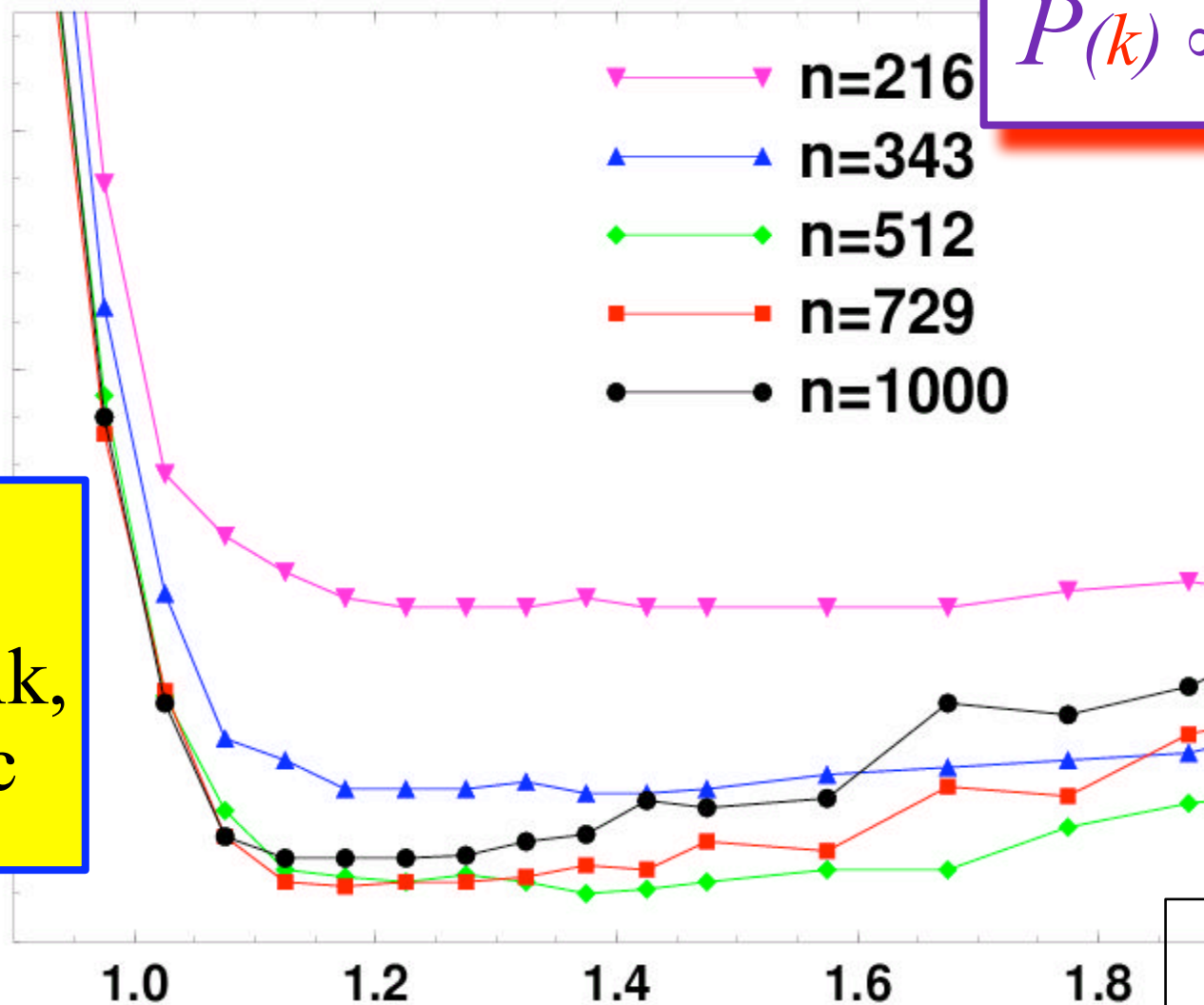
$\langle \text{Cost} \rangle$

$$P(k) \propto k^{-\tau}$$

-1.75

-1.76

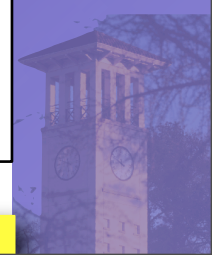
- $\nabla$  n=216
- $\blacktriangle$  n=343
- $\blacklozenge$  n=512
- $\blacksquare$  n=729
- $\bullet$  n=1000

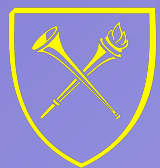


$$0 \leftarrow \tau$$

random walk,  
too ergodic

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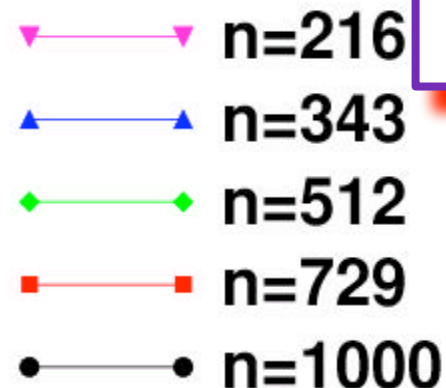
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$$\tau \rightarrow \infty$$

greedy + frozen,  
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1.0

1.2

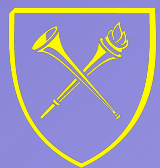
1.4

1.6

1.8

$\tau$



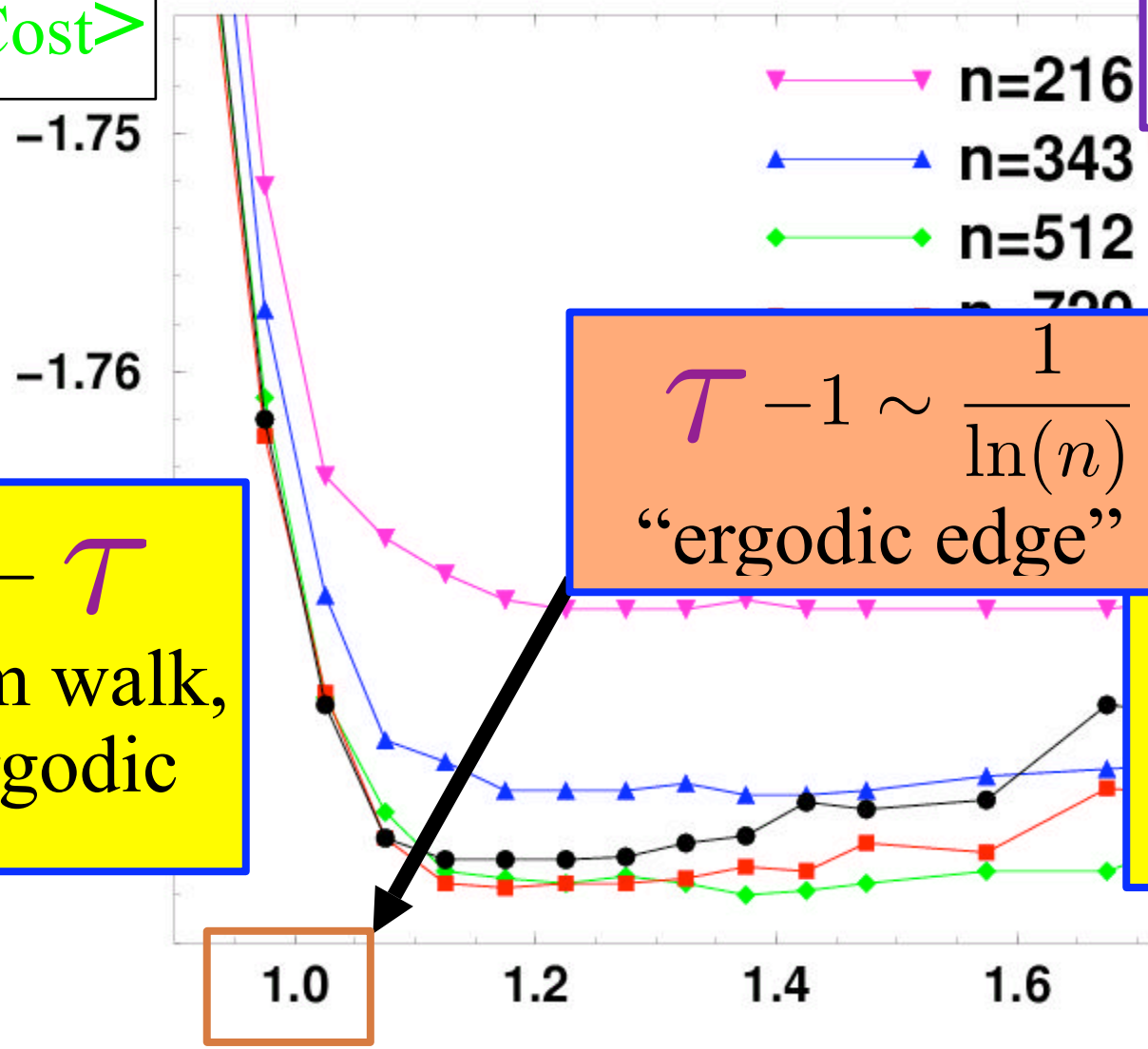


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$$\tau - 1 \sim \frac{1}{\ln(n)}$$

“ergodic edge”

$0 \leftarrow \tau$   
random walk,  
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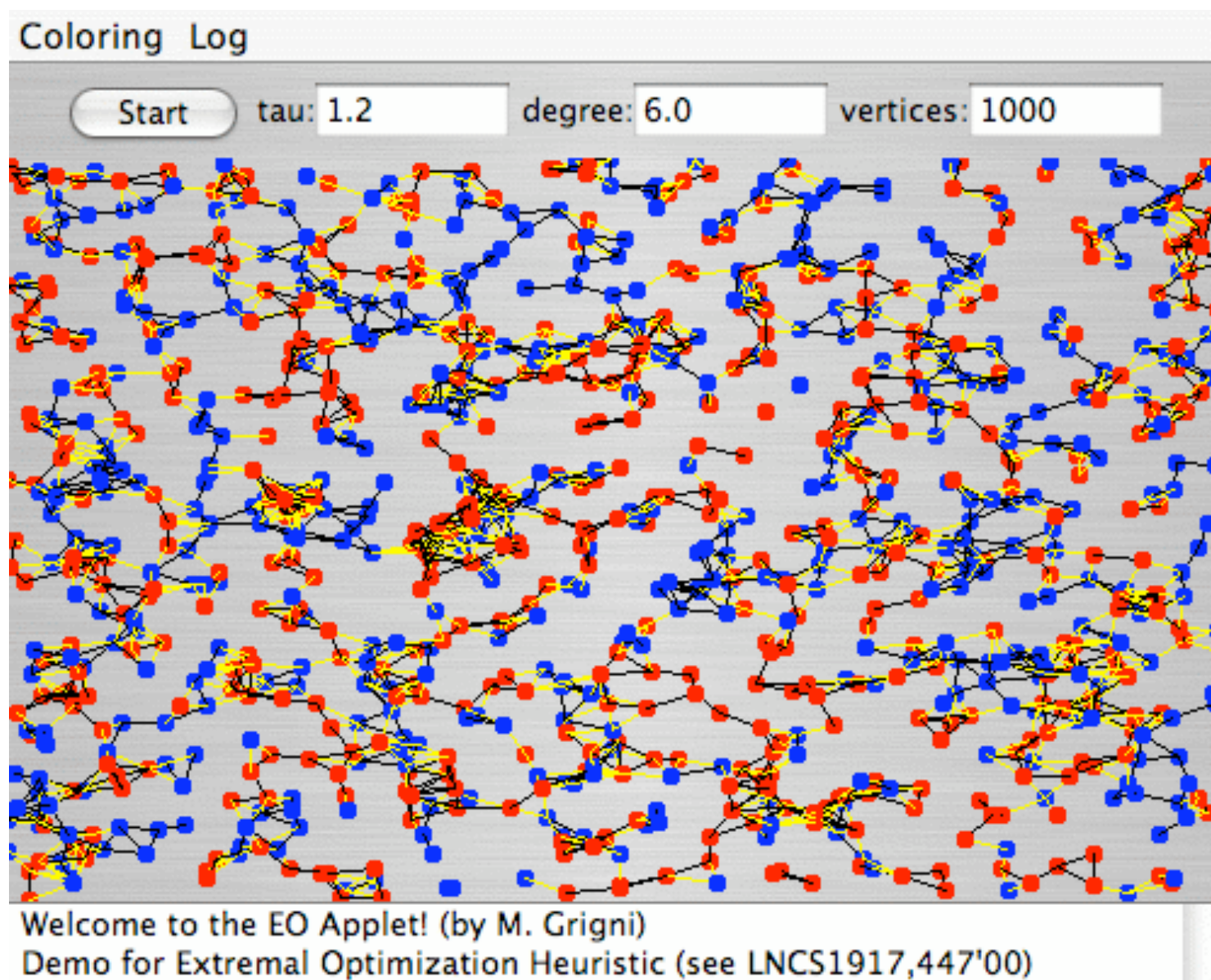
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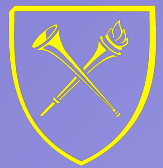
$\tau$



# Animation of $\tau$ -EO for Graph-Partitioning

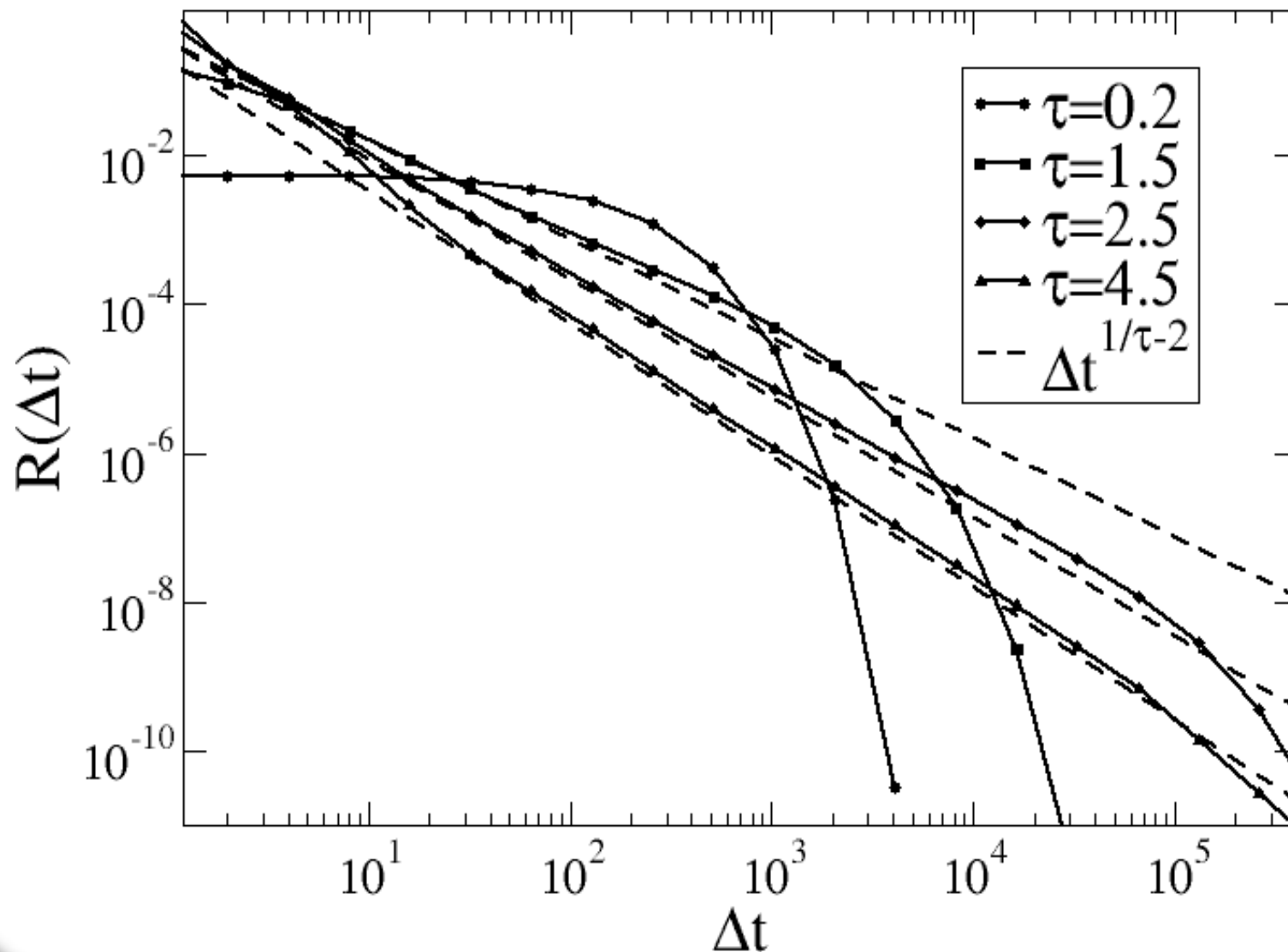


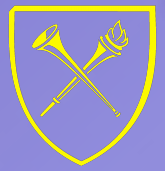




# Dynamics of $\tau$ -EO: (for $\pm J$ -Spin Glass on 3-reg. Graph, $N=256$ )

First-return time distribution  $R(\Delta t)$ :





# Dynamics of $\tau$ -EO:

## Derivation of First-return time distribution $R(\Delta t)$ :

Have: Total number of updates  $T$

Then, number of updates at rank  $k$  is:

$$n(k) = T P(k)$$

Typical lifetime of variable with rank  $k$  is:

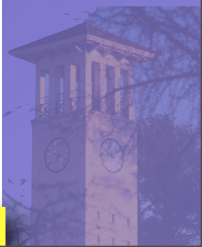
$$\Delta t(k) \sim \frac{T}{n(k)} = \frac{1}{P(k)}$$

With  $R(\Delta t)d(\Delta t) = P(k)dk$ :

$$R(\Delta t) = P(k) \frac{dk}{d\Delta t} \sim -\frac{P(k)^3}{P'(k)}$$

With  $P(k) \sim k^{-\tau}$  :

$$R(\Delta t) \sim \Delta t^{\frac{1}{\tau}-2}$$

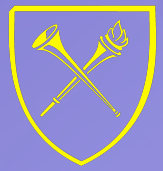




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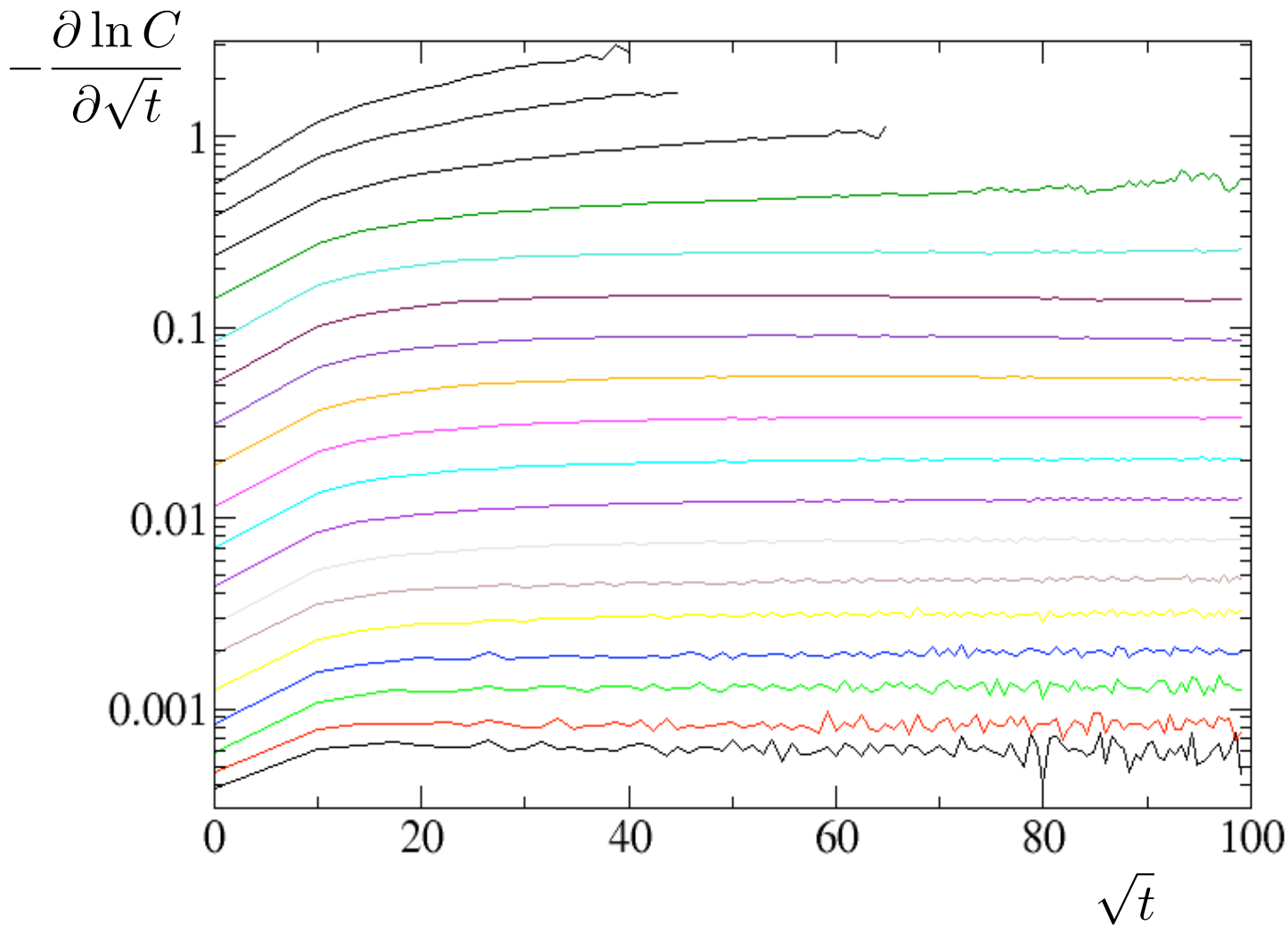
Stretched-exponential Autocorrelations:  $C(t) \sim \exp \left[ -B_\tau \sqrt{t} \right]$

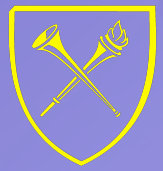




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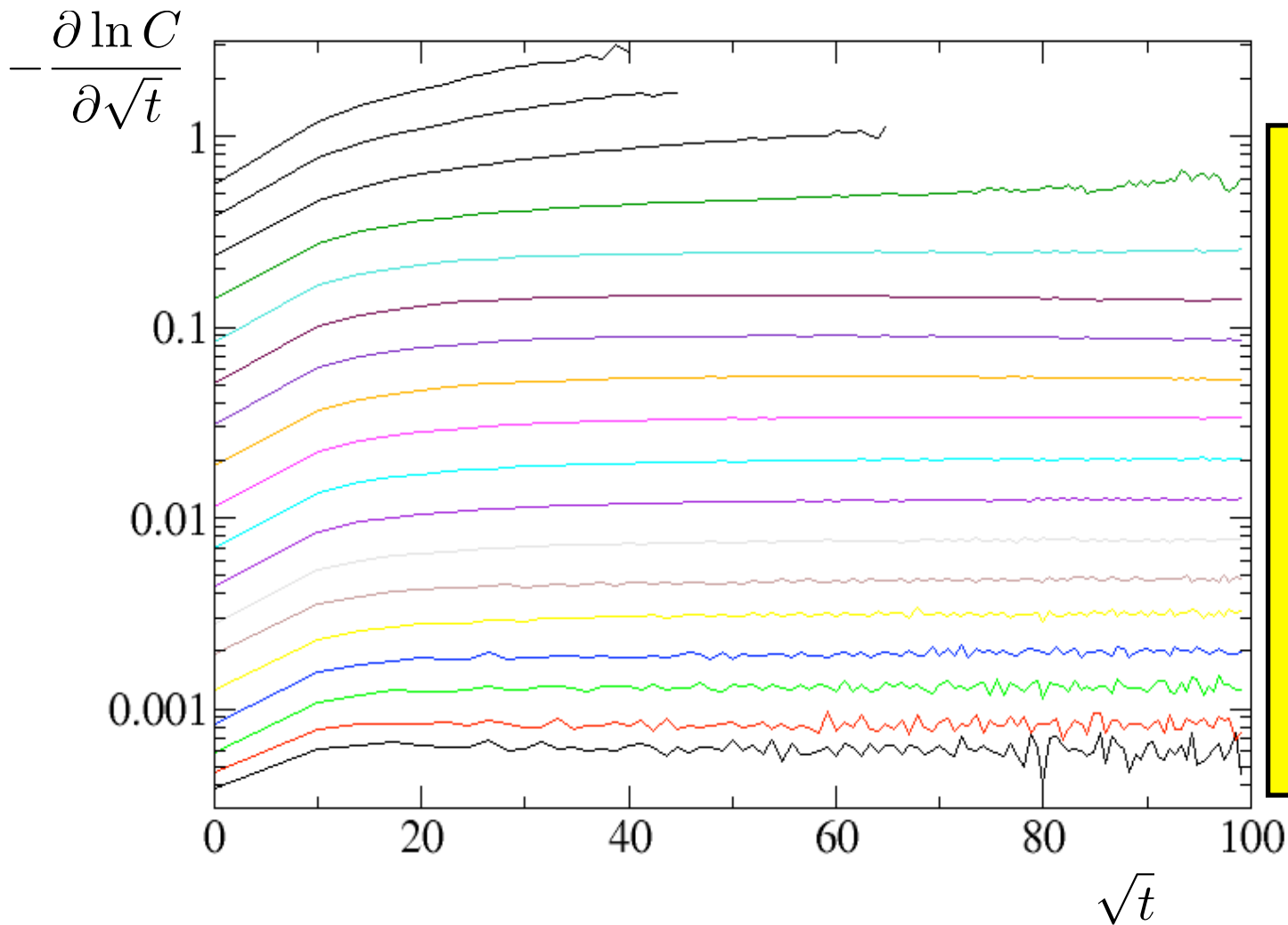
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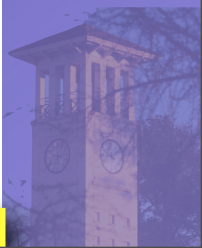
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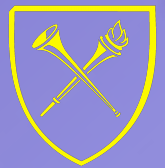


$$\tau = 1.1$$

$$B_\tau \sim -\frac{\partial \ln C(t)}{\partial \sqrt{t}} \\ \sim 1.6 e^{-2.4\tau}$$

$$\tau = 3.9$$





# Jamming Model for $\tau$ -EO :

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Let: Only 3 states  $s$  for each  $x_i$ ,

$$\lambda_i = -s, \quad s \in \{0, 1, 2\},$$

density of variables  $x_i$  in state  $s$ :

$$\rho_s(t) = \frac{1}{n} |\{i | \lambda_i = -s\}|,$$

Cost function:

$$e(t) = \sum_{s=0}^2 s \rho_s(t),$$

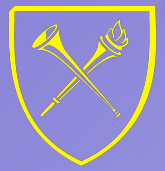
Annealed Flow Equation:

$$\rho_r(t+1) = \rho_r(t) + \sum_{s=0}^2 T_{r,s} Q_s,$$

where

- $Q_s(\{\rho(t)\})$  = Prob. to update variable in state  $s$ ,
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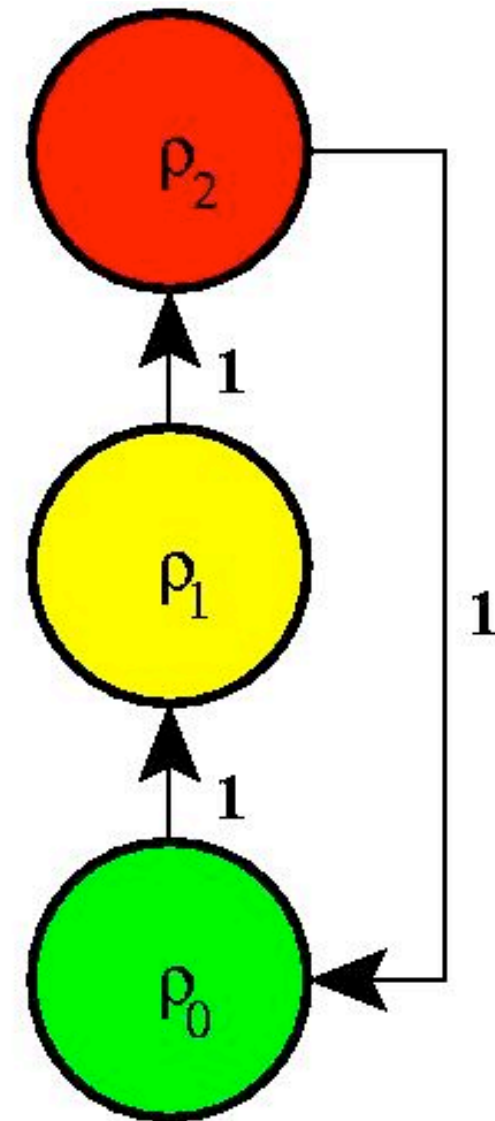
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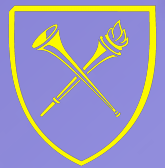
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**Flow up**



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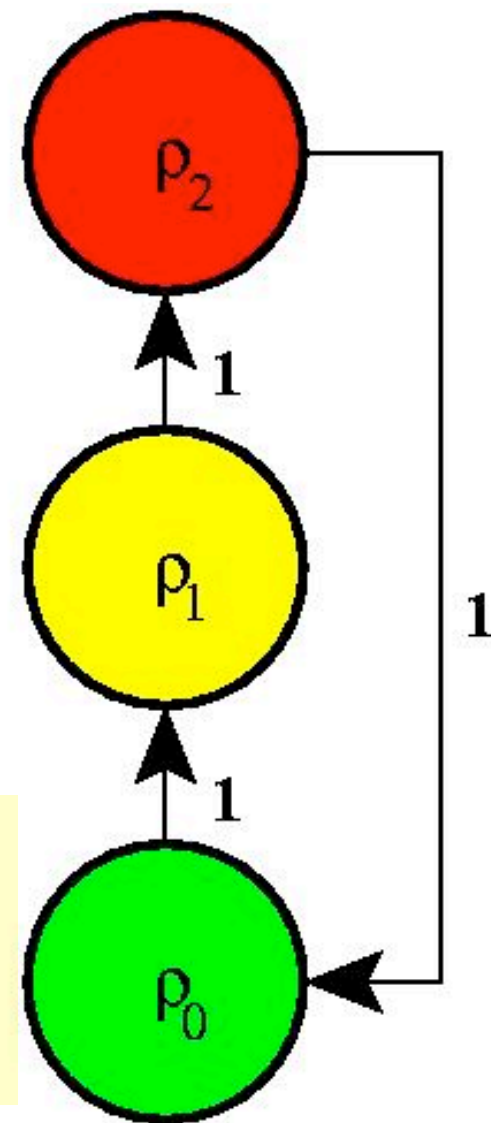
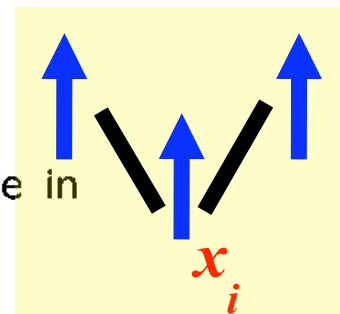
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$$\rho_r(t+1) = \rho_r(t) + \sum_{s=0}^2 T_{r,s} Q_s,$$

where

- $Q_s(\{\rho(t)\})$  = Prob. to update variable in state  $s$ ,
- $T_{r,s}(\{\rho(t)\})$  = Flow of variables to state  $r$ , if variable in state  $s$  is updated.



