Bifurcation and Patterns in Compromise Processes

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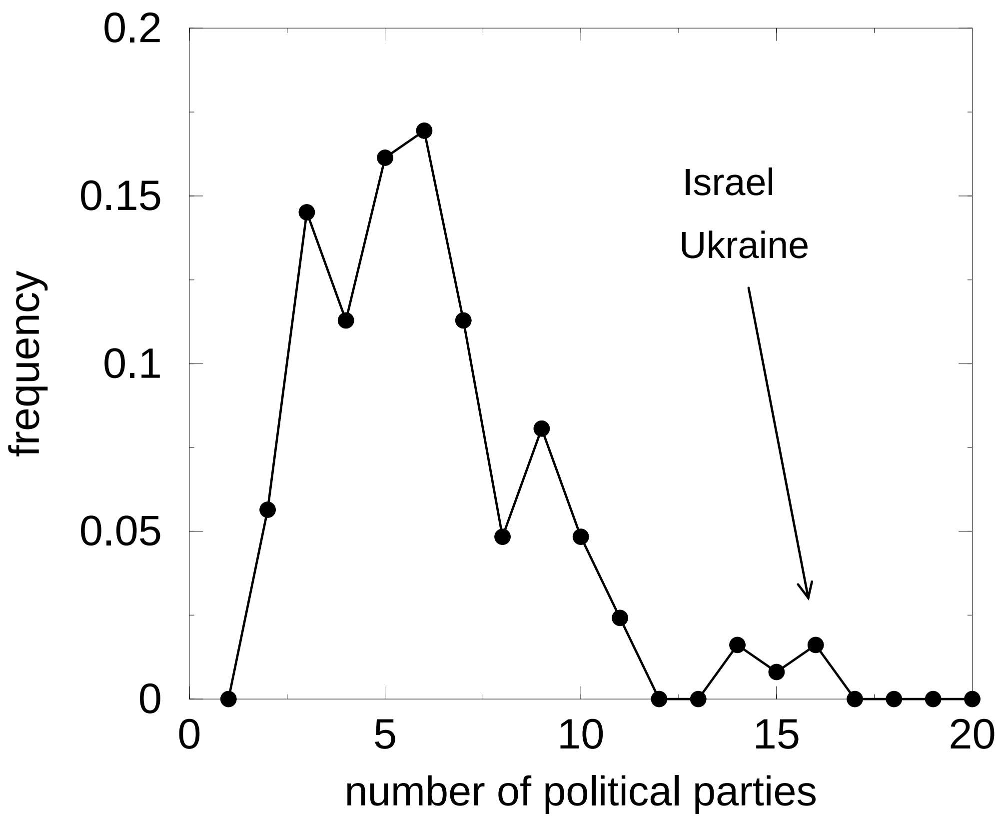
I Motivation

II Continuum: Numerics & Scaling

III Discrete: Theory & General Features

with: Paul Krapivsky, Sidney Redner (Boston)

How many political parties?



- Data: CIA world factbook 2002
- 120 countries with