**Flow up**



# Jamming Model for $\tau$ -EO :

## Jamming Model for $\tau$ -EO

Let: Only 3 states  $s$  for each  $x_i$ ,

$$\lambda_i = -s, \quad s \in \{0, 1, 2\},$$

density of variables  $x_i$  in state  $s$ :

$$\rho_s(t) = \frac{1}{n} |\{i | \lambda_i = -s\}|,$$

Cost function:

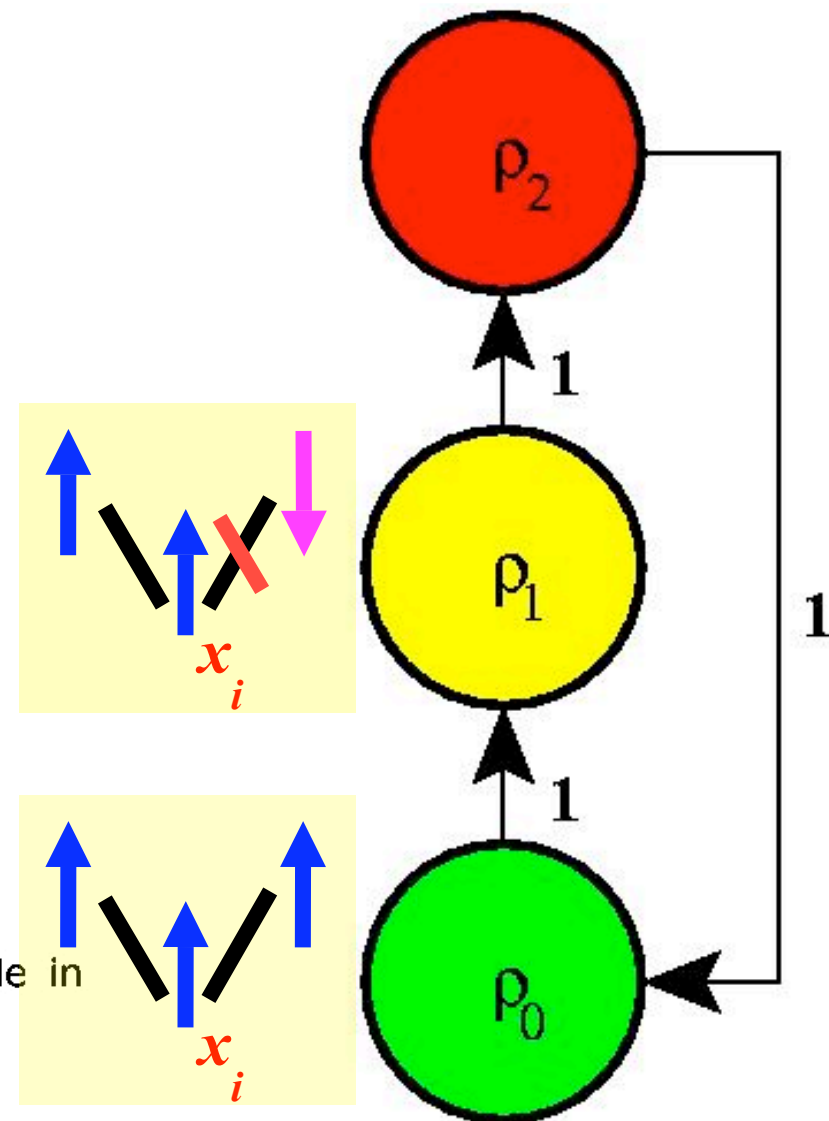
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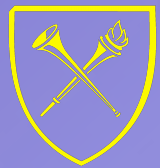
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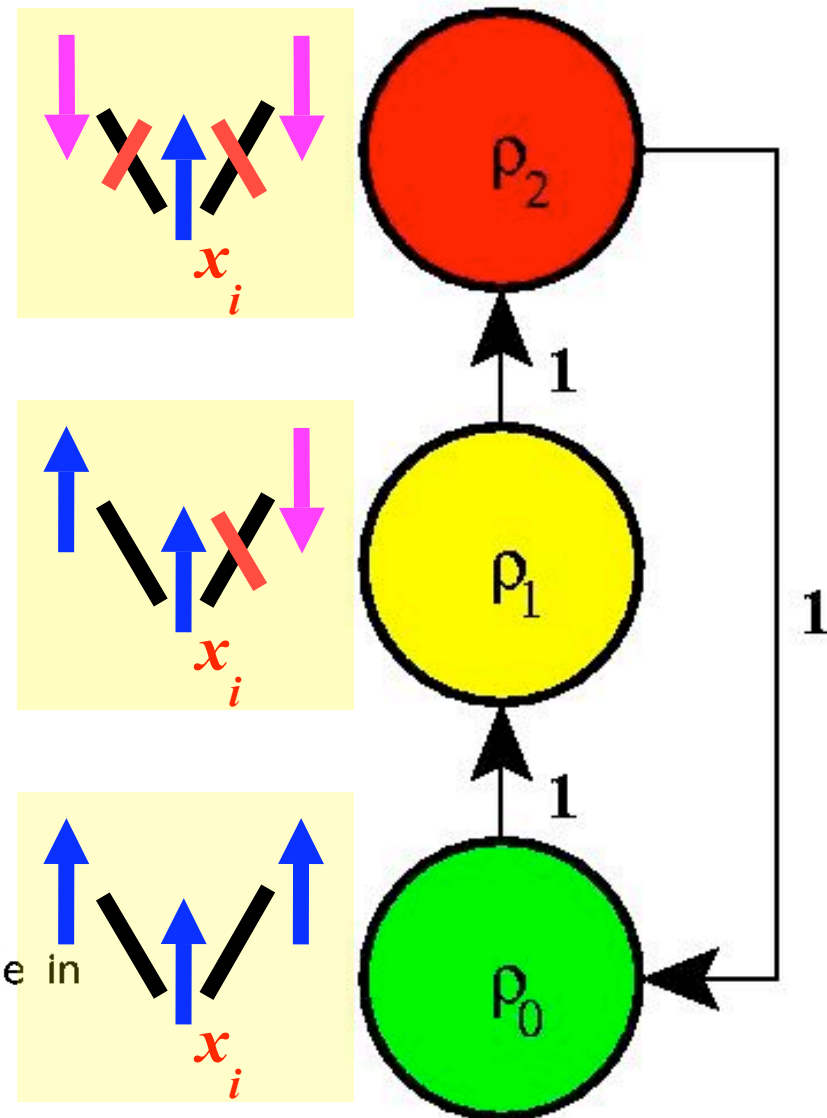
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**Flow up**



# $\tau$ -EO Eq. for Jammed Flow:

## Flow Equation for a Jam

$$\dot{\rho}_0 = \frac{1}{n} \left[ -Q_0 + \frac{1}{2} Q_1 \right],$$

$$\dot{\rho}_1 = \frac{1}{n} \left[ \frac{1}{2} Q_0 - Q_1 + (\theta - \rho_1) Q_2 \right],$$

$$\dot{\rho}_2 = \frac{1}{n} \left[ \frac{1}{2} Q_0 + \frac{1}{2} Q_1 - (\theta - \rho_1) Q_2 \right],$$

$$1 = \rho_0 + \rho_1 + \rho_2.$$

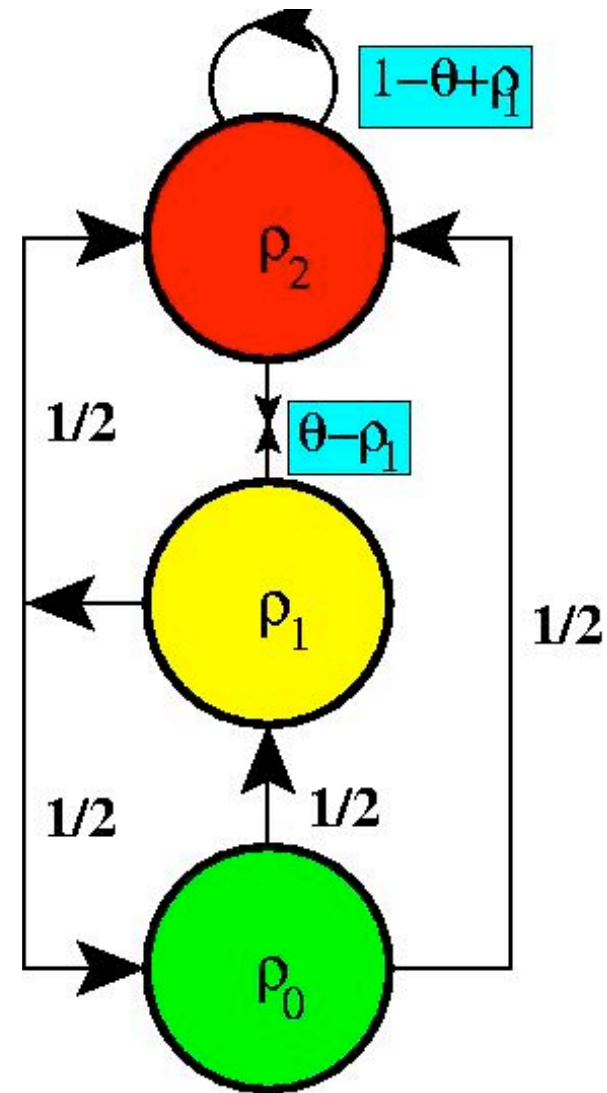
For  $\tau$ -EO:

$$Q_2 = \int_{1/n}^{\rho_2} dx \frac{\tau - 1}{n^{\tau-1} - 1} x^{-\tau}$$

$$= \frac{1}{1 - n^{\tau-1}} \left[ \rho_2^{1-\tau} - n^{\tau-1} \right]$$

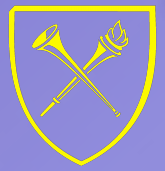
$$Q_1 = \frac{1}{1 - n^{\tau-1}} \left[ (1 - \rho_0)^{1-\tau} - \rho_2^{1-\tau} \right]$$

$$Q_0 = \frac{1}{1 - n^{\tau-1}} \left[ 1 - (1 - \rho_0)^{1-\tau} \right]$$

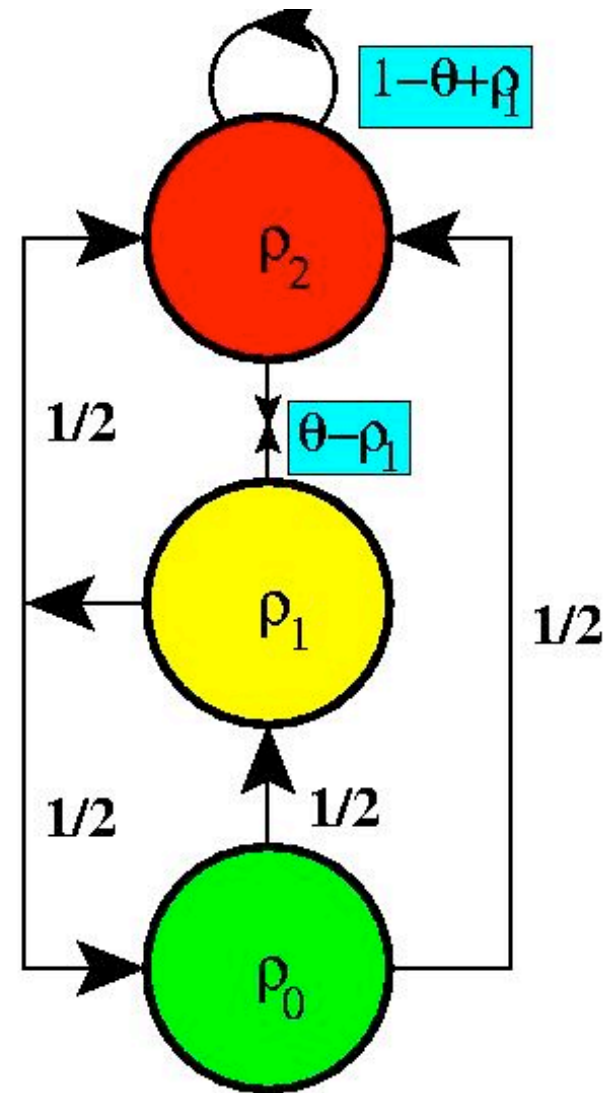


Flow jam

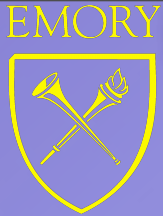




# $\tau$ -EO Evolution for Jammed Flow:



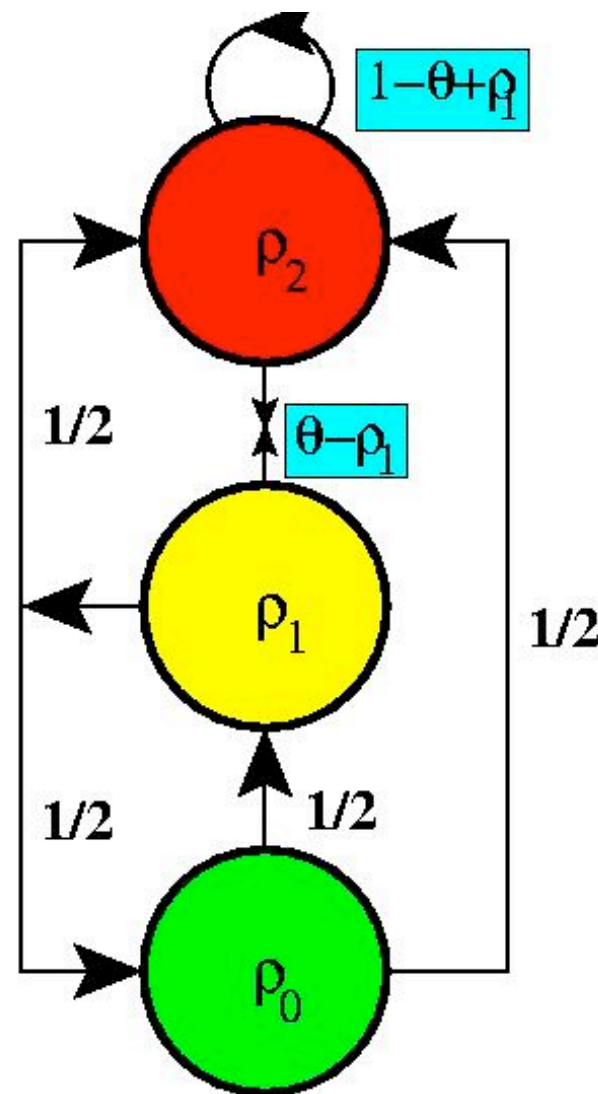
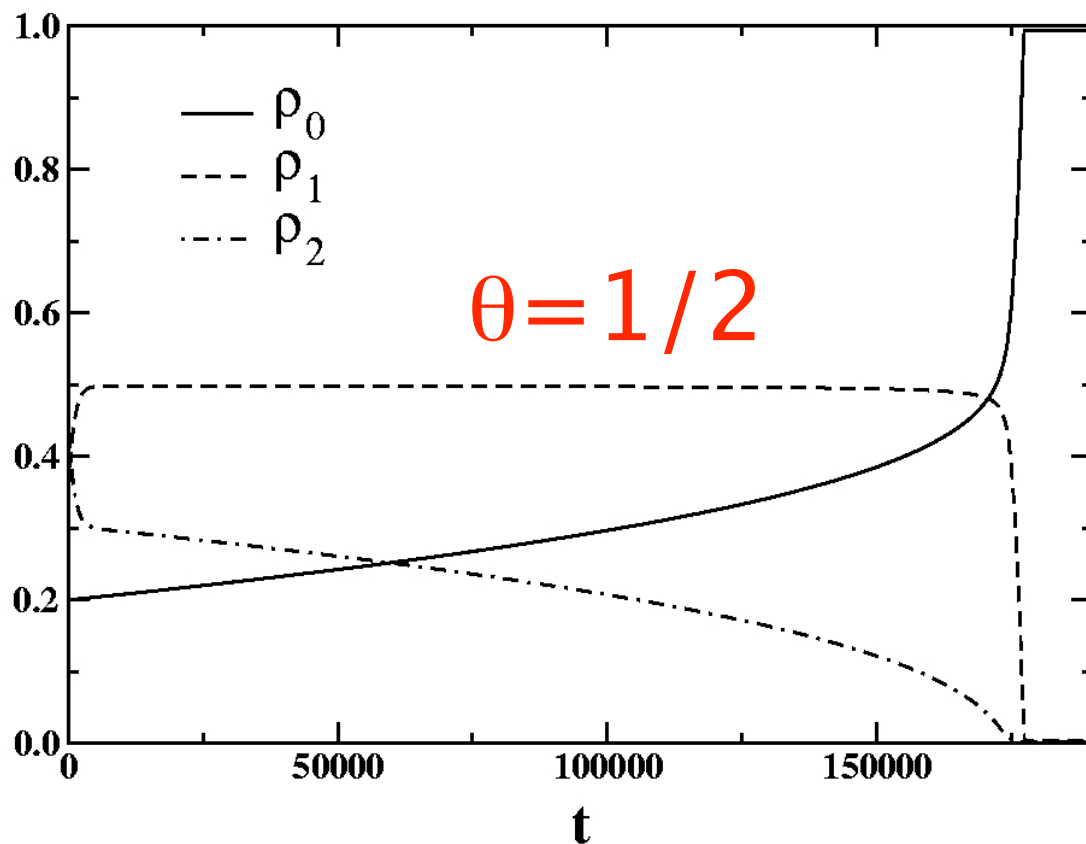
Flow jam



# $\tau$ -EO Evolution for Jammed Flow:

$\tau = 2.5$

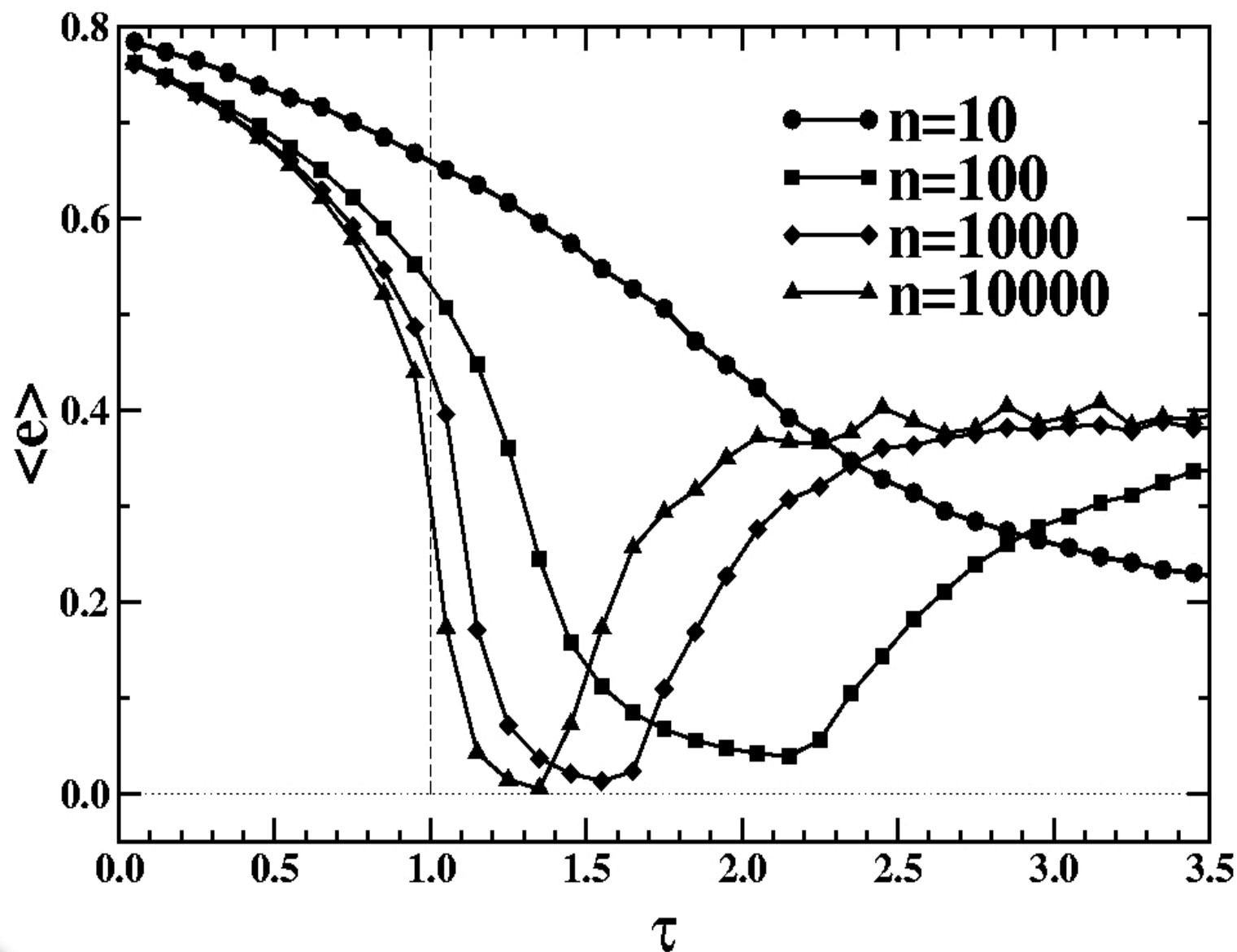
$\theta = 1/2$

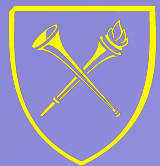


Flow jam

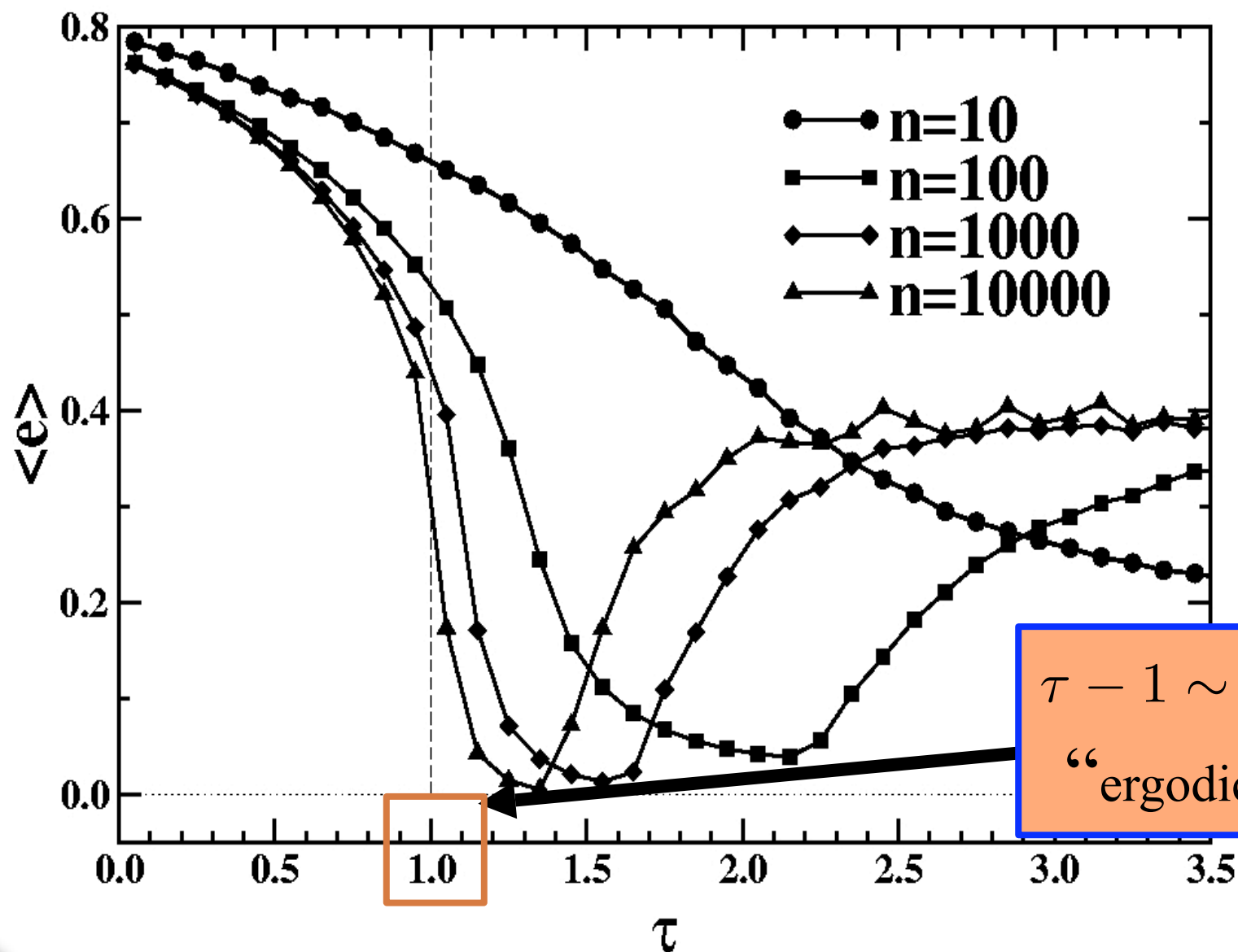


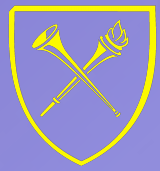
# Optimal Choice for Jammed $\tau$ -EO:





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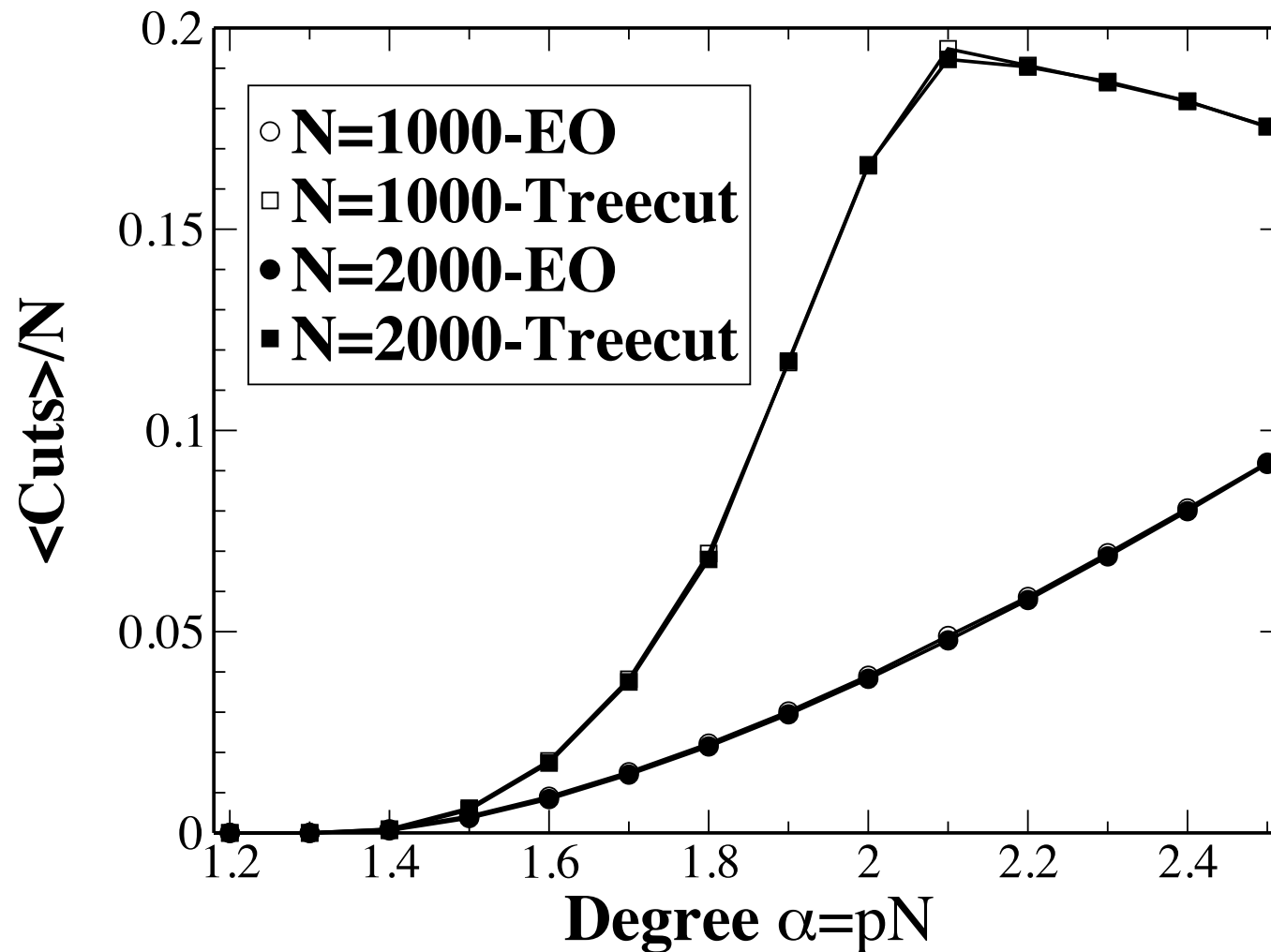


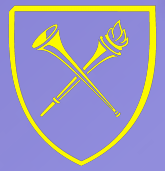


## Results for $\tau$ -EO:

### • For Graph Bi-Partitioning:

Random Graph Bi-Partitioning

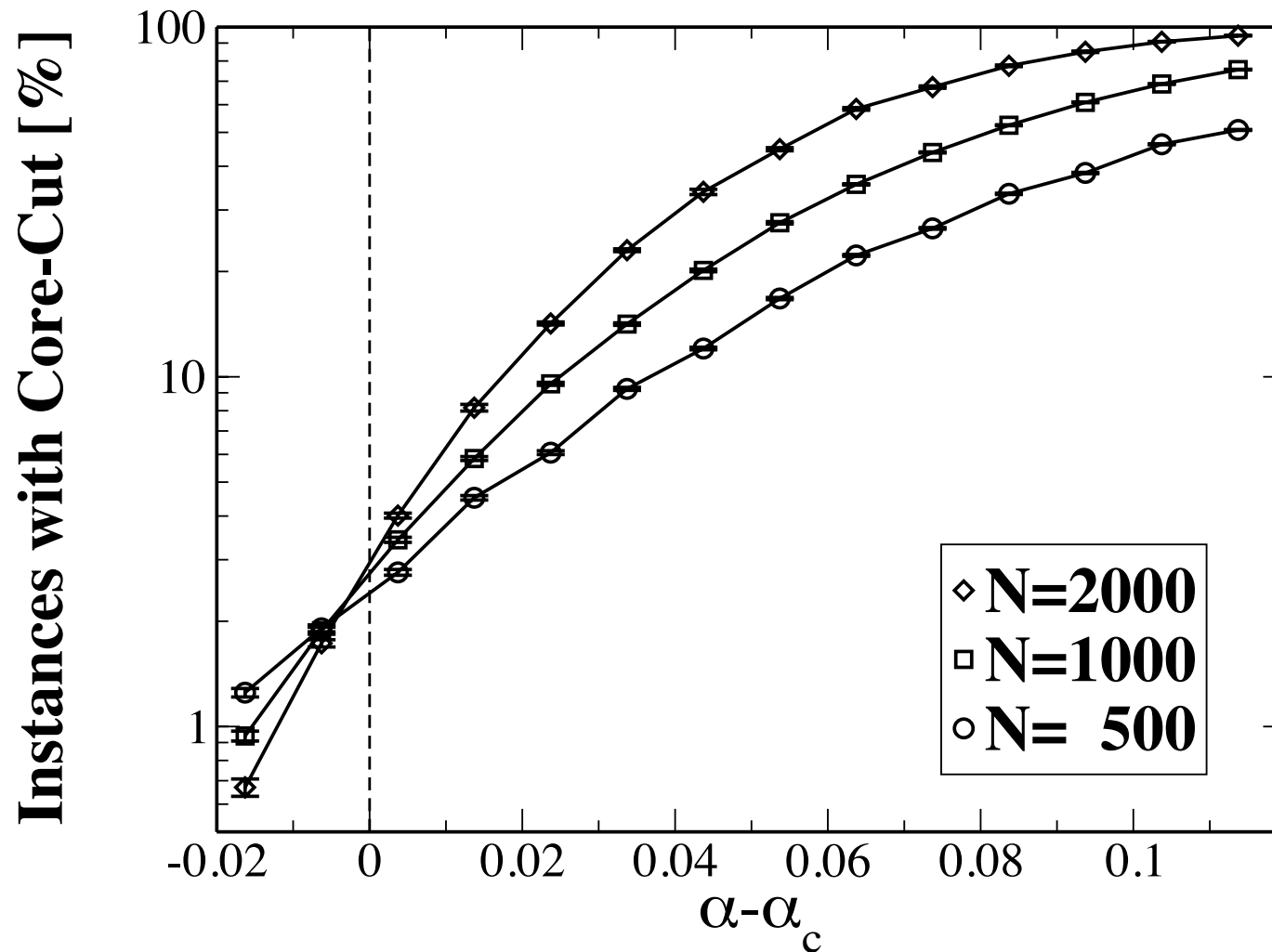


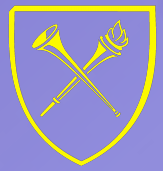


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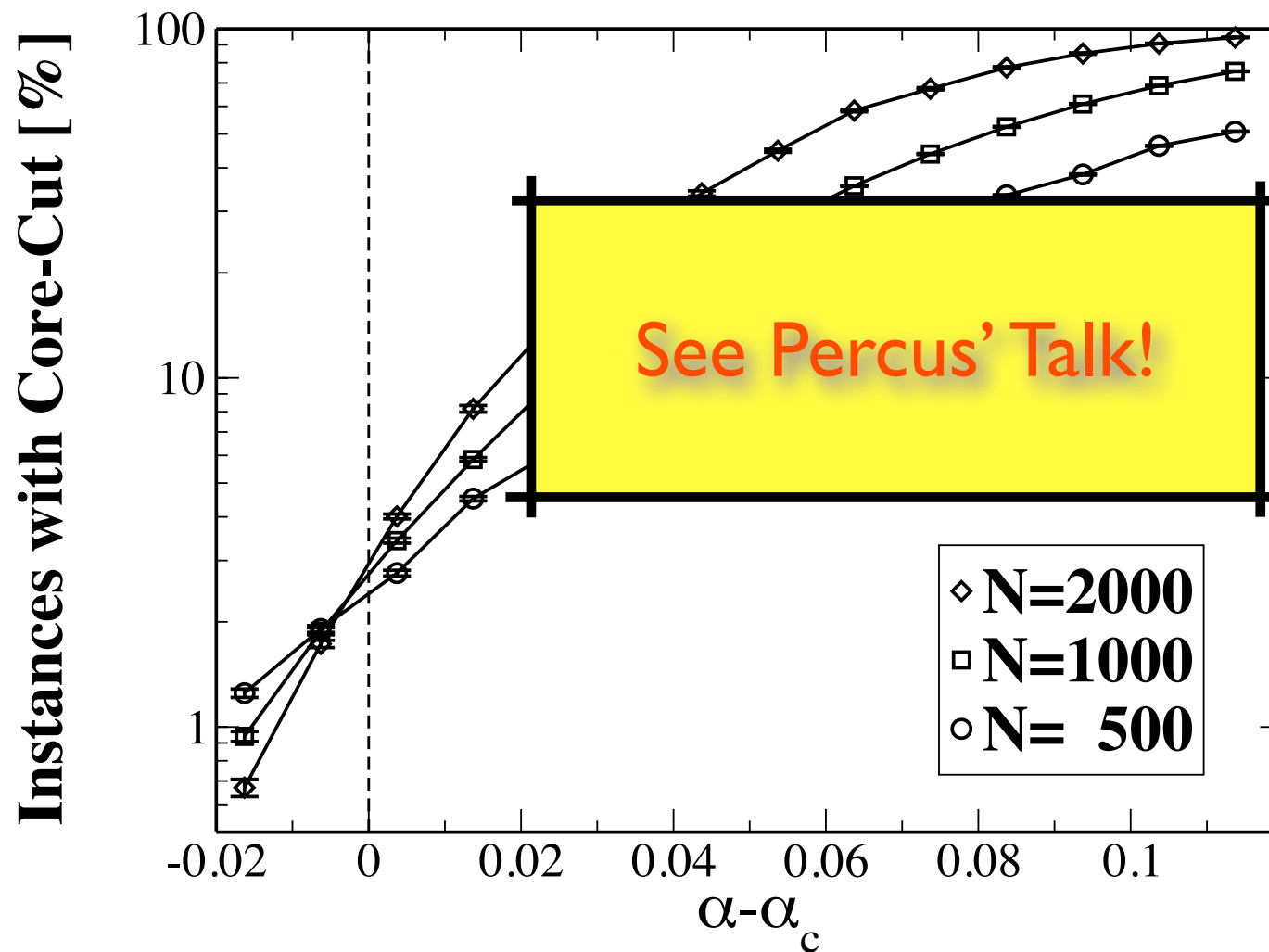




# Results for $\tau$ -EO:

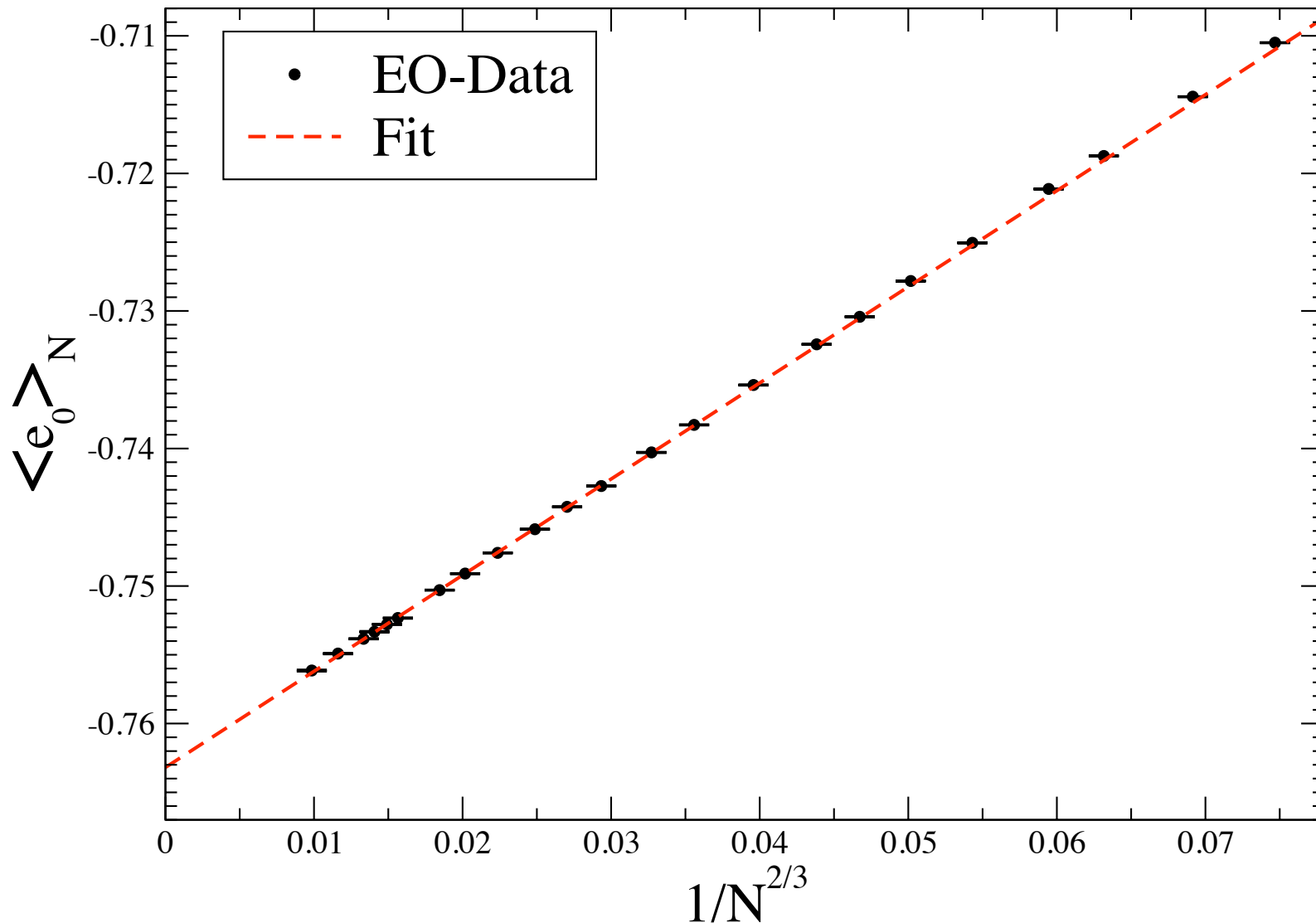
- For Graph Bi-Partitioning:

Random Graph Bi-Partitioning



# $\tau$ -EO for Sherrington-Kirkpatrick

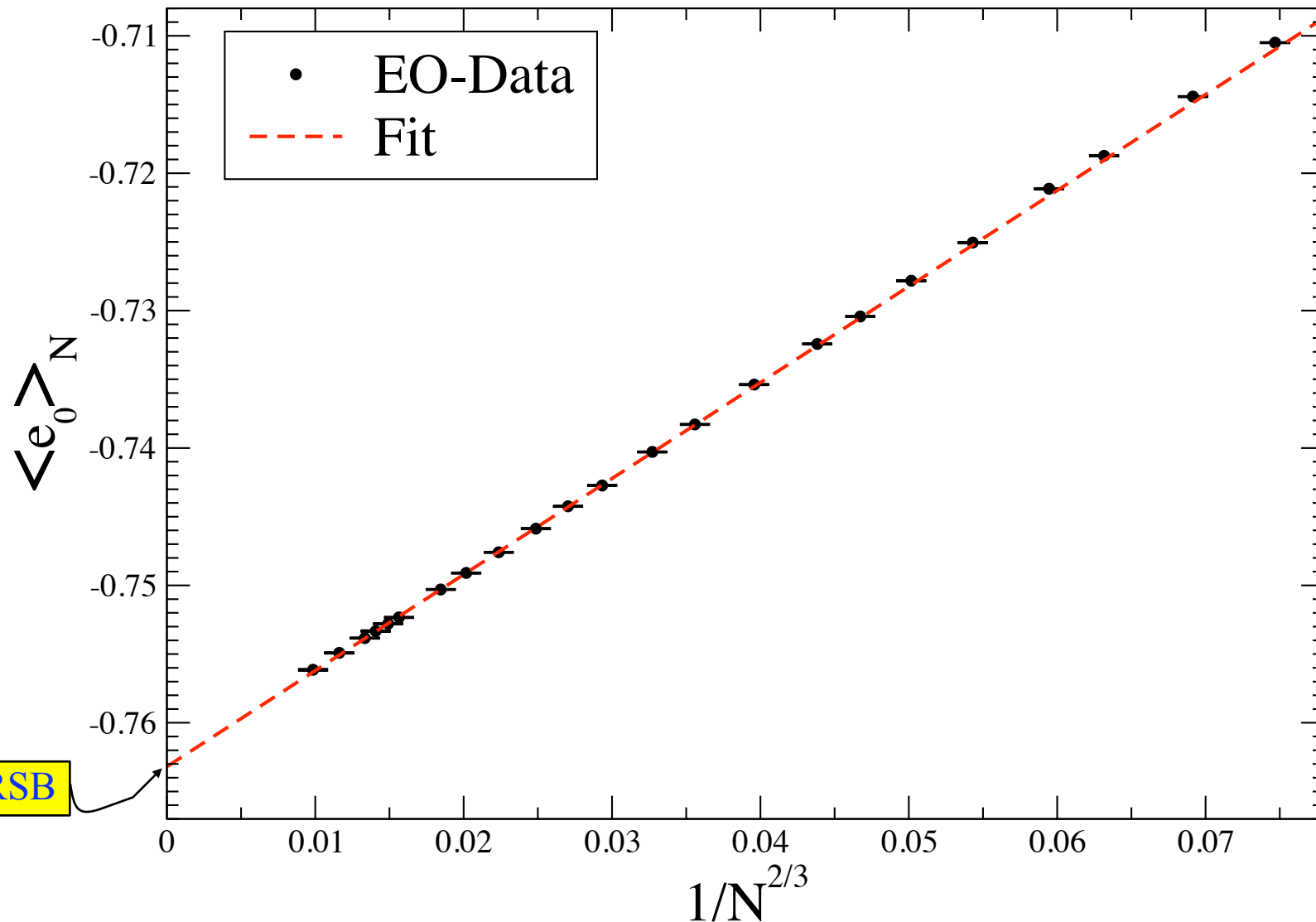
- Mean-Field ( $d \rightarrow \infty$ ) Spin Glasses:





# $\tau$ -EO for Sherrington-Kirkpatrick

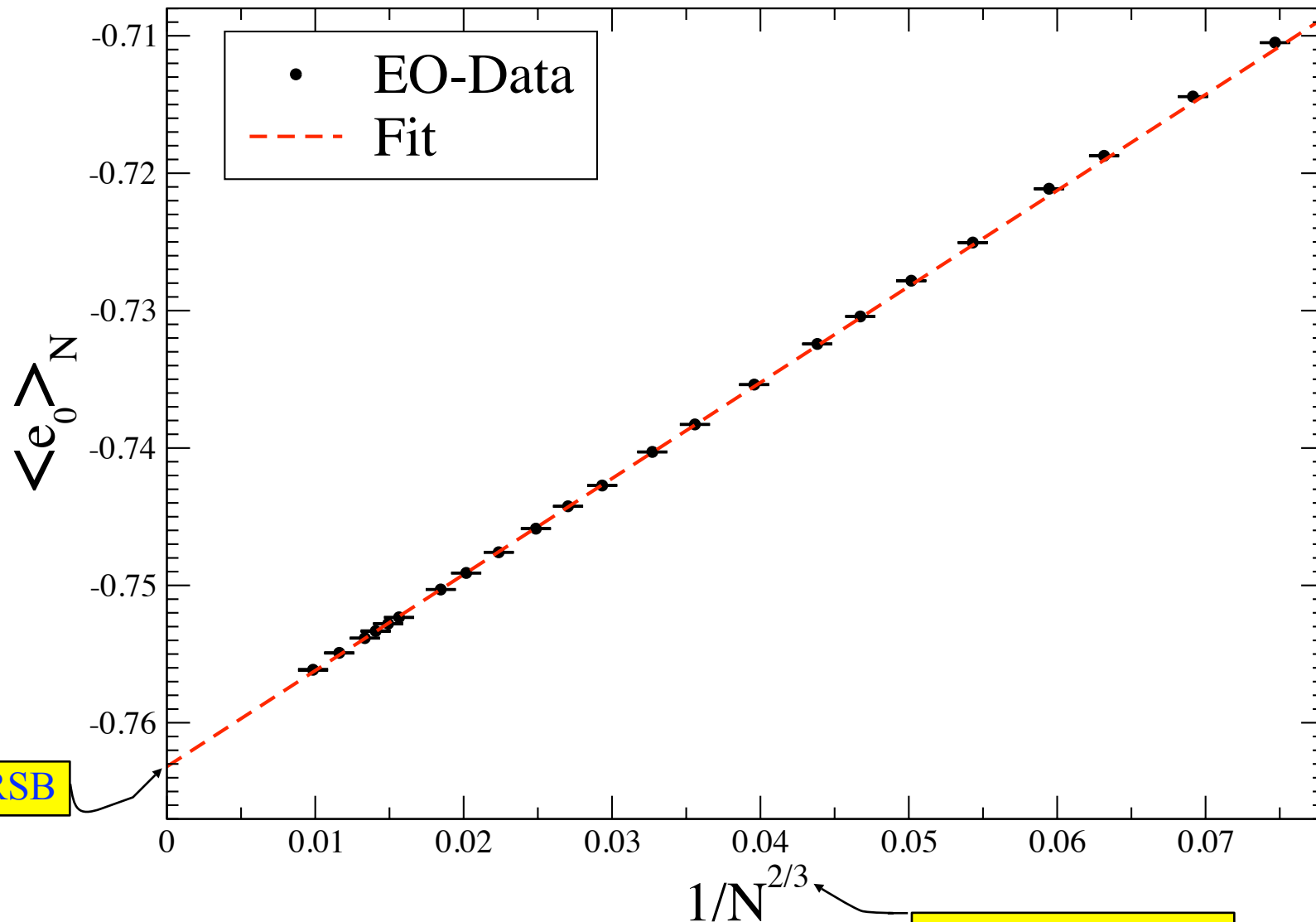
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Parisi's RSB

# $\tau$ -EO for Sherrington-Kirkpatrick

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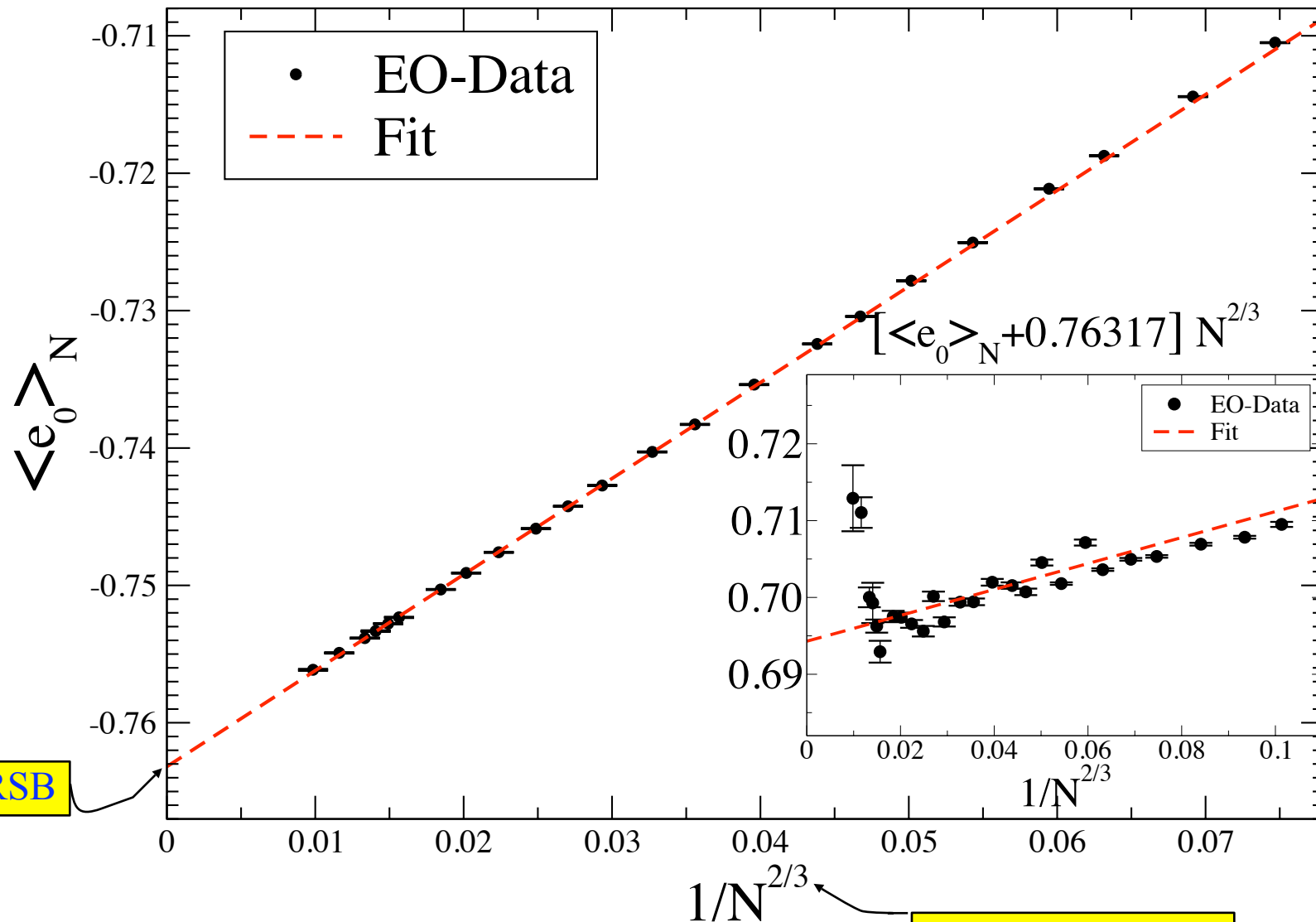


Parisi's RSB

Scaling Correction  
 $\omega/d=2/3$

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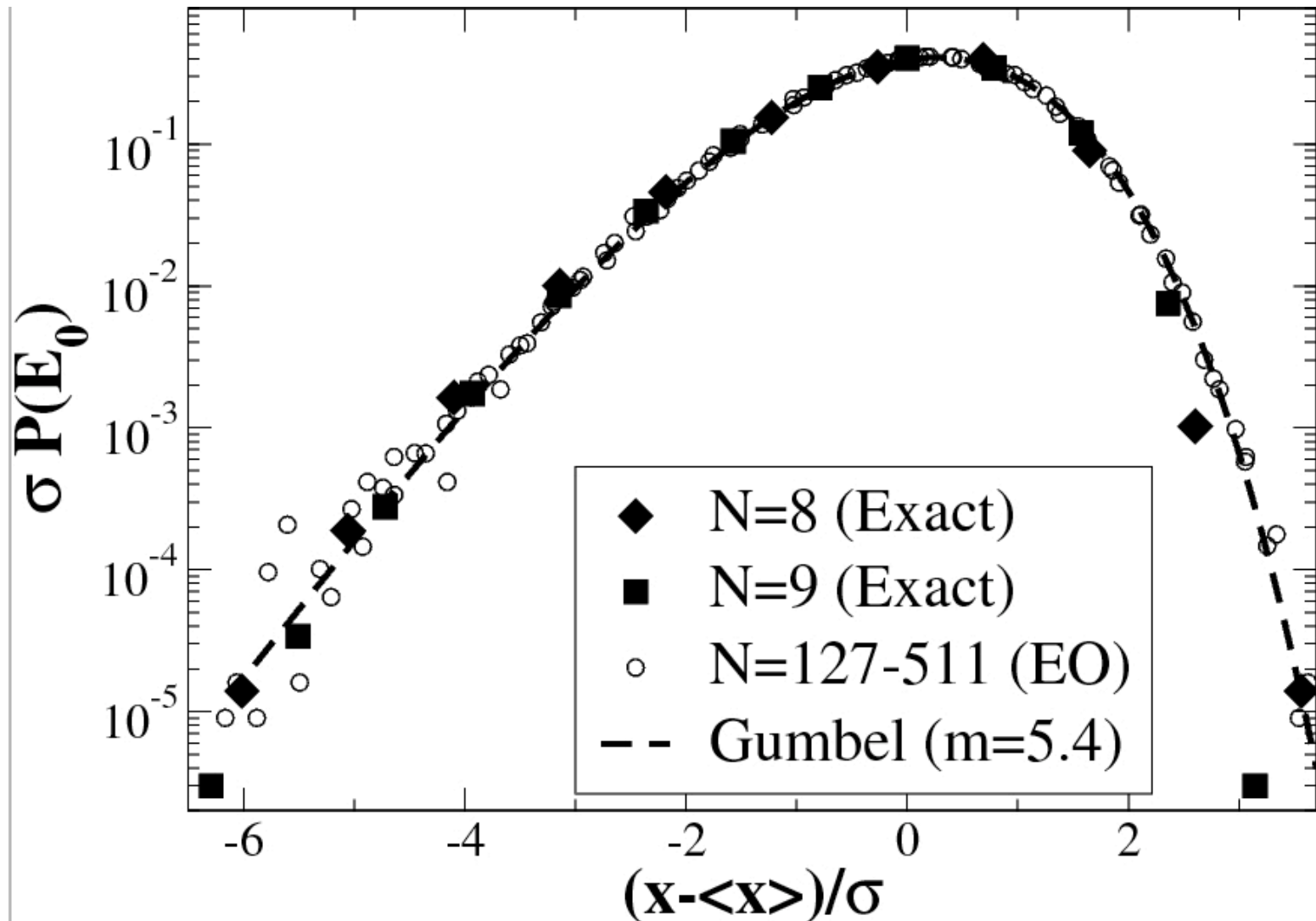


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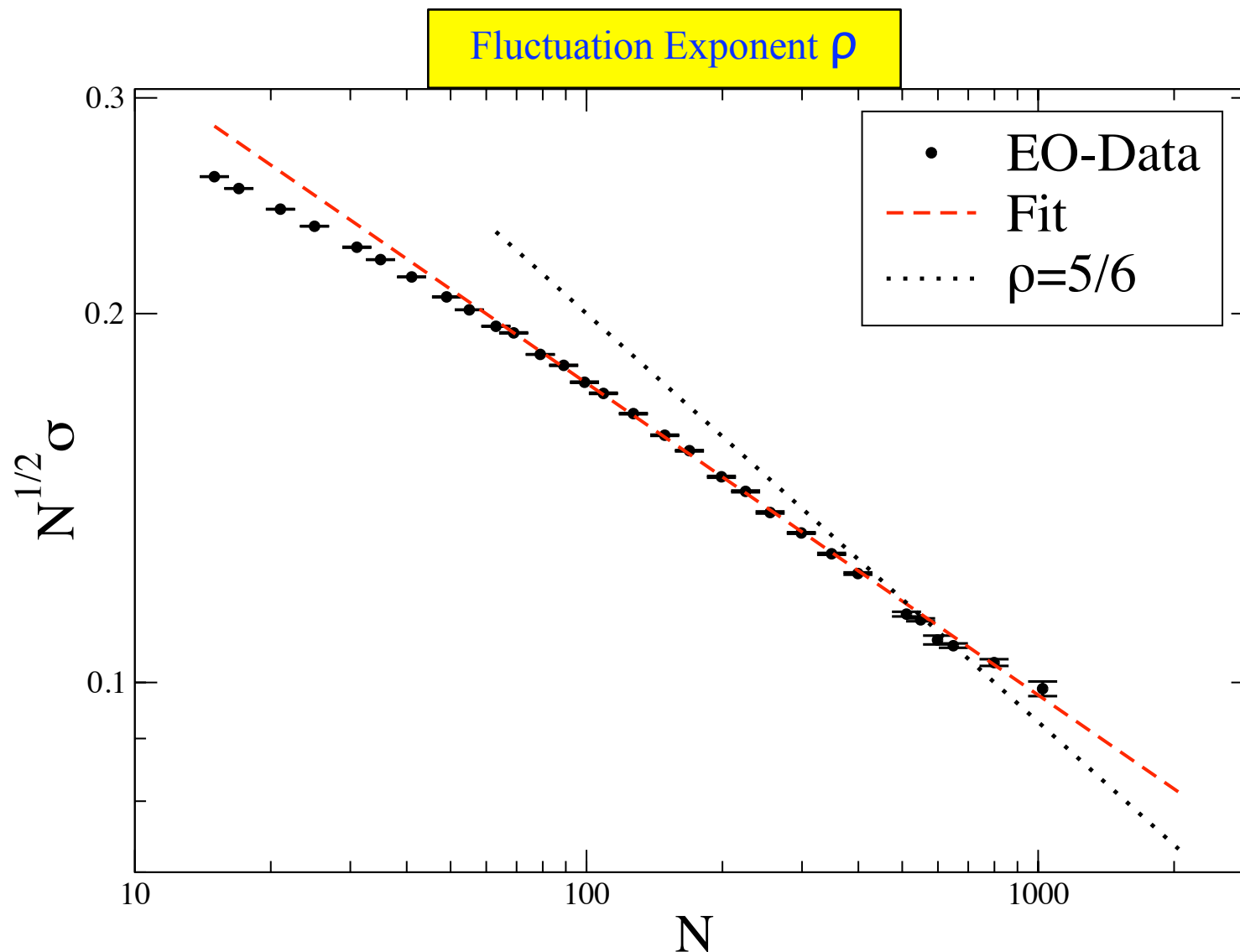
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# $\tau$ -EO for Sherrington-Kirkpatrick

- Mean-Field ( $d \rightarrow \infty$ ) Spin Glasses:



## $\tau$ -EO for Sherrington-Kirkpatrick

- “Width”  $\sigma$  of the GS-Energy:

$$\begin{aligned}\sigma &= \sqrt{\langle e_0^2 \rangle - \langle e_0 \rangle^2}, \\ &\sim A \frac{1}{N^\rho} + B \frac{1}{N^\alpha}, \quad (\alpha > \rho),\end{aligned}$$

$$\ln \sigma \sim -\rho \ln(N) + \ln(A) + \ln \left( 1 + \frac{B}{A} N^{\rho-\alpha} \right),$$

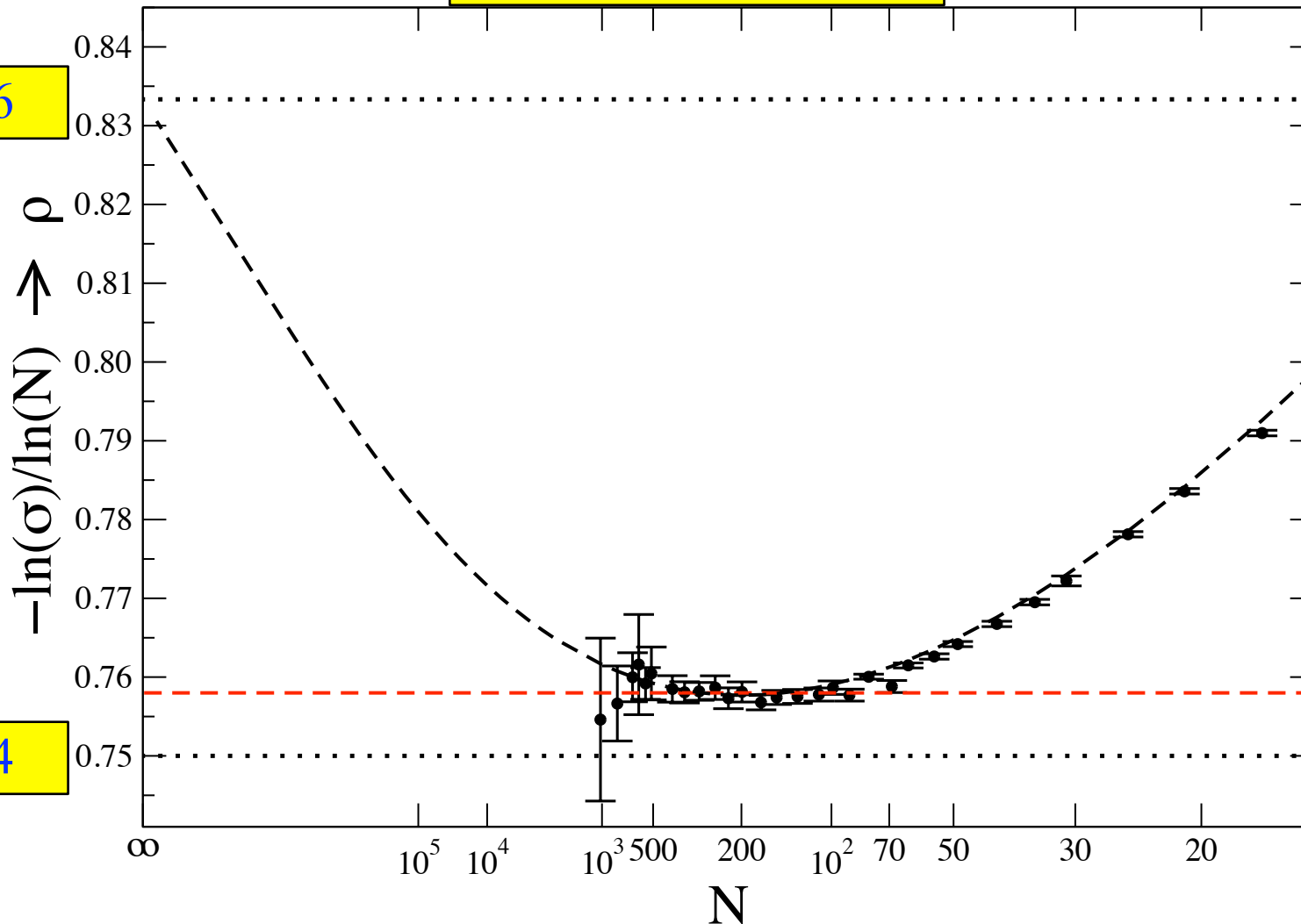
$$\begin{aligned}-\frac{\ln \sigma}{\ln N} &\sim \rho + a x + b x \exp \left[ \frac{\rho - \alpha}{x} \right], \\ &\quad \left( x = \frac{1}{\ln N} \rightarrow 0 \right).\end{aligned}$$

# $\tau$ -EO for Sherrington-Kirkpatrick

- Mean-Field ( $d \rightarrow \infty$ ) Spin Glasses:

Fluctuation Exponent  $\rho$

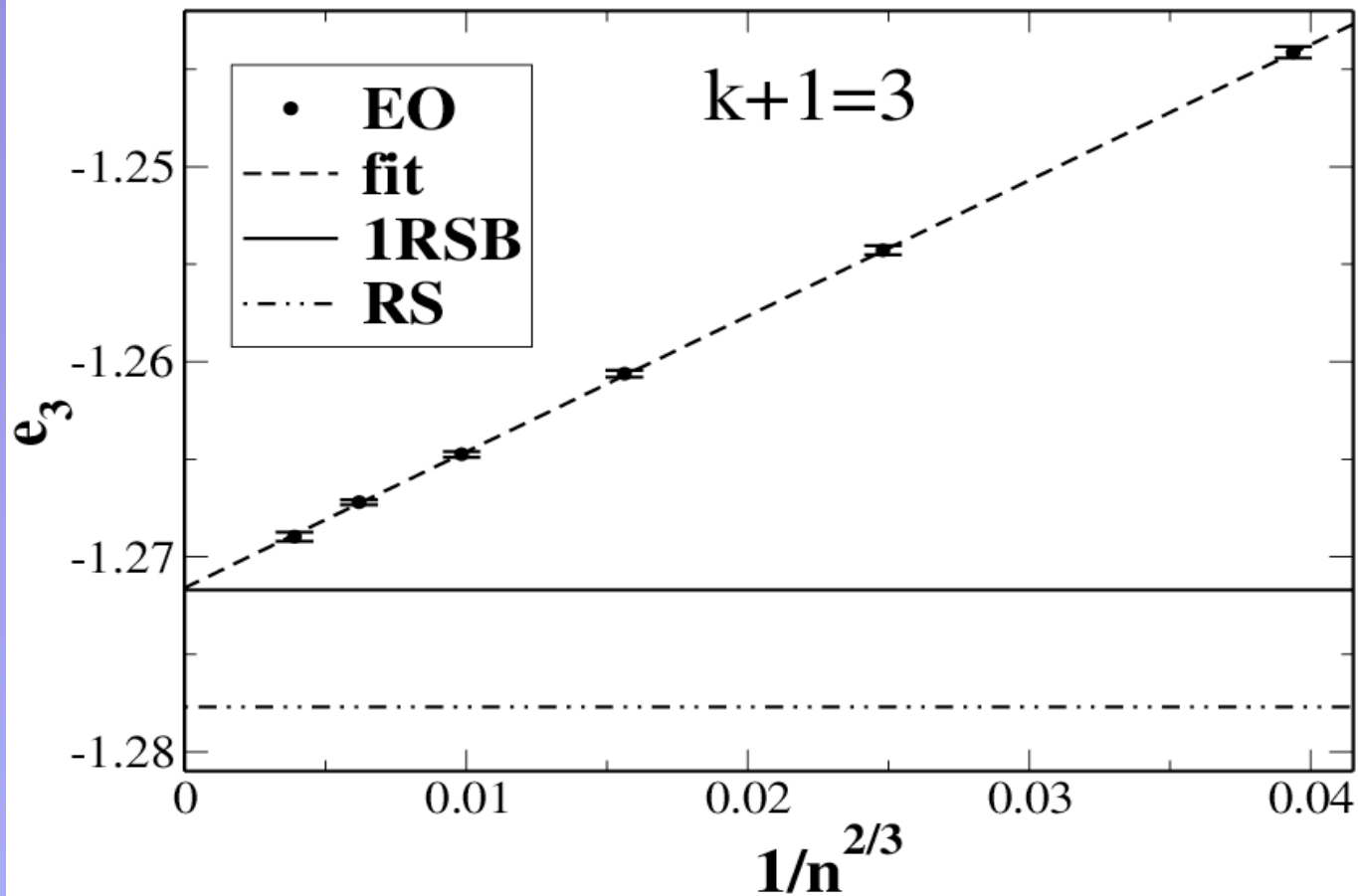
$\rho = 5/6$



$\rho = 3/4$

# $\tau$ -EO for Bethe Lattices:

EO for 3-connected Bethe Lattice Glass w/ Replica Sym. Breaking:

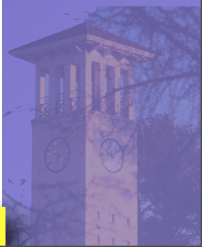
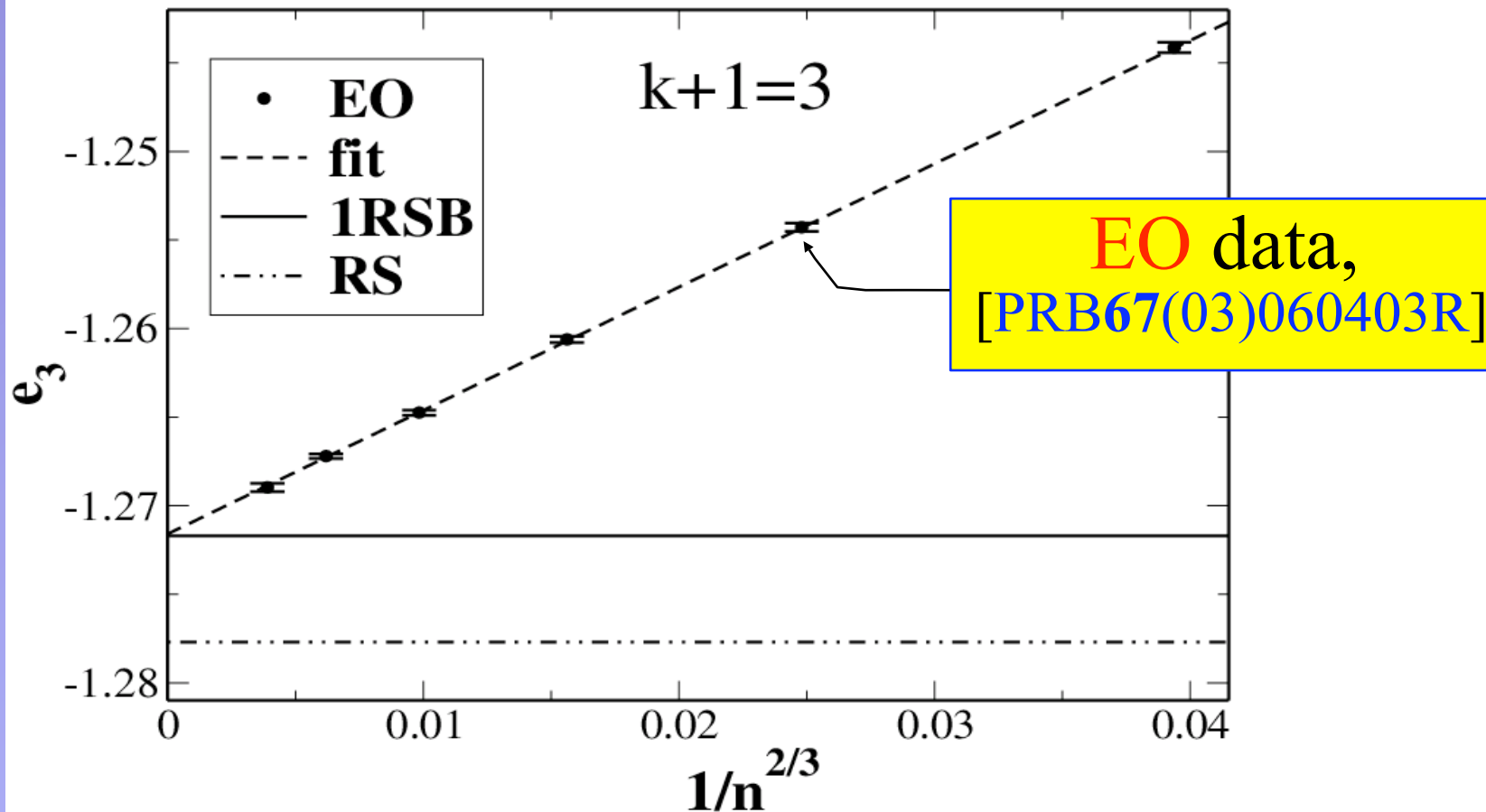


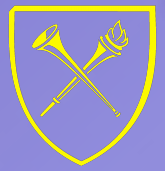




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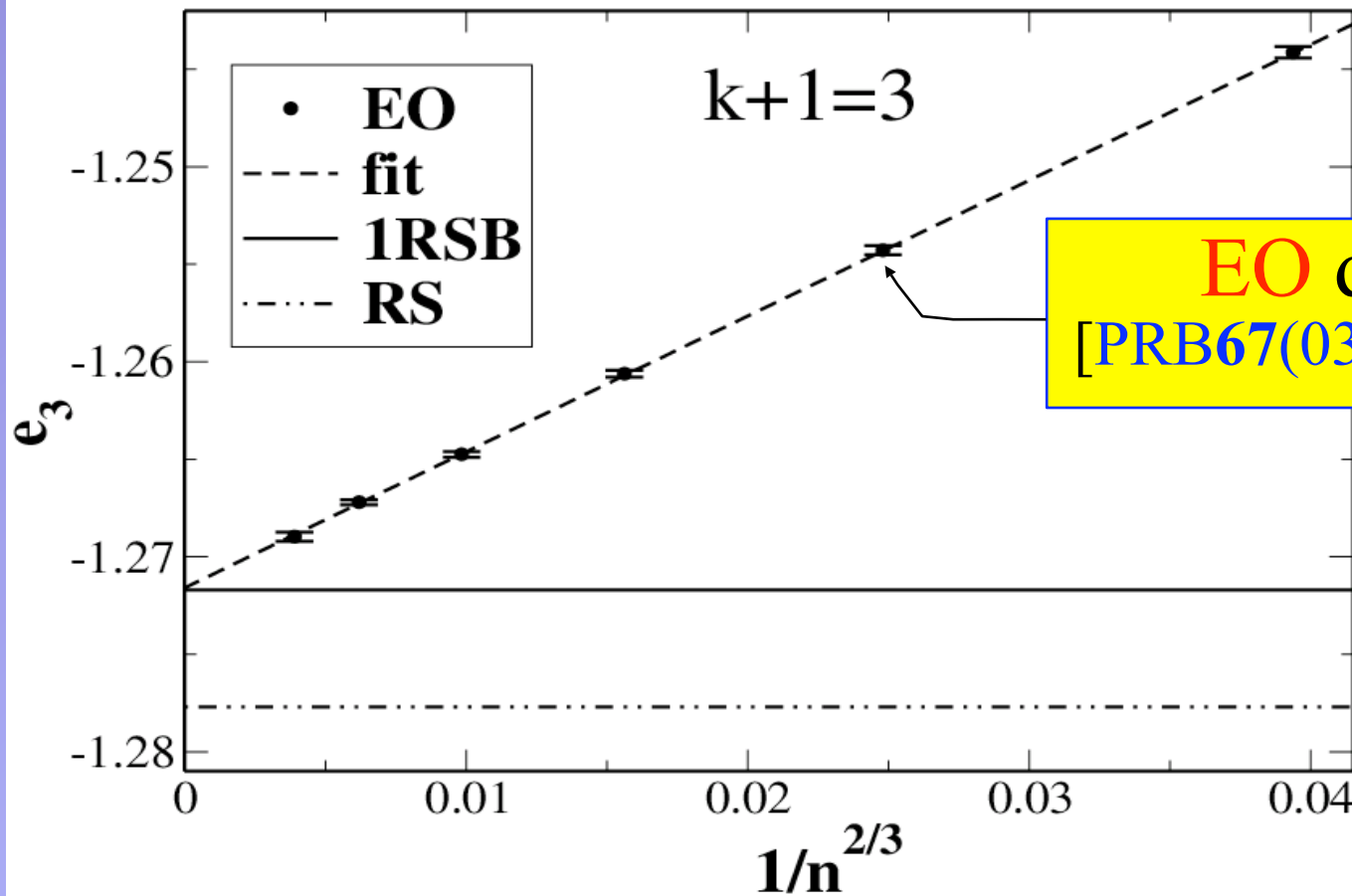
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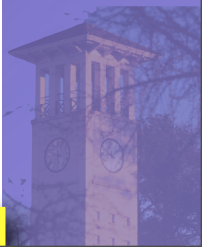


Replica Theory:

$\Leftarrow$  1RSB,

$\Leftarrow$  no RSB,

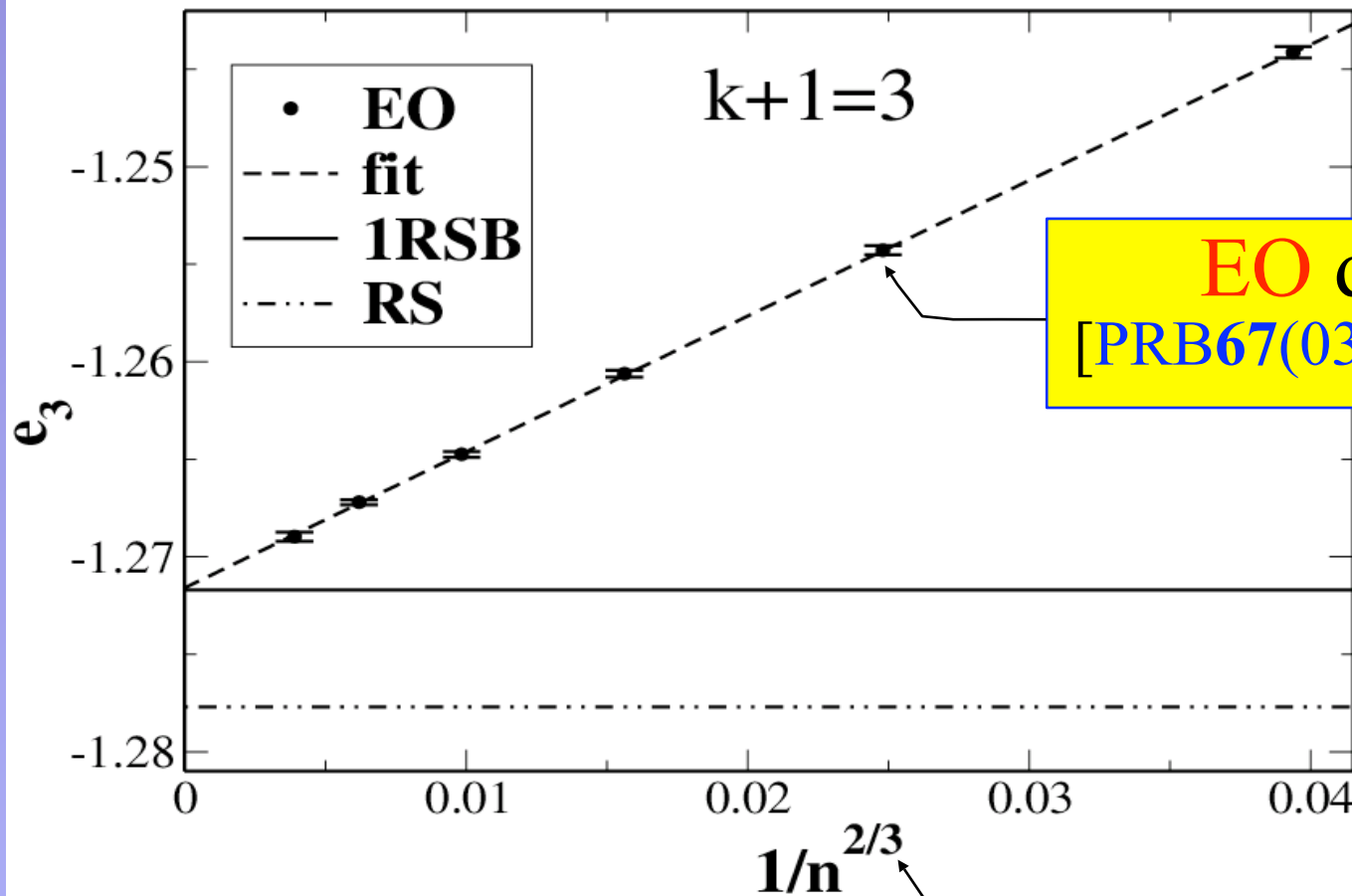
[J.Stat.Phys.111(03)1]

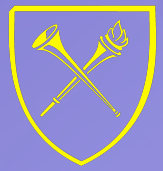




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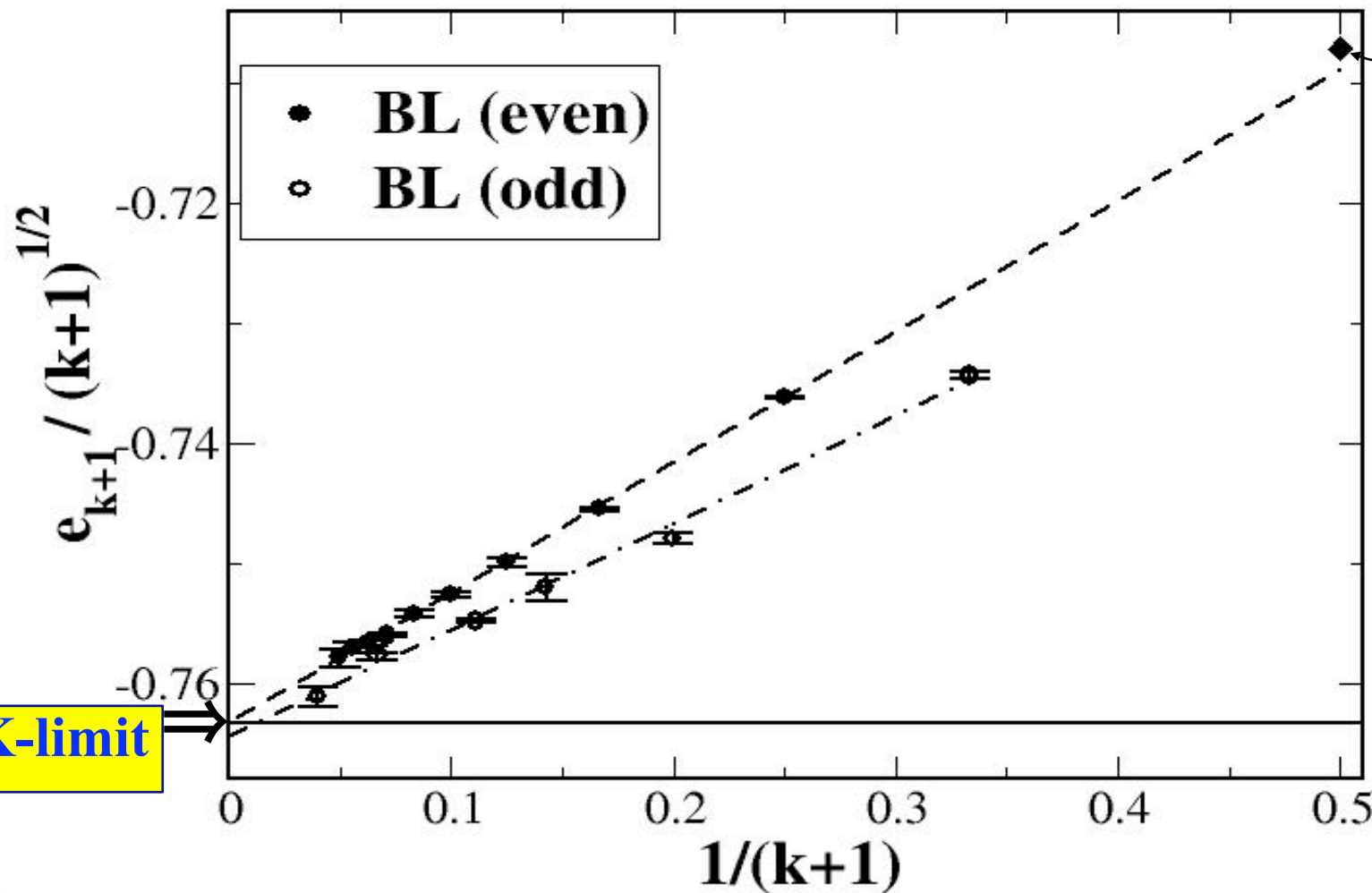


# $\tau$ -EO for Bethe Lattices:

EO for  $(k+1)$ -connected Bethe Lattice Glasses for  $(k+1) \rightarrow \infty$ :

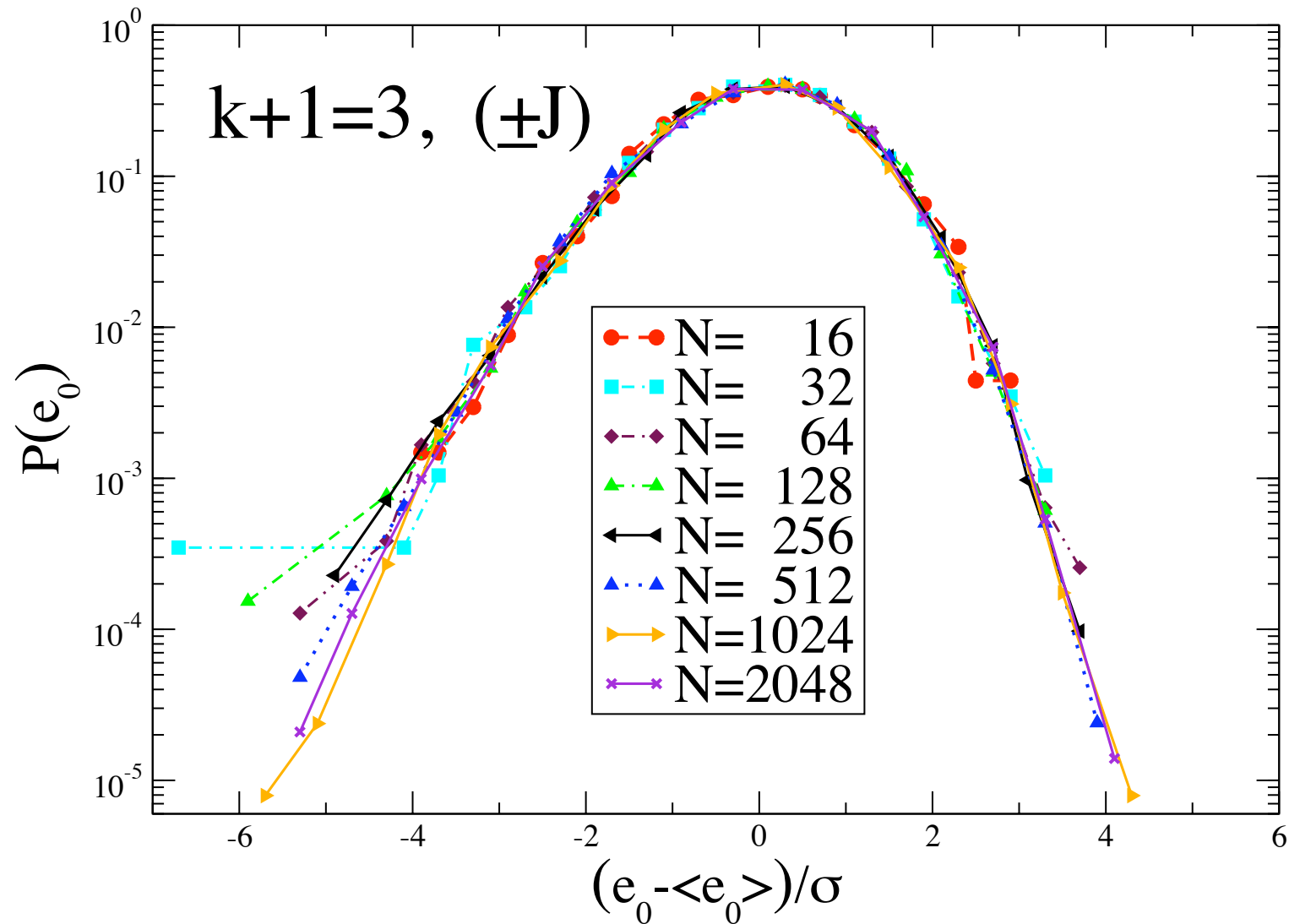
## Energies

$k=1$ , exact



# $\tau$ -EO for Bethe Lattices:

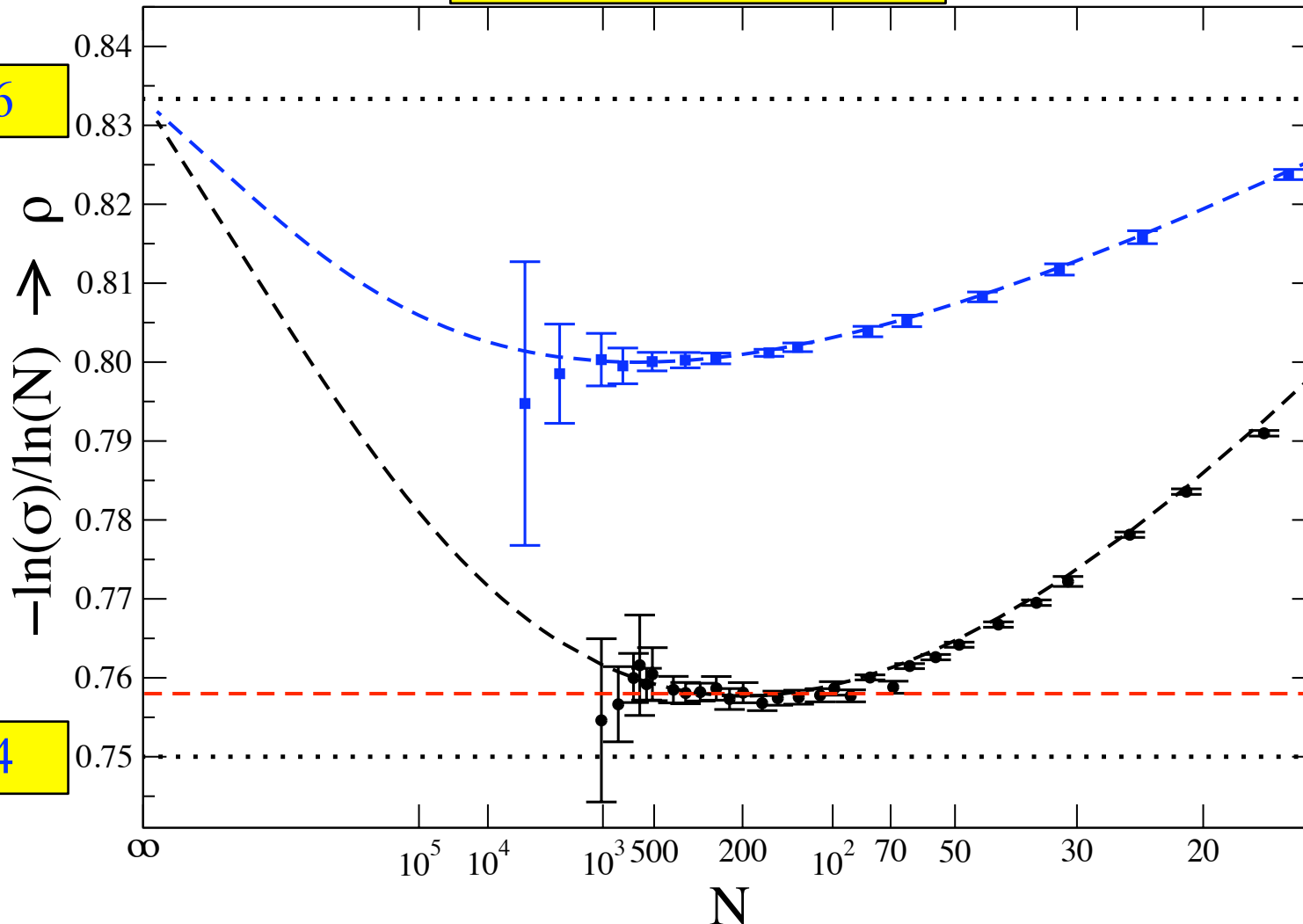
EO for 3-connected Bethe Lattice Glass:



# $\tau$ -EO for Sherrington-Kirkpatrick

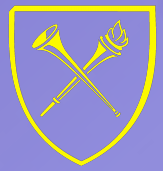
- Mean-Field ( $d \rightarrow \infty$ ) Spin Glasses:

Fluctuation Exponent  $\rho$



$\rho=5/6$

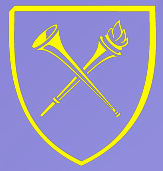
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# Lattice Spin Glasses (at $T=0$ ):

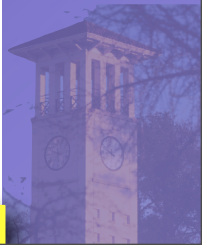
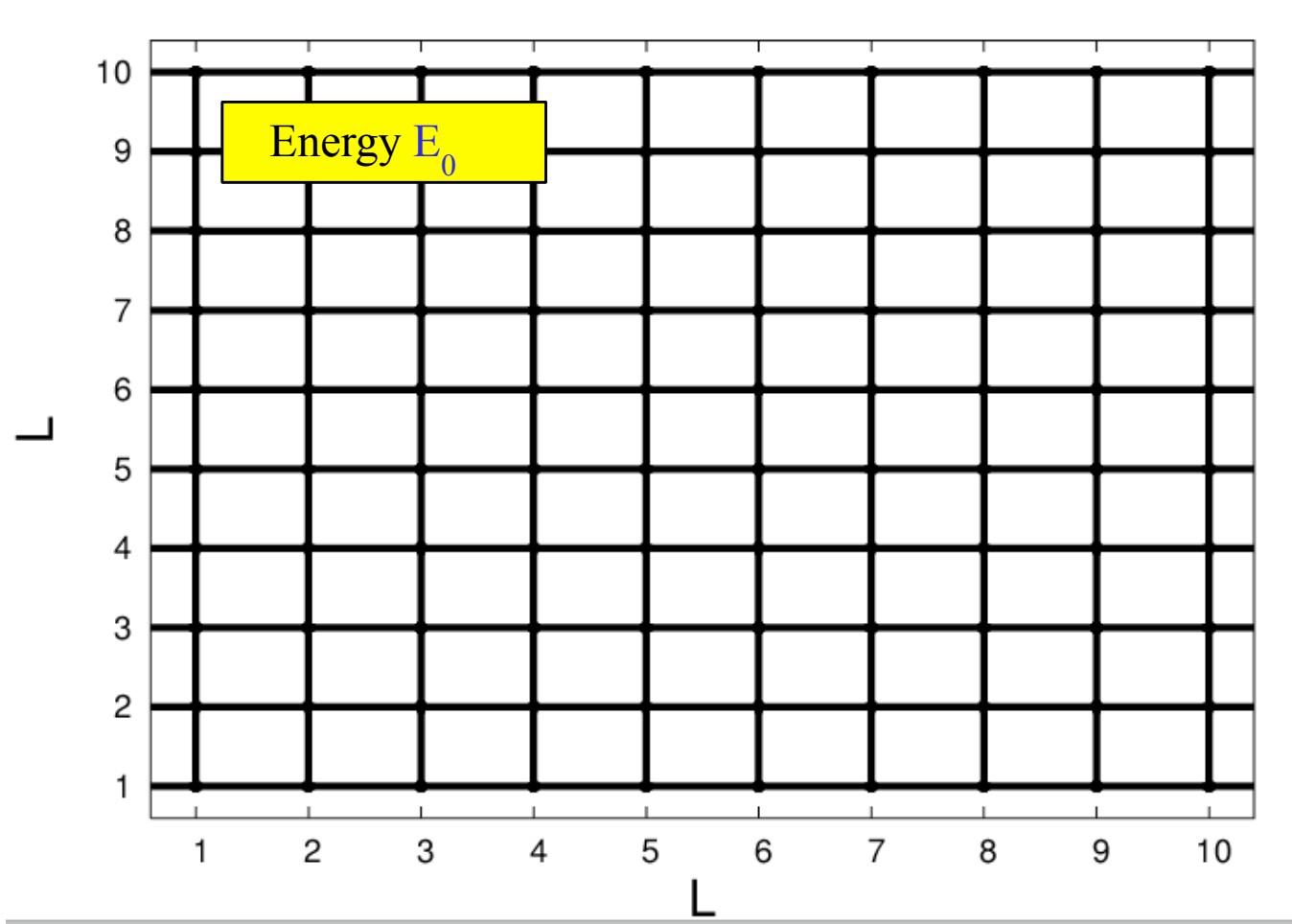
Defect-Energy:





# Lattice Spin Glasses (at $T=0$ ):

## Defect-Energy:

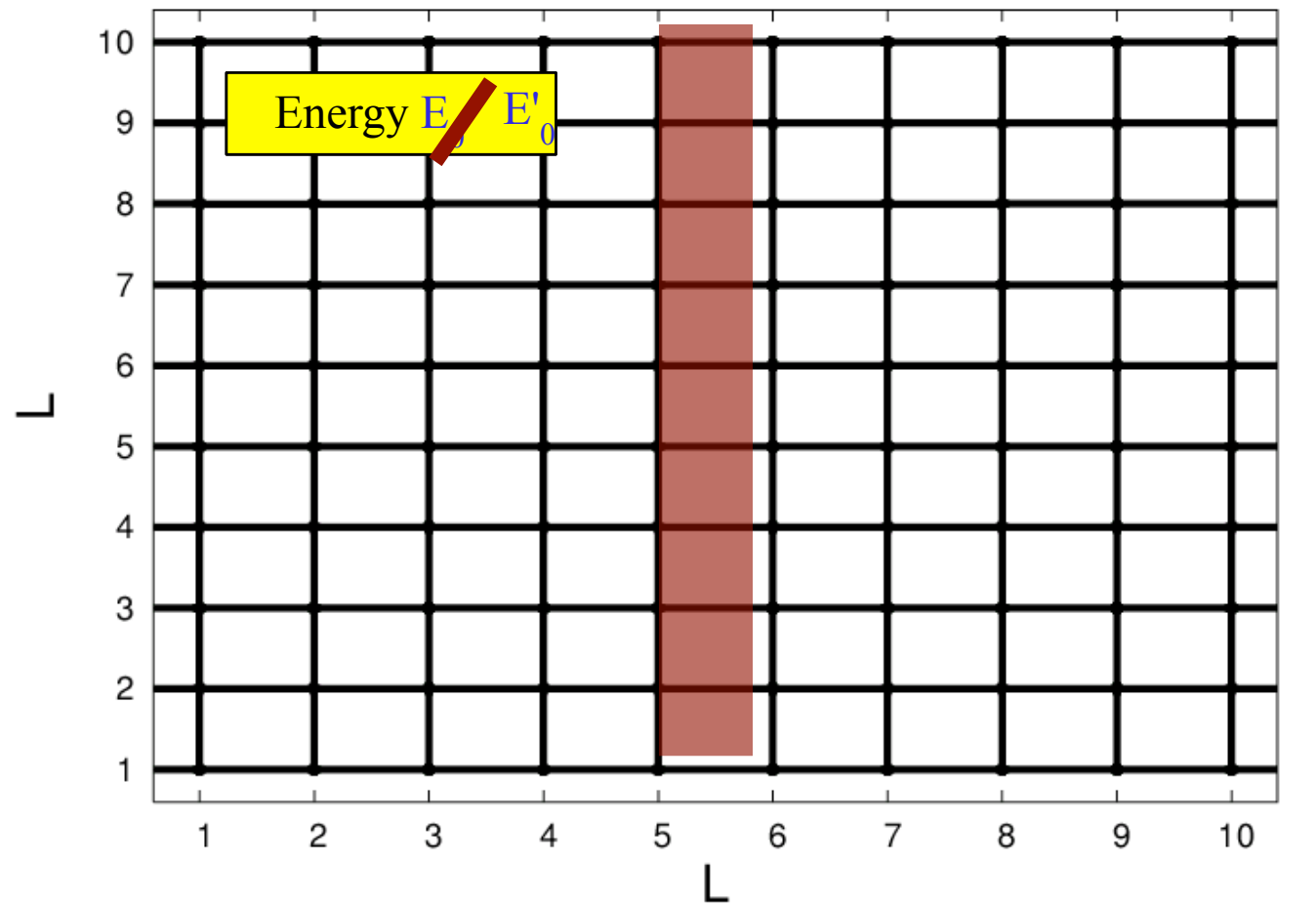




# Lattice Spin Glasses (at $T=0$ ):

## Defect-Energy:

Measure Defect Energy  $\Delta E = E_0 - E'_0$

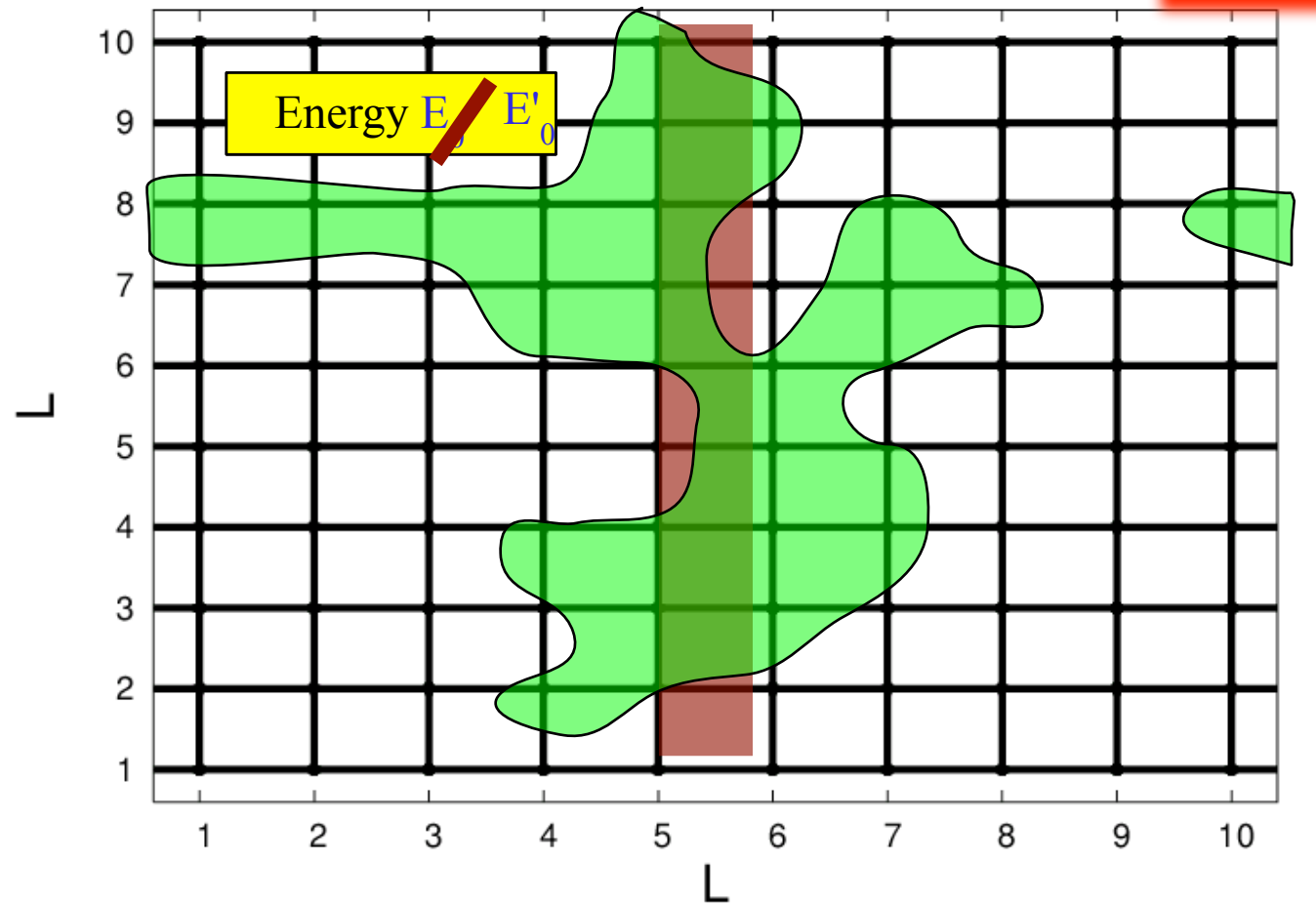


# Lattice Spin Glasses (at $T=0$ ):

## Defect-Energy:

Measure Defect Energy  $\Delta E = E_0 - E'_0$

$$\Rightarrow \sigma(\Delta E) \sim L^y$$

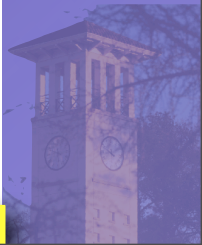
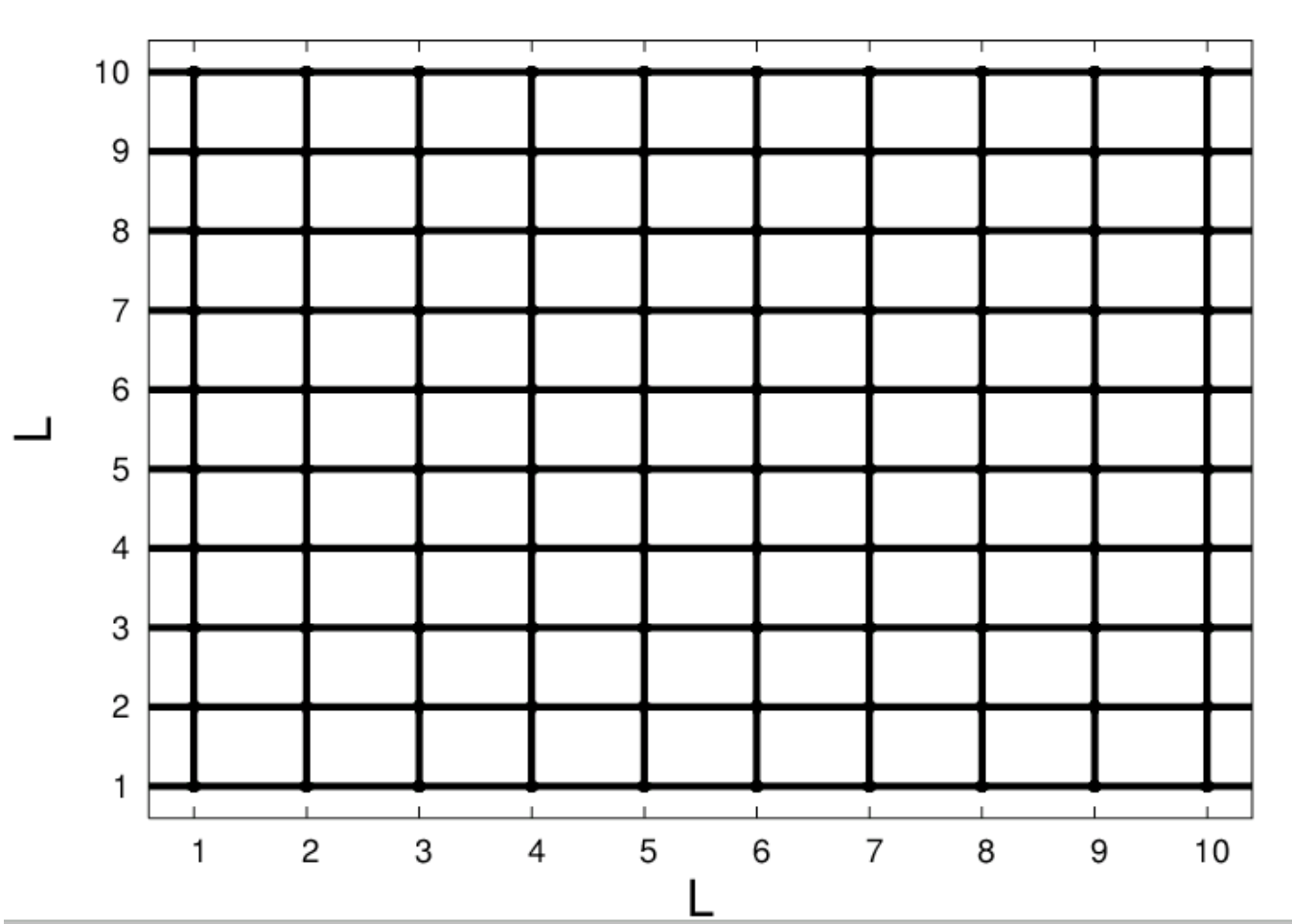


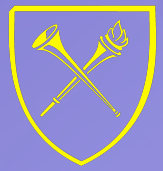
$\Rightarrow$  Low Energy Excitations (like “small oscillations”)



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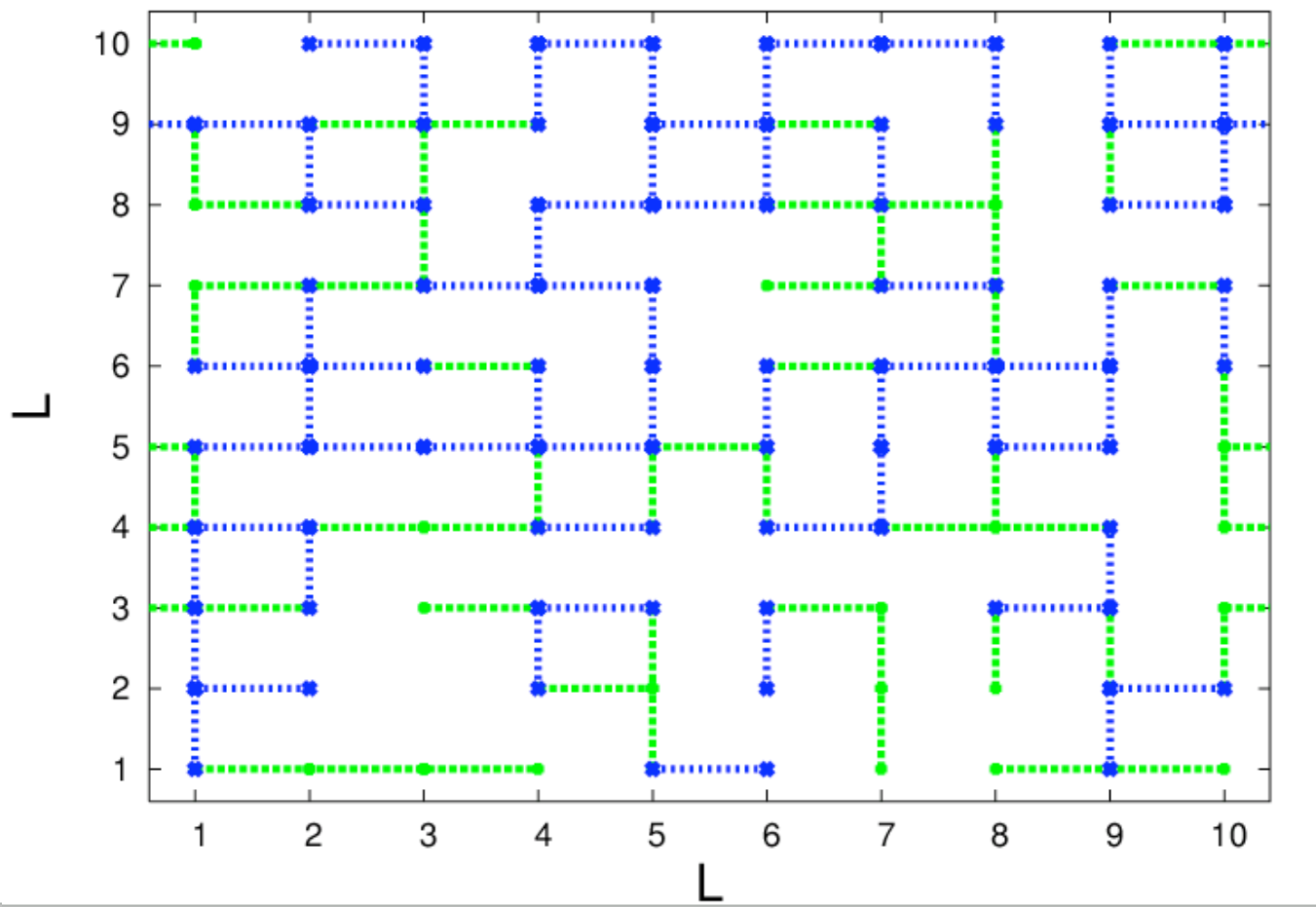
## Defect-Energy:





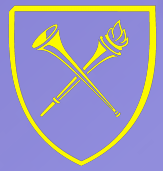
# Lattice Spin Glasses (at $T=0$ ):

## Defect-Energy:



Before:  
100 Spins



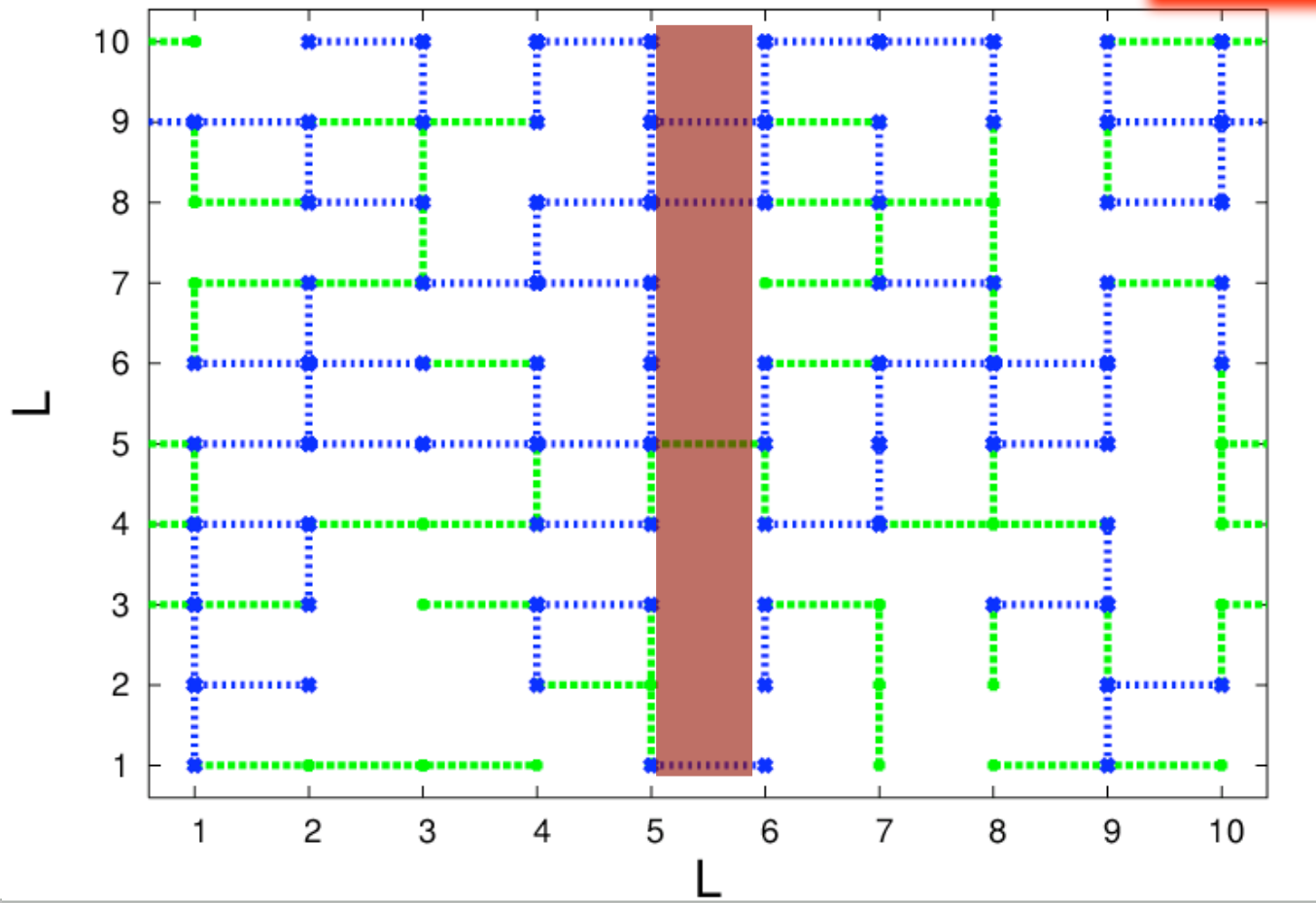


# Lattice Spin Glasses (at $T=0$ ):

## Defect-Energy:

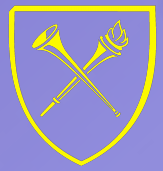
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Before:  
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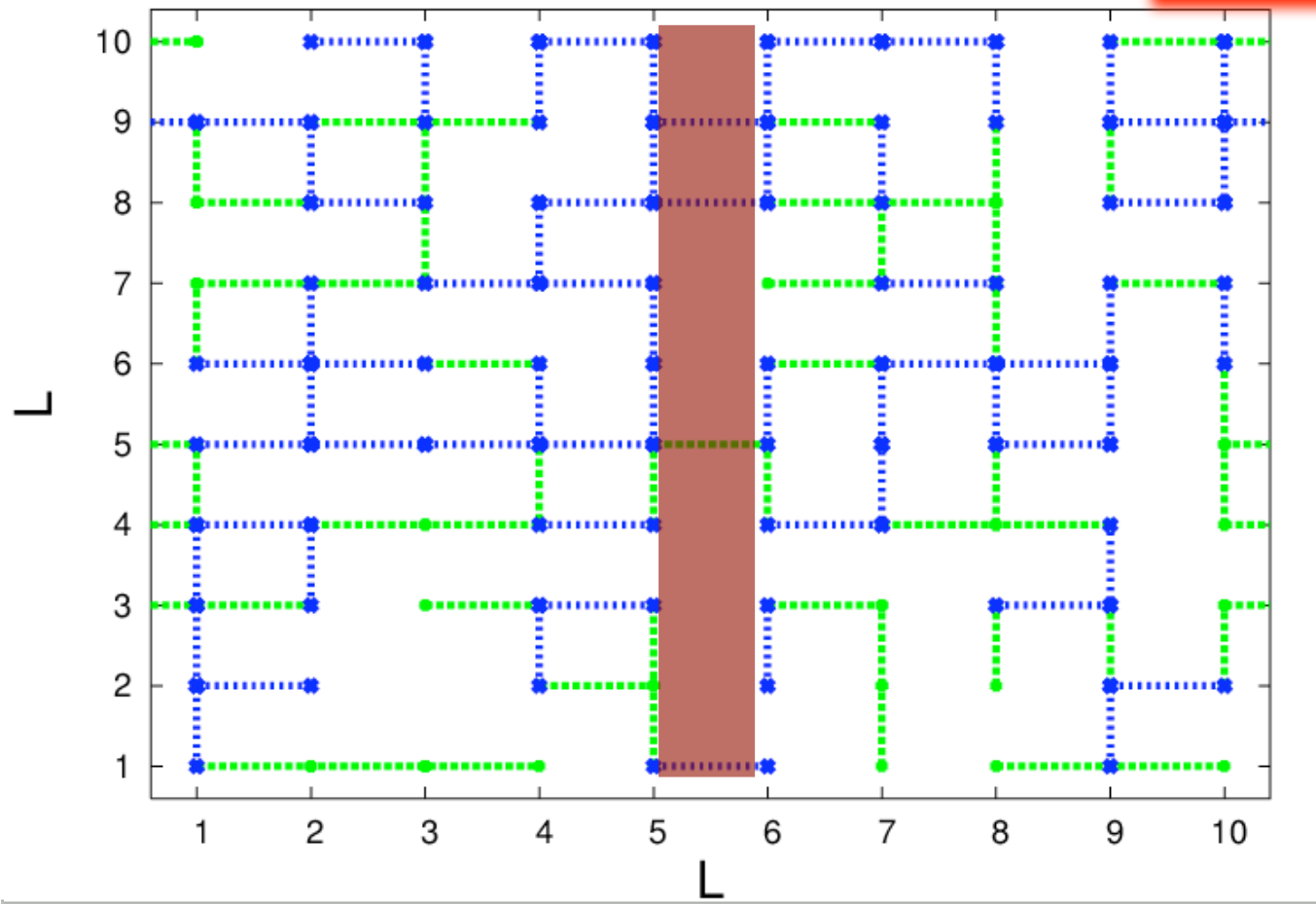


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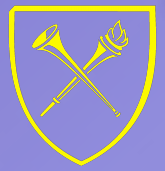
$$\Rightarrow \sigma(\Delta E) \sim L^y$$



Before:  
100 Spins

$\Rightarrow$  Low Energy Excitations of **bond-diluted** Lattices



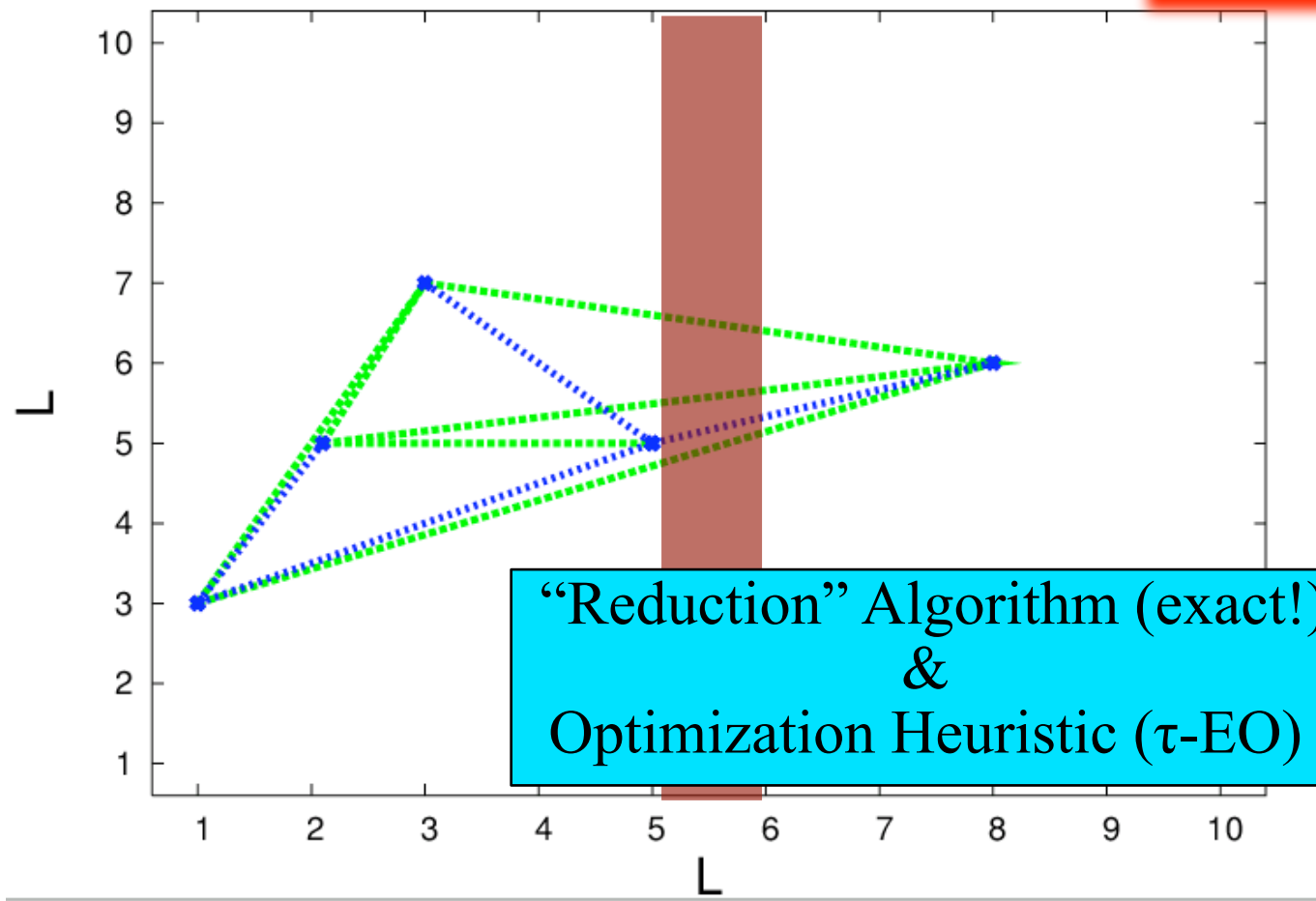


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Measure Defect Energy  $\Delta E = E_0 - E'_0$

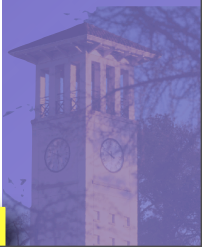
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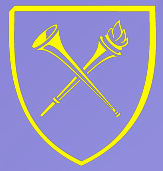


Before:  
100 Spins

After:  
5 Spins

$\Rightarrow$  Low Energy Excitations of **bond-diluted** Lattices





# Lattice Spin Glasses (at $T=0$ ):

Defect-Energy: Measure “Stiffness”:  $\sigma(\Delta E) \sim L^y$



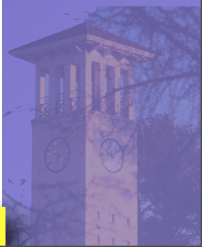




# Lattice Spin Glasses (at $T=0$ ):

Defect-Energy: Measure “Stiffness”:  $\sigma(\Delta E) \sim L^y$

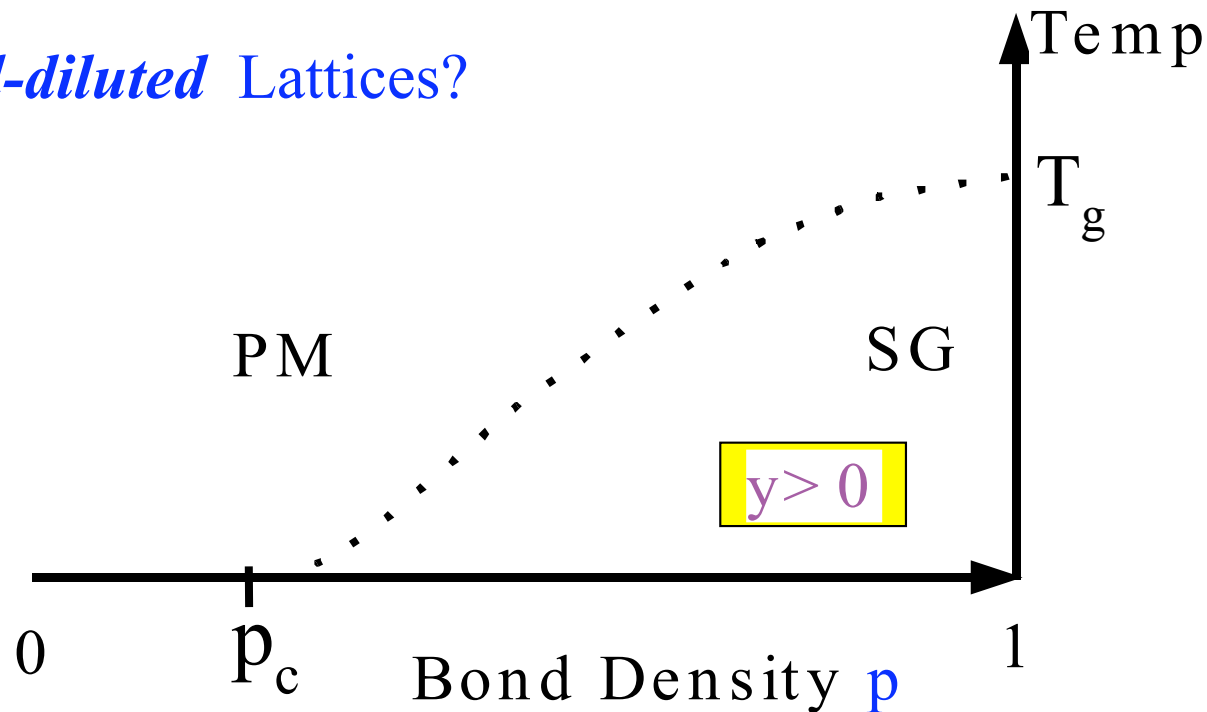
How *bond-diluted* Lattices?



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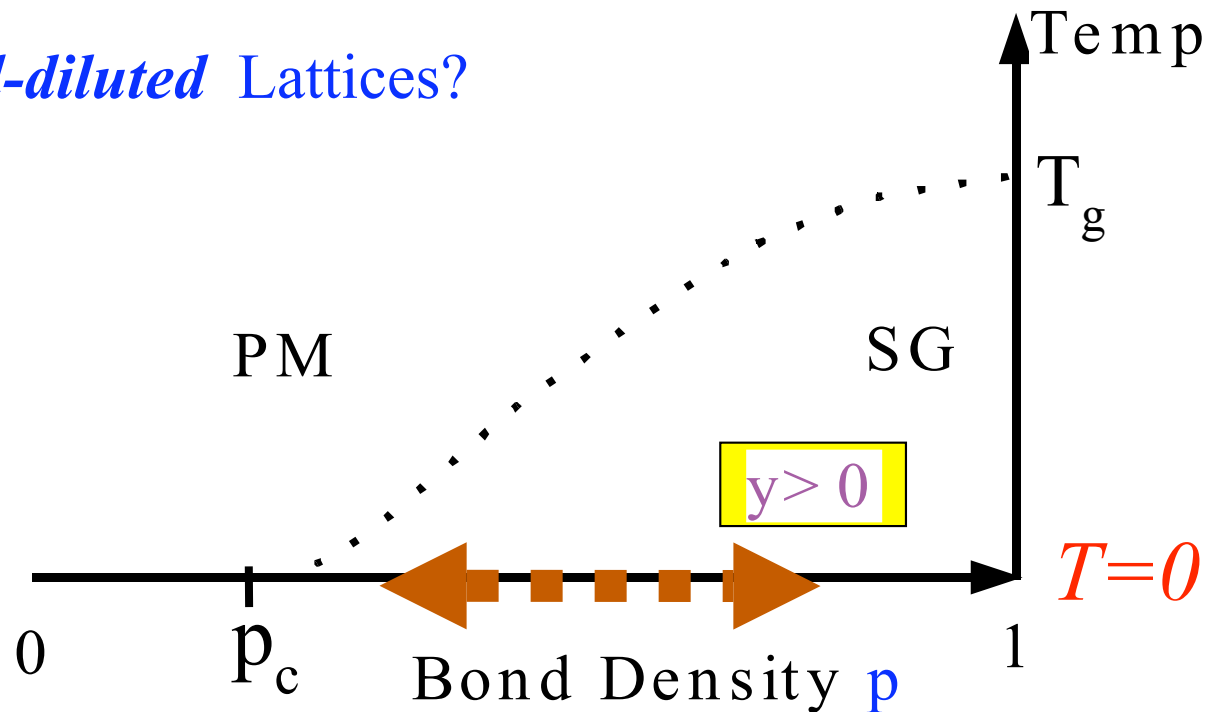
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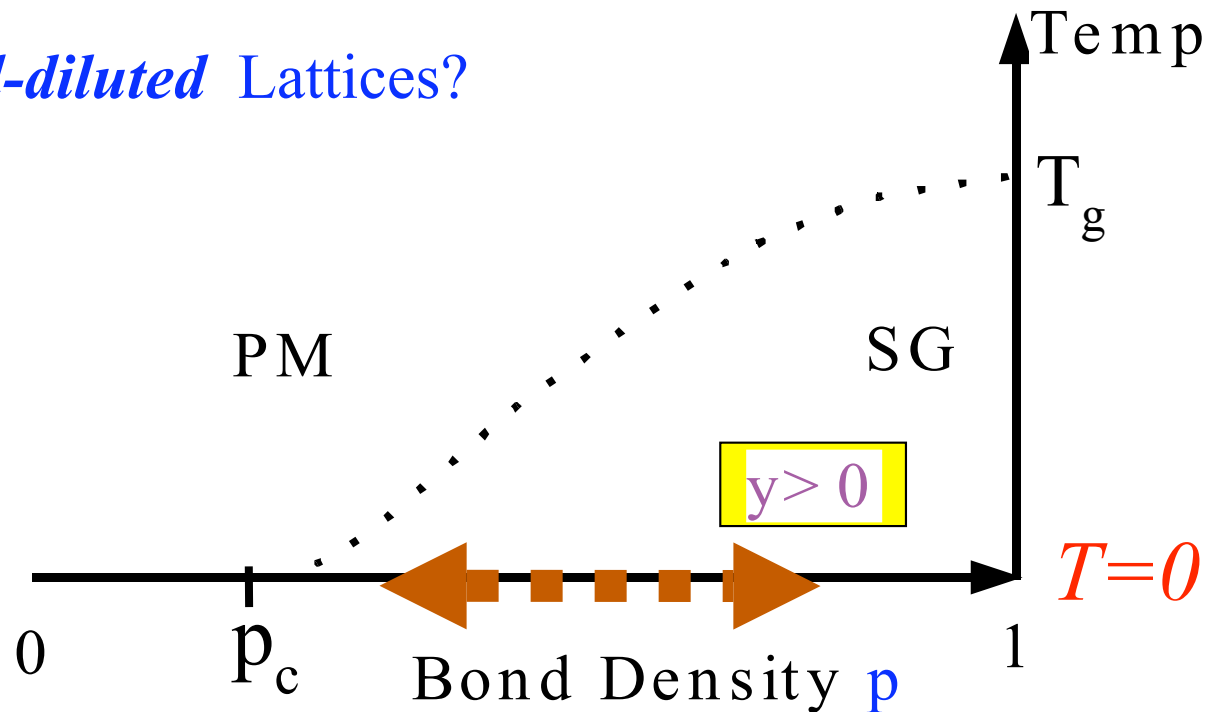
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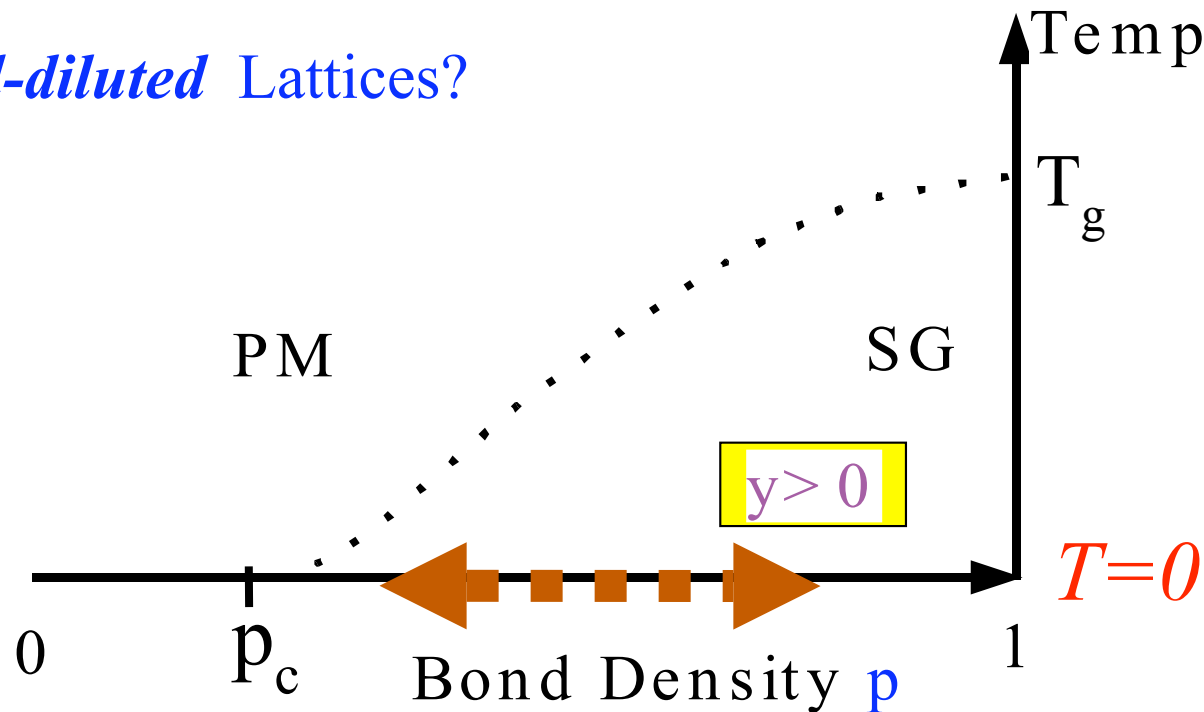


Why *bond-diluted* Lattices?

# Lattice Spin Glasses (at $T=0$ ):

Defect-Energy: Measure “Stiffness”:  $\sigma(\Delta E) \sim L^y$

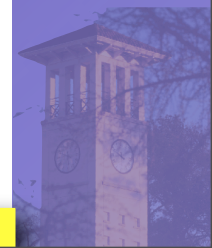
How bond-diluted Lattices?

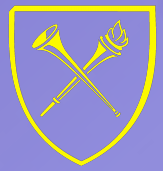


Why bond-diluted Lattices?

- Simpler Problem
- Larger Sizes  $L$
- Better Scaling

# Defect-Energy of diluted Lattices:





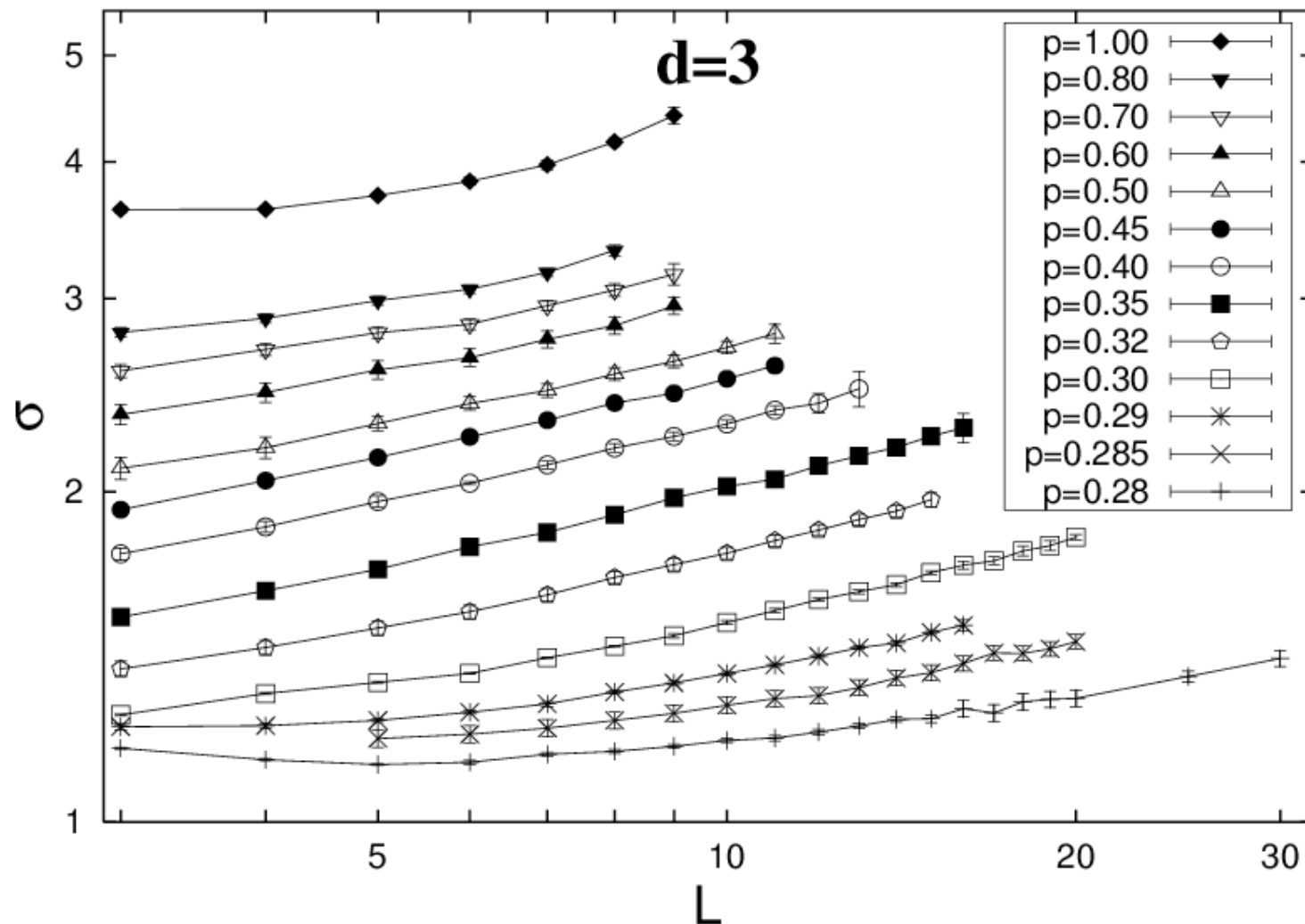
# Defect-Energy of diluted Lattices:

- $\pm J$ -Glasses on Lattices of size  $L$  and density  $p$ .
- Defect-Energy  $\sigma(\Delta E)$  with Reduction & Heuristic ( $\tau$ -EO).



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- $\pm J$ -Glasses on Lattices of size  $L$  and density  $p$ .
- Defect-Energy  $\sigma(\Delta E)$  with Reduction & Heuristic ( $\tau$ -EO).







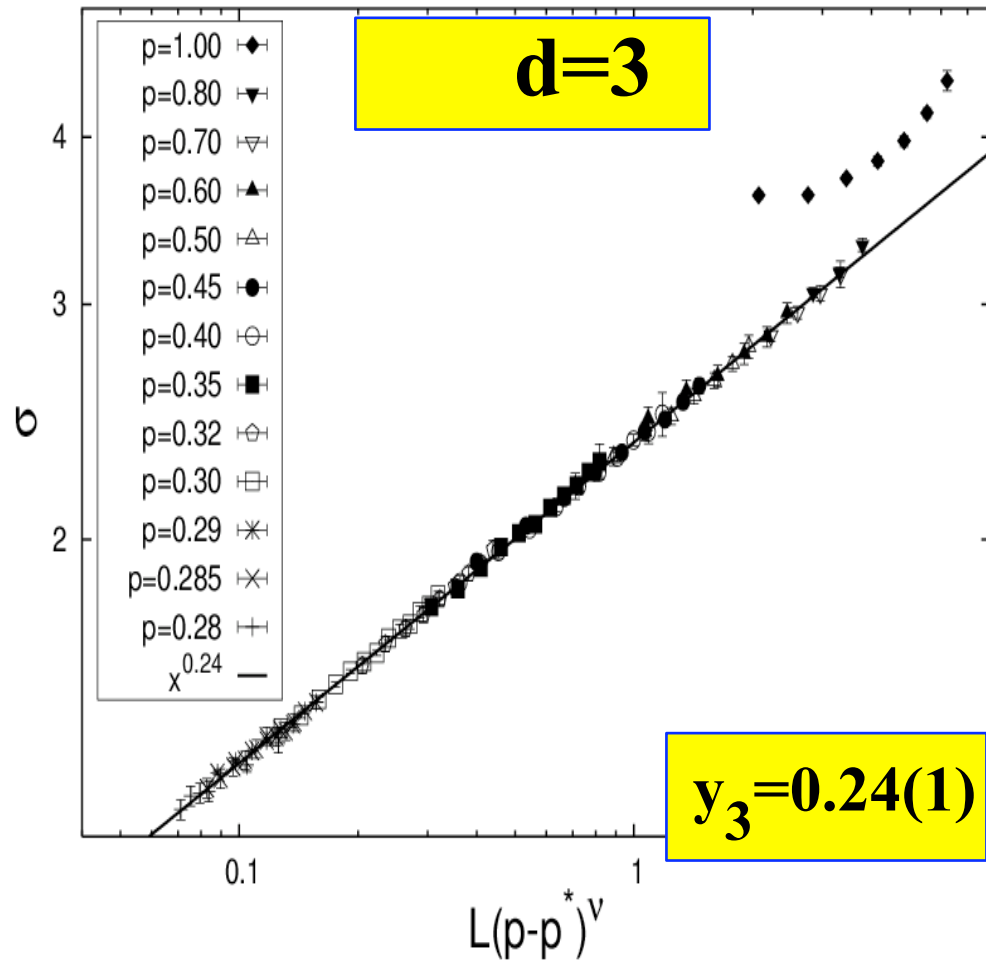
# Stiffness Exponent $\gamma$ for Lattice Glasses:

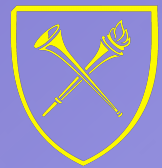
“Stiffness”:  $\sigma(\Delta E) \sim L^\gamma$



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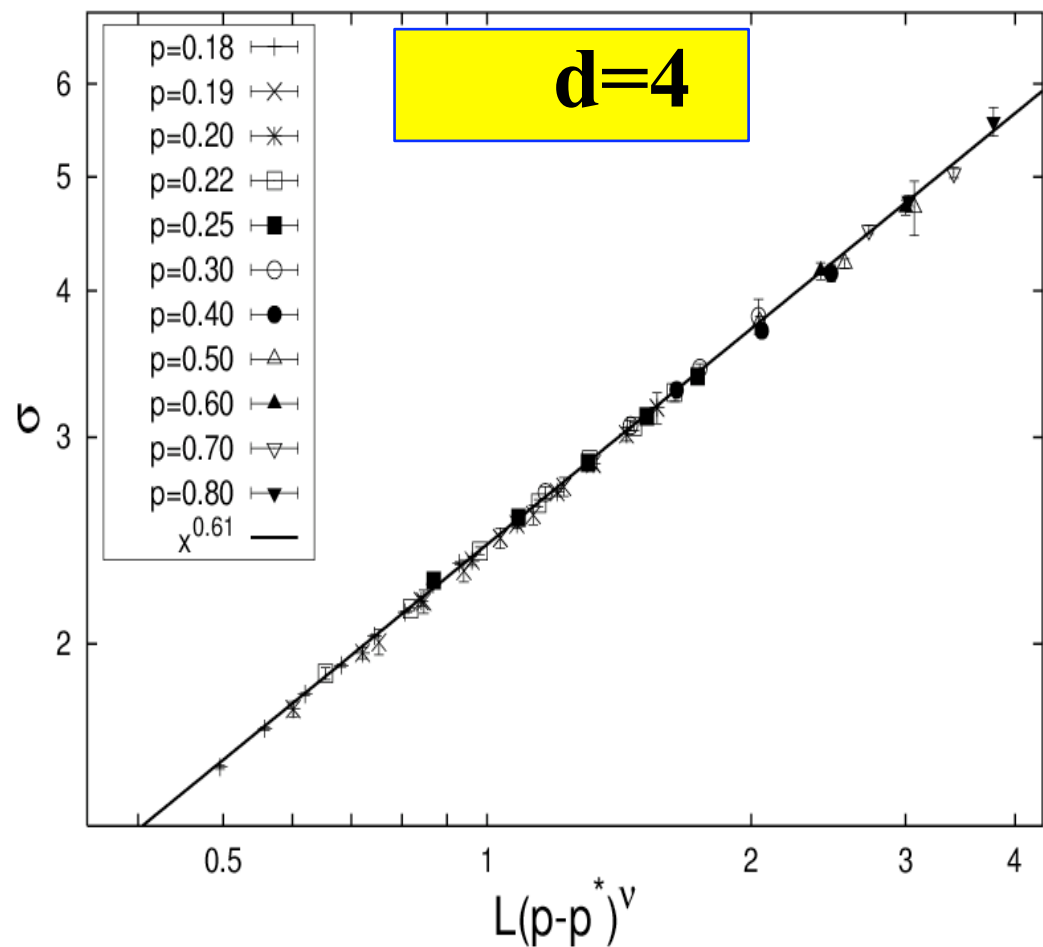
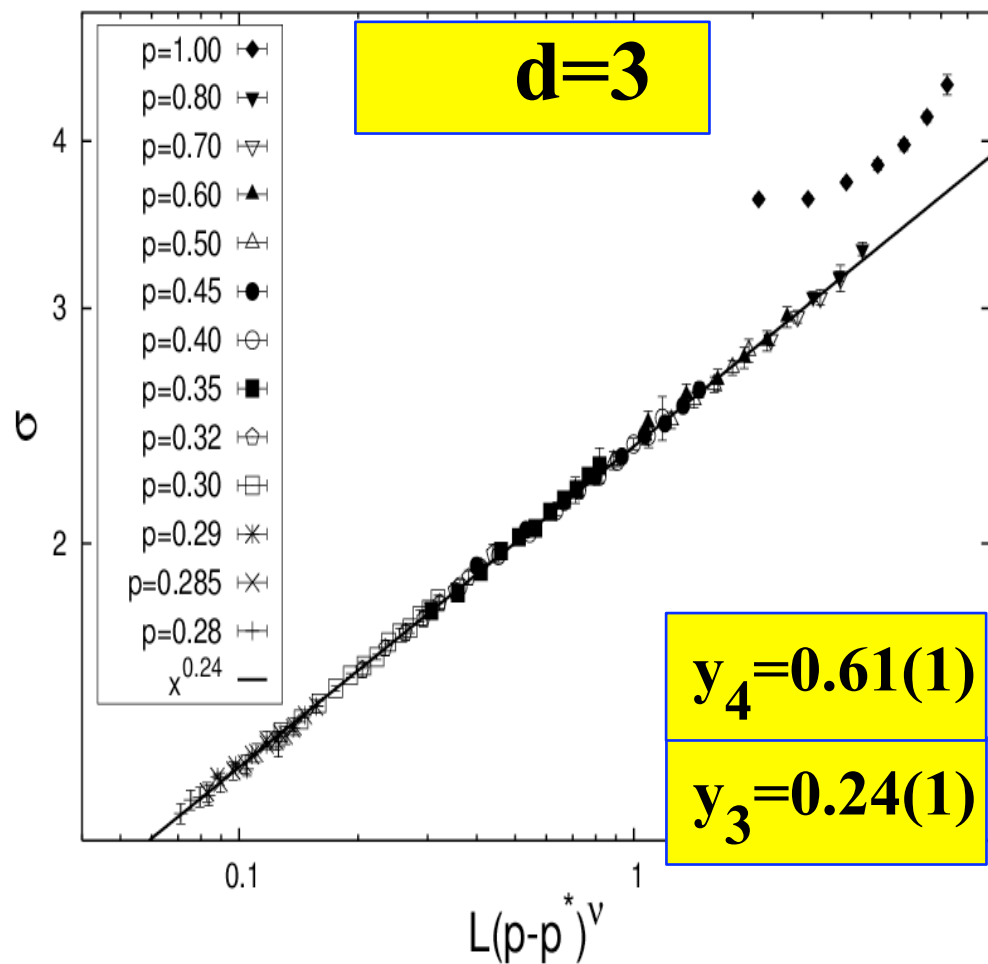
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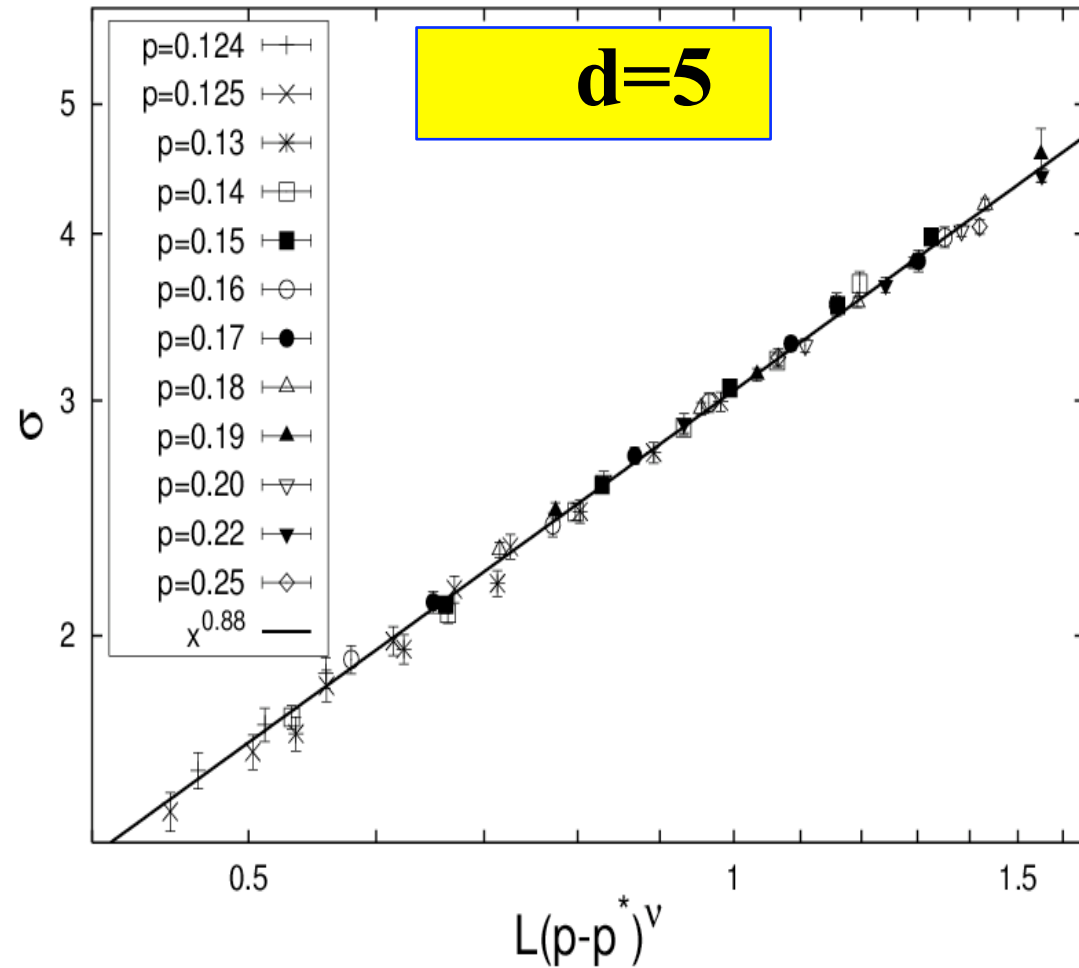
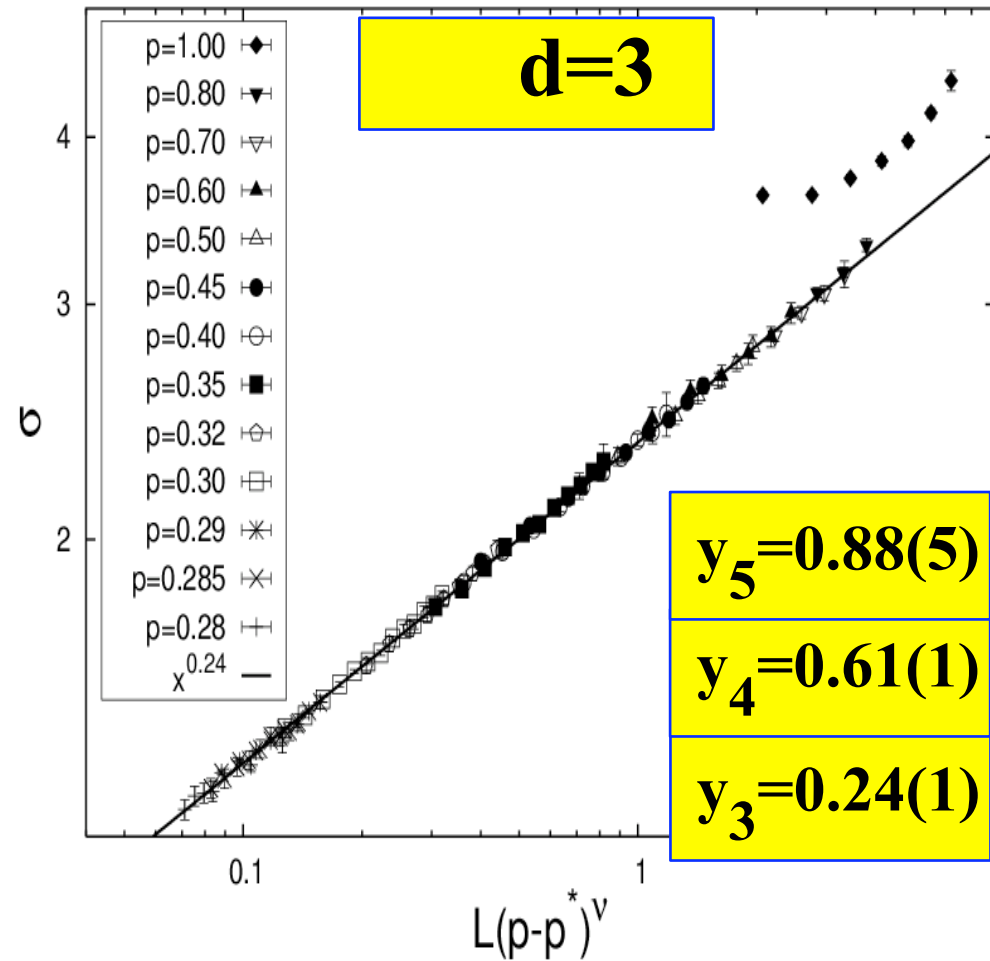
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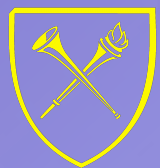
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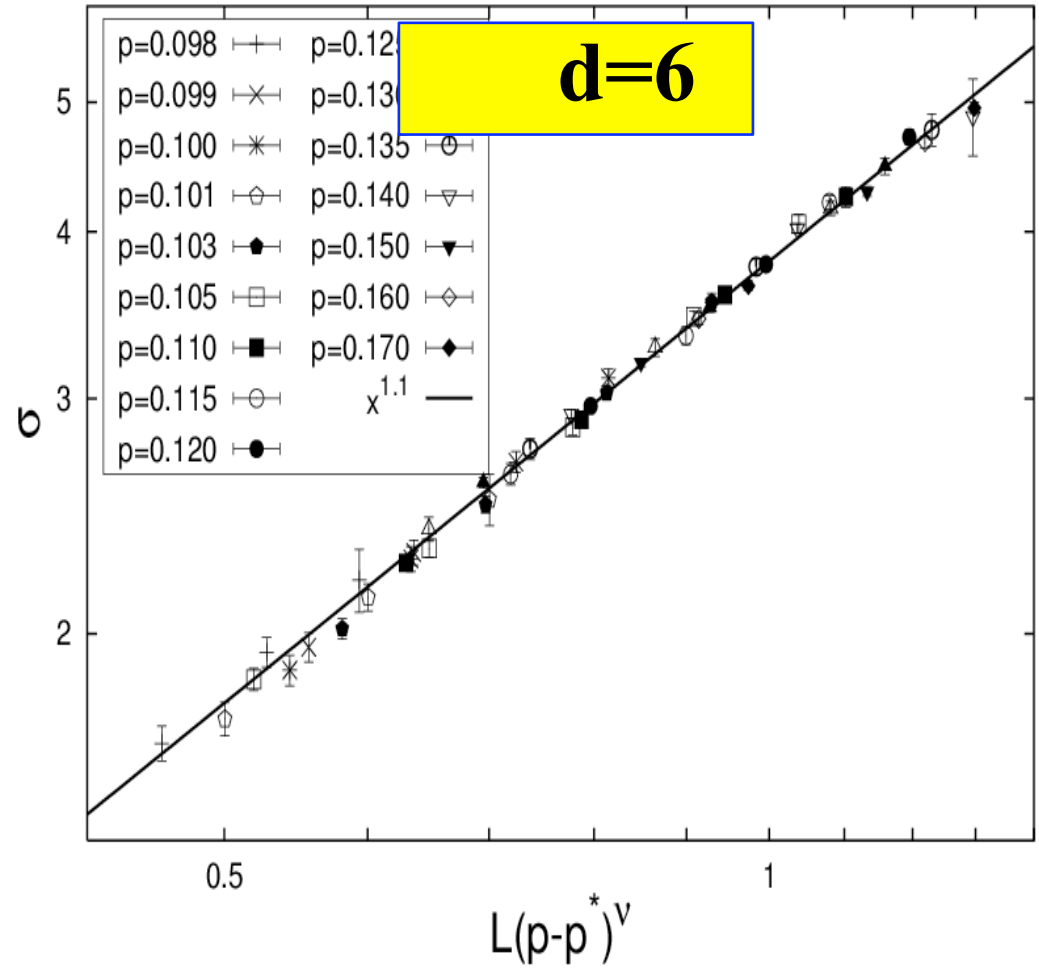
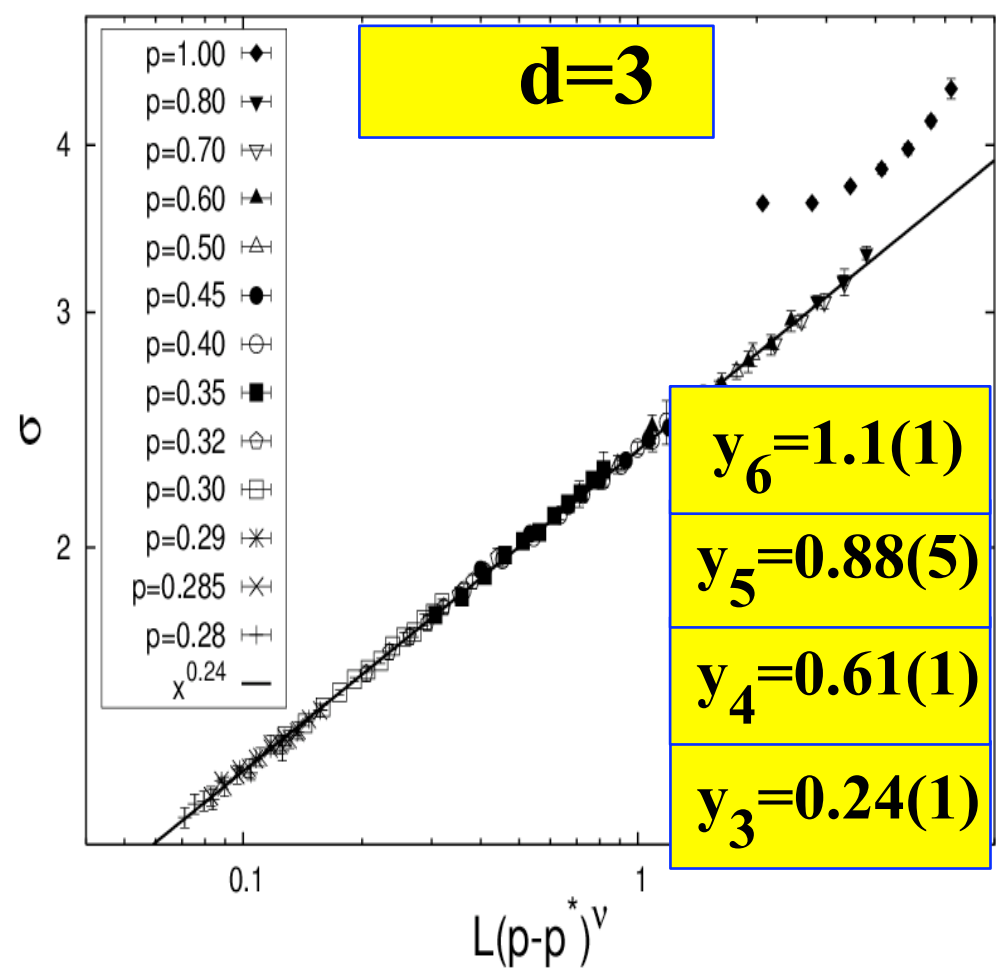
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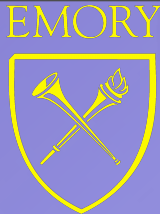




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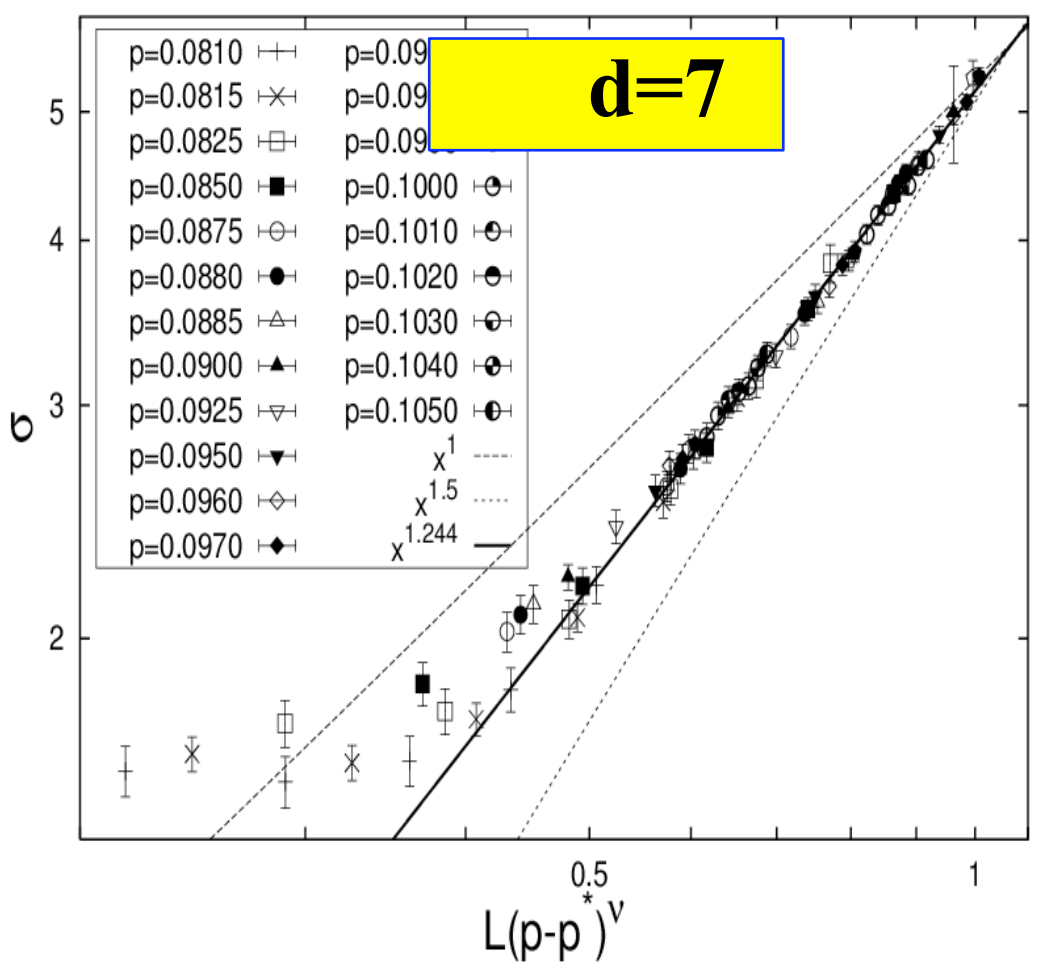
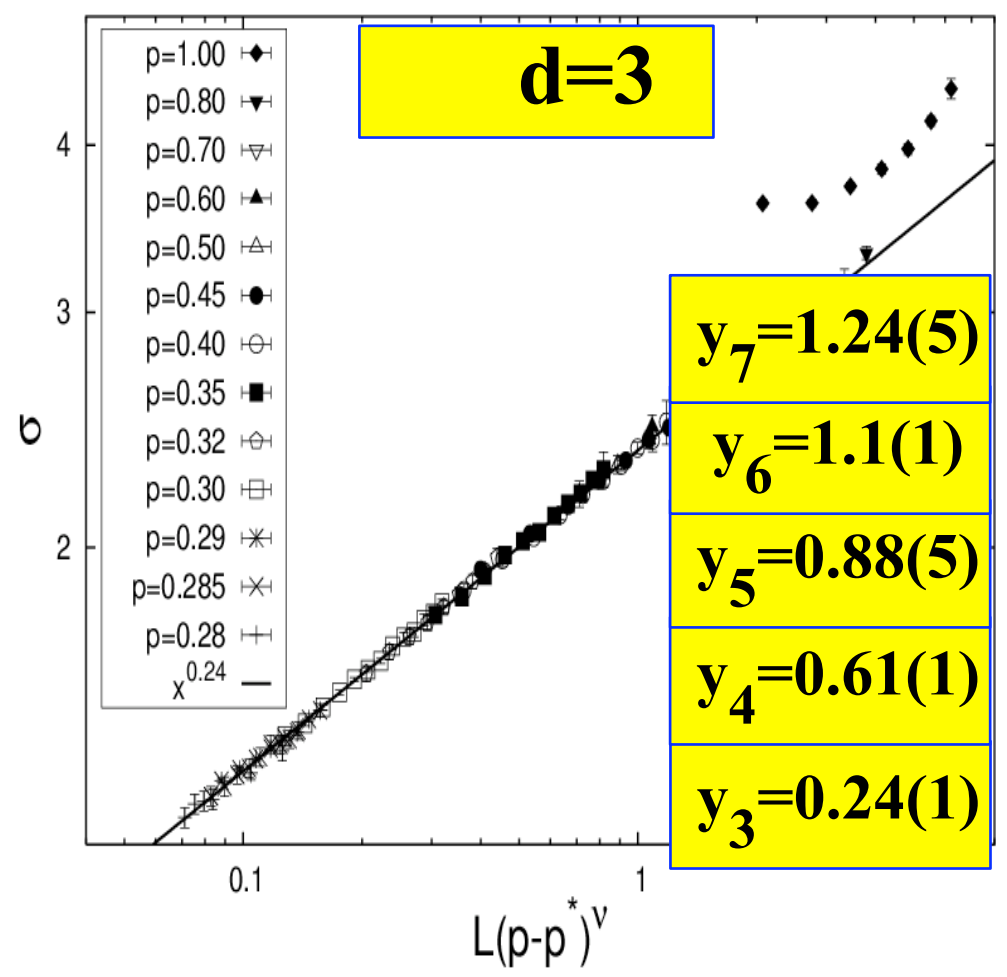
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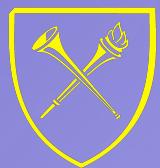




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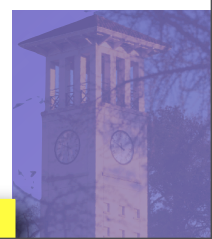
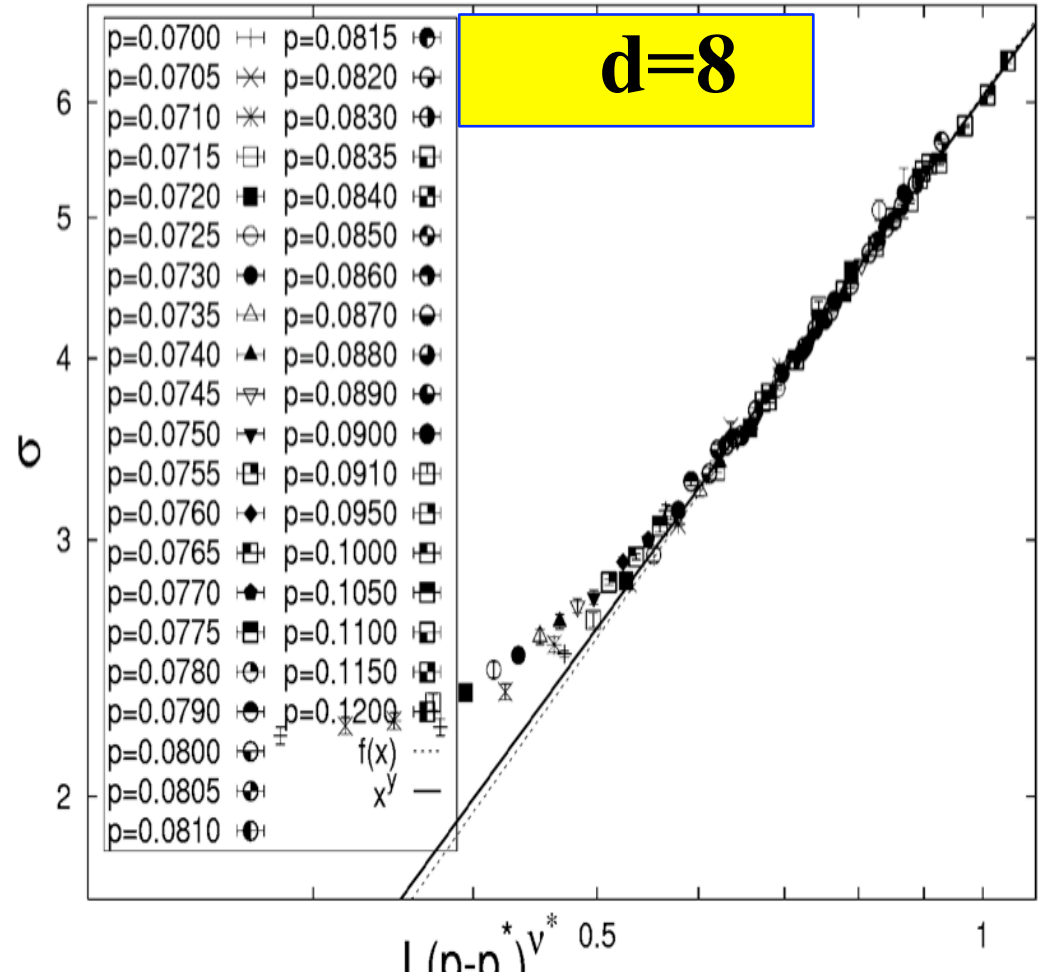
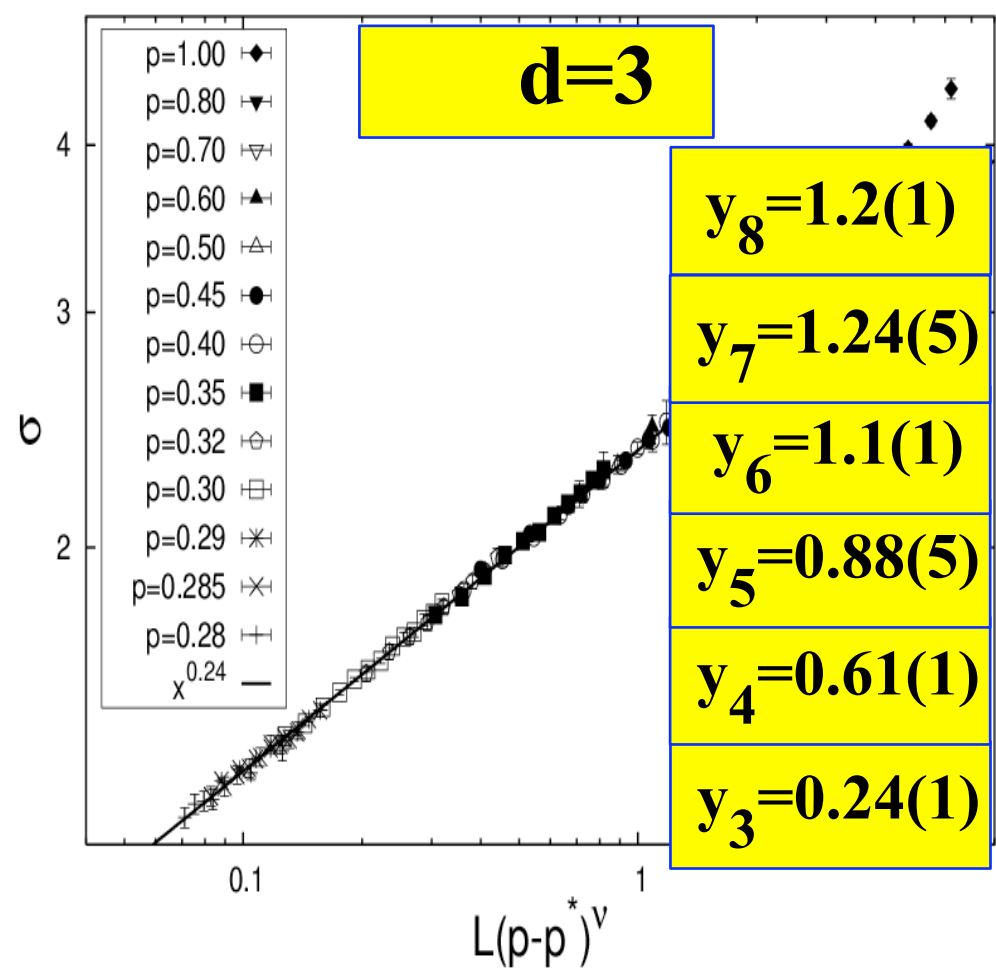
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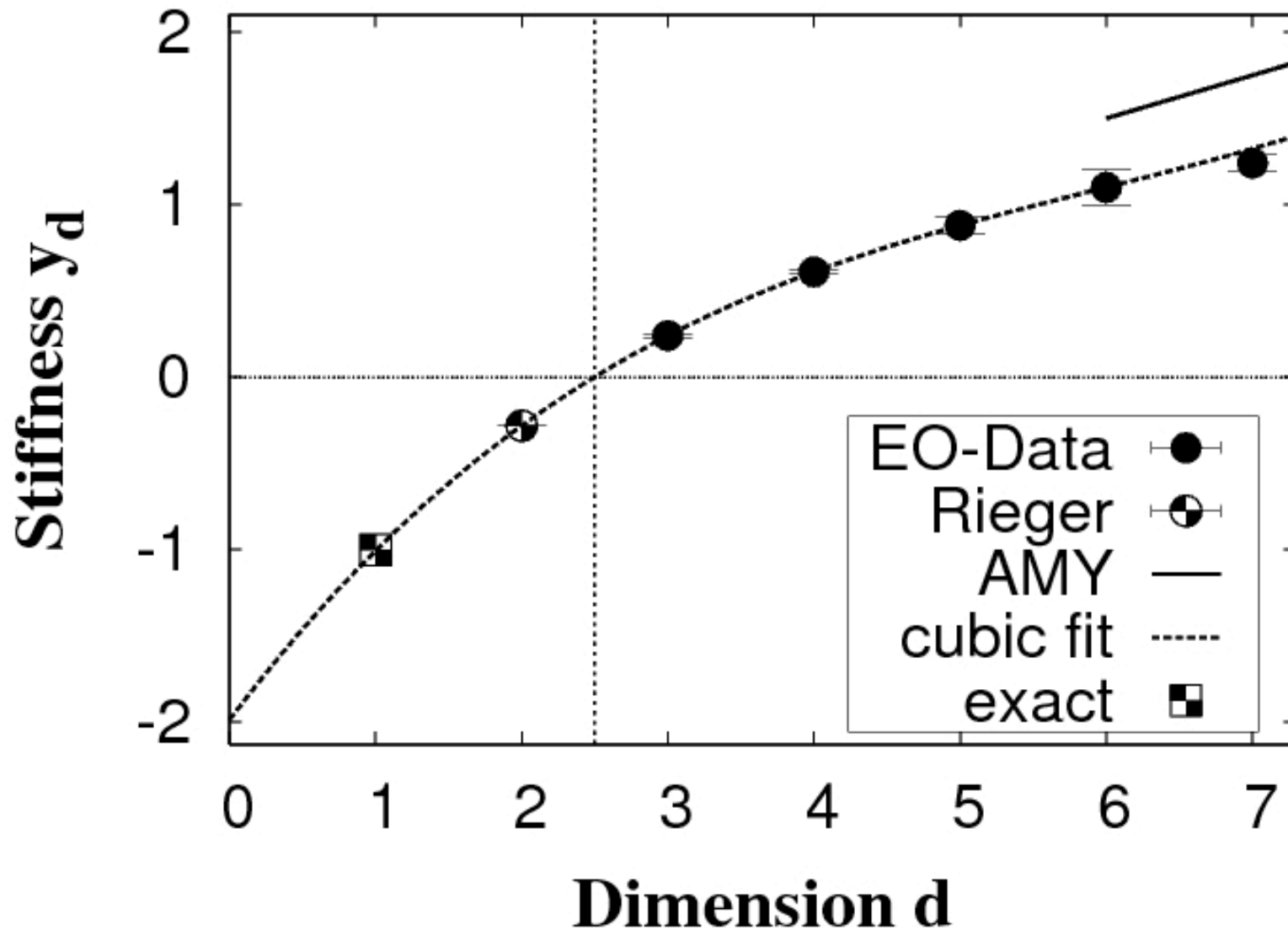
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# Comparing with Theory:

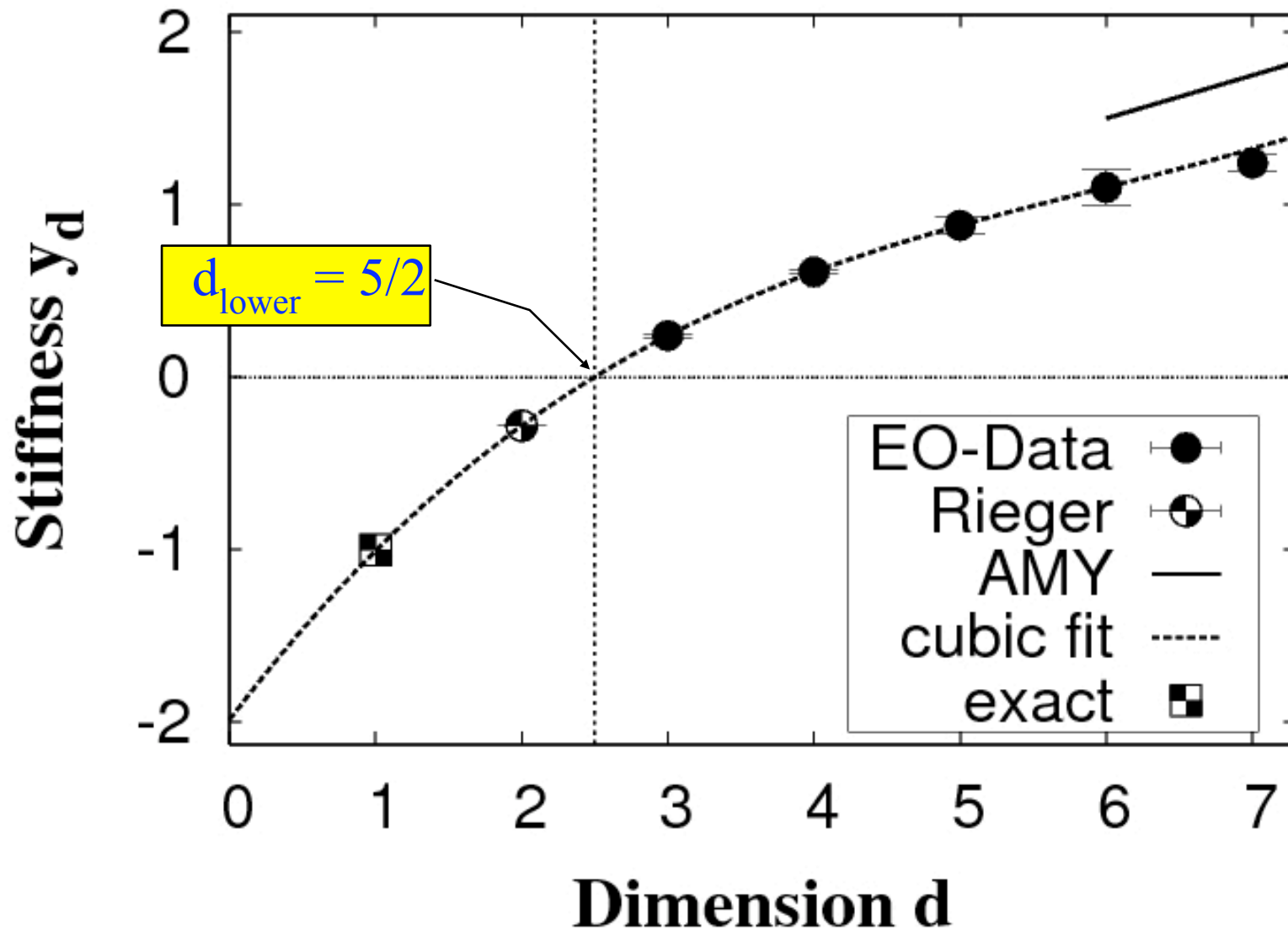
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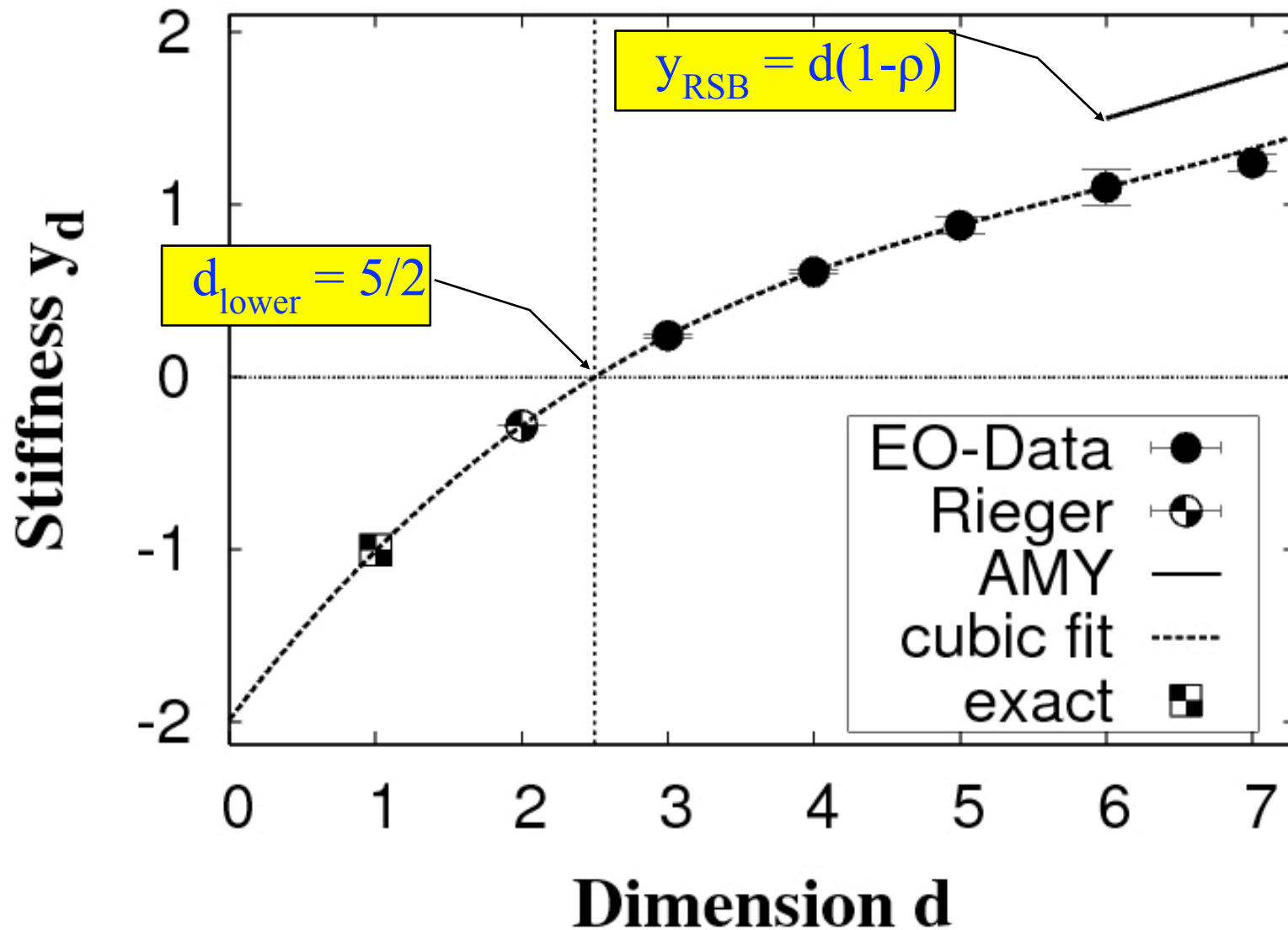
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# Other Evidence for $d_1=5/2$ :

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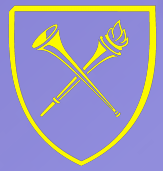




## Other Evidence for $d_l=5/2$ :

- From Theory: (Franz, Parisi & Virasoro, J. Phys. I **4**, 1657, '94)  
Effective Mean Field calculation near  $T_g$ , where Replica Symmetry Breaking (RSB) disappears (ie.  $T_g \rightarrow 0$ ) for  $d_l=5/2$ .





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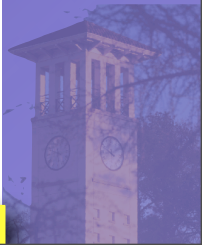
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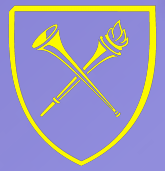
- From Numerics:

Know:

$$T_g \approx \sqrt{2d} \quad (d \rightarrow \infty)$$

$$T_g \approx \sqrt{2d - d_l} \quad (d \rightarrow d_l)$$





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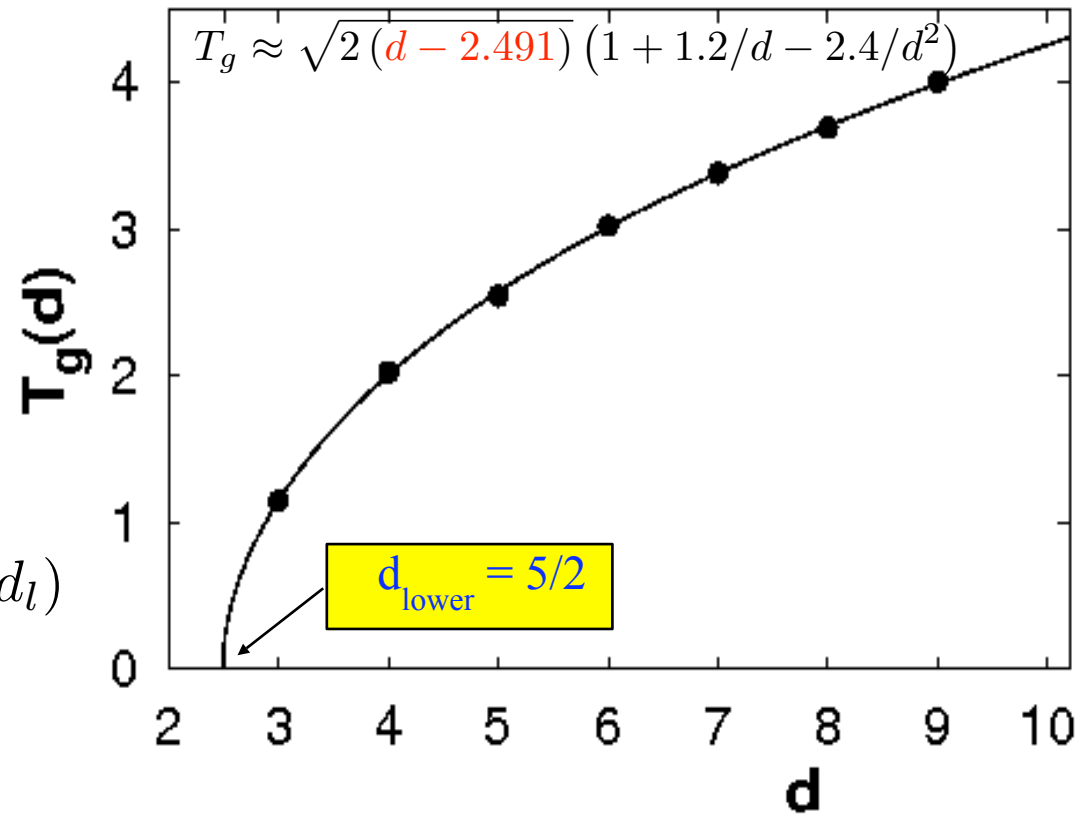
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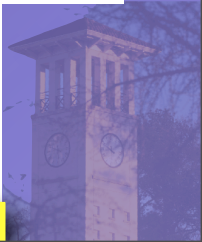
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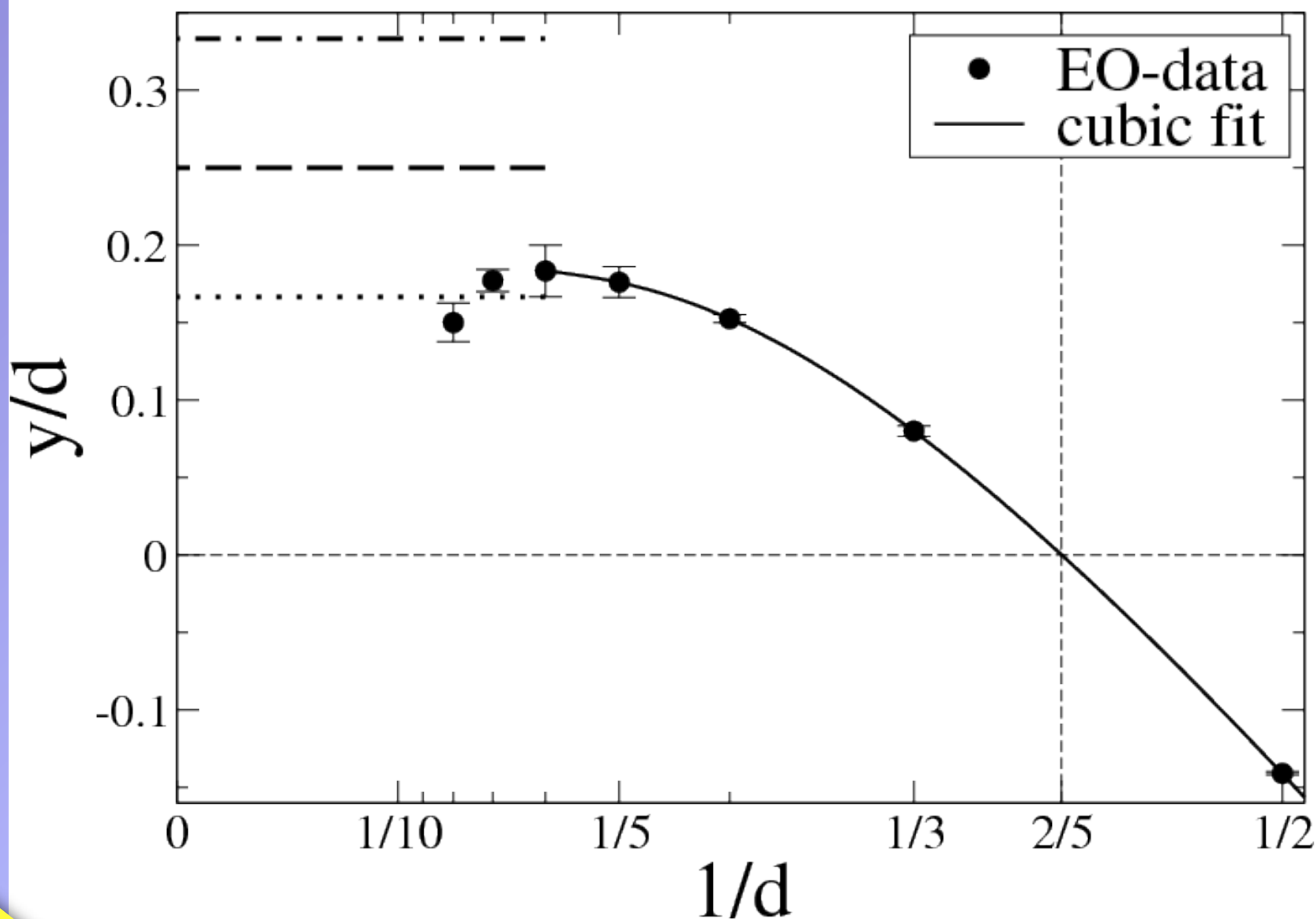
Data from:

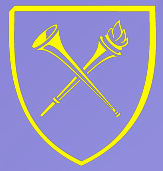
- MC (Ballesteros et al) for  $d=3, 4$
- High-T Series (Klein et al) for  $d \geq 5$



## Comparing with Theory:

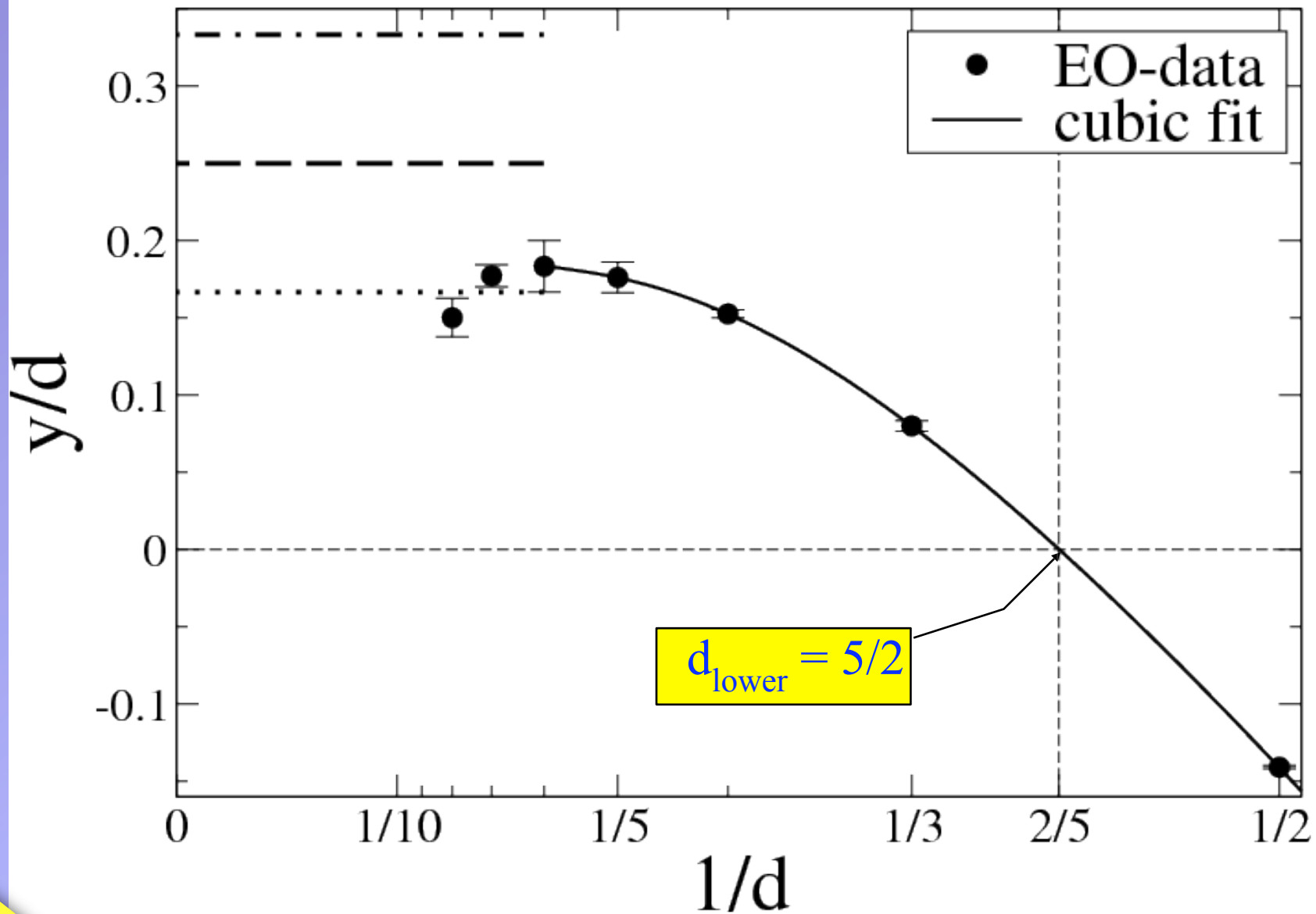
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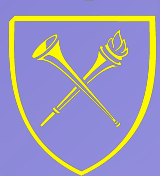


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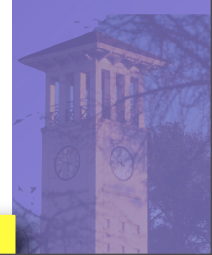
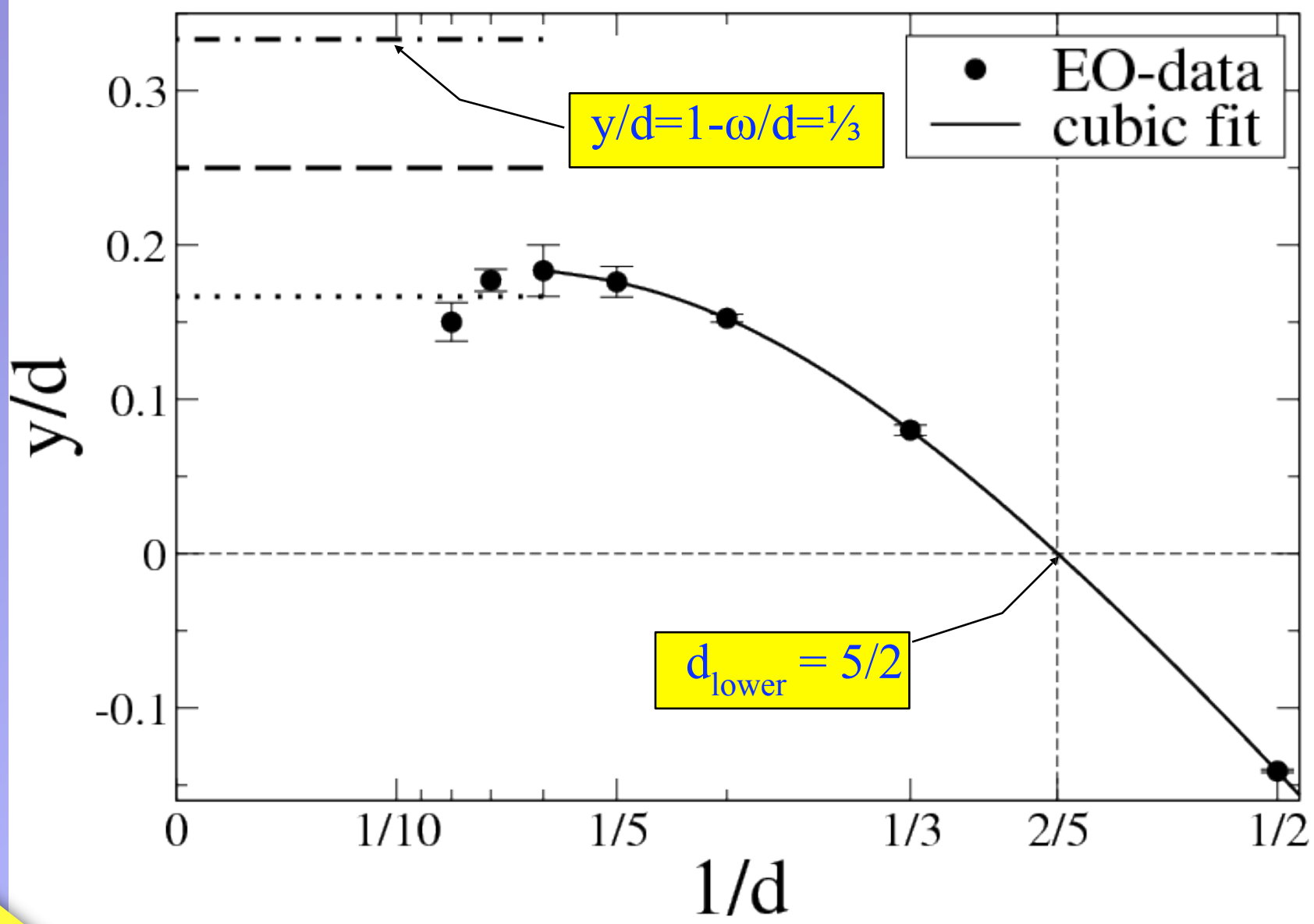


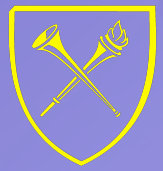




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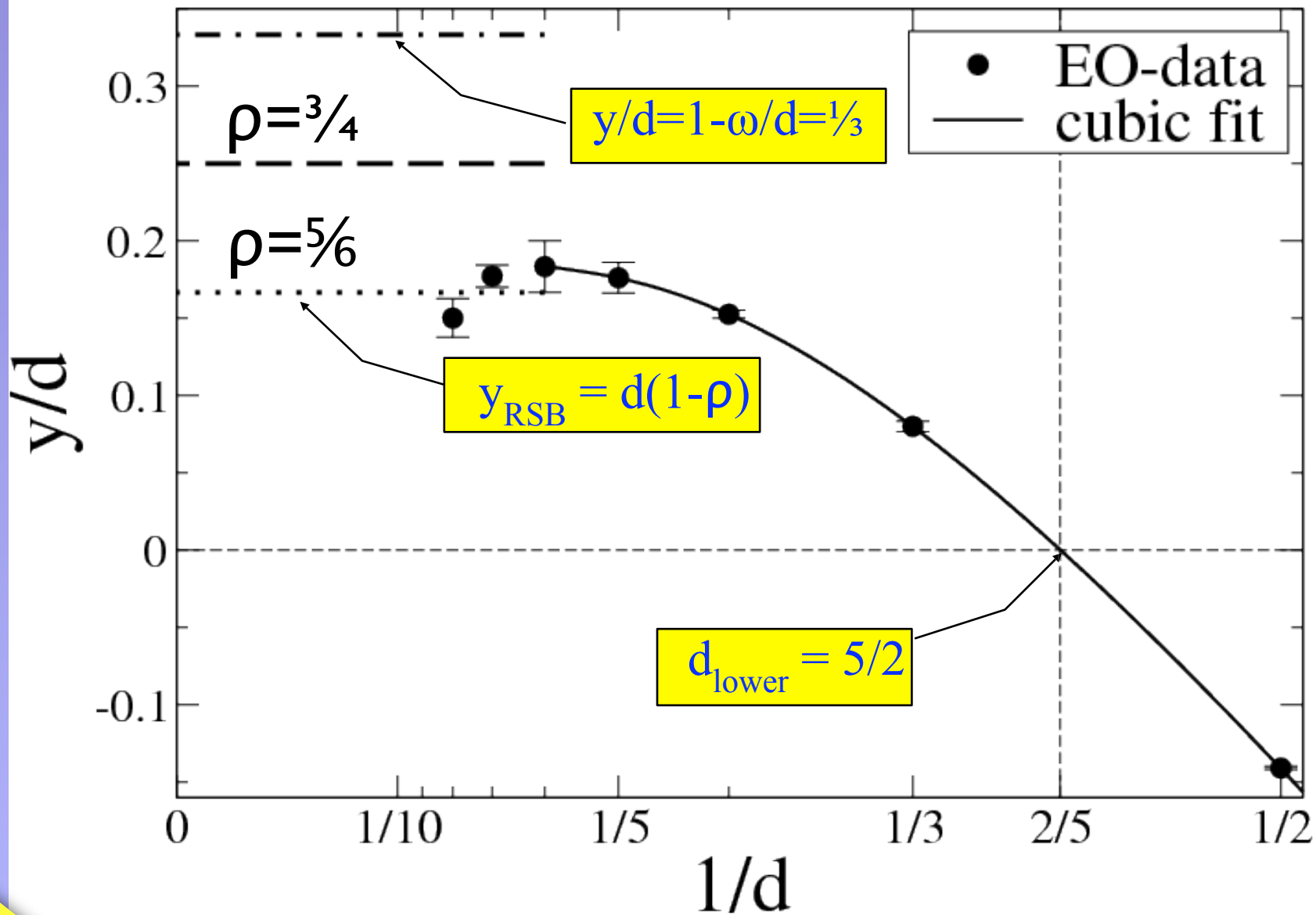
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## Corrections-to-Scaling in EA:

Ground State Energy:  $E(L) \sim e_0 L^d + AL^y \quad (L \rightarrow \infty)$





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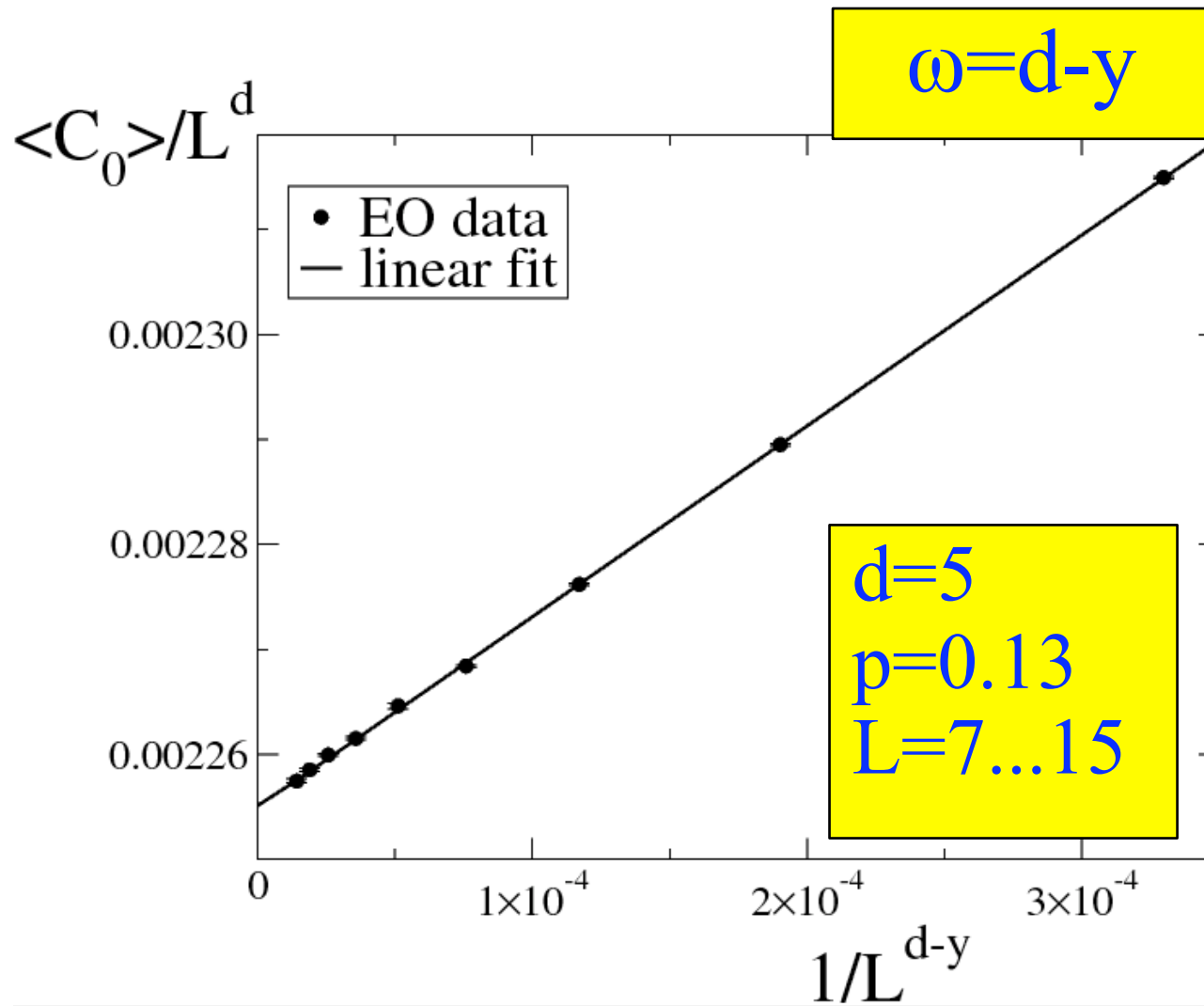
Ground State Energy:  $E(L)/L^d \sim e_0 + A/L^{d-y} \quad (L \rightarrow \infty)$

$$\omega = d - y$$



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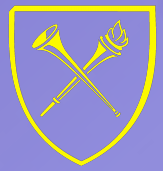
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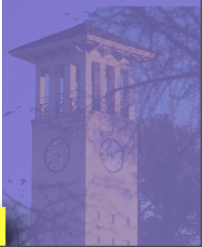


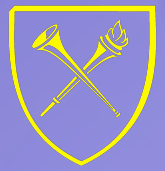


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## ◉ Extremal Optimization:

- Selection *against* extremely *Bad*  $\Rightarrow$  Greedy!
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- *Single, fixed* Parameter ( $\tau$ )  $\Rightarrow$  Simple!
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## ◉ Results:

- Works well for Partitioning, Coloring, Spin Glasses, Satisfiability, Pattern Recognition (at least!).
- Works poorly for TSP, Polymer Folding, ie. strongly connected problems!
- Theory: “*Jamming*” Model, predicting  $\tau_{opt} \rightarrow 1^+$ , always remains super-cooled, not frozen!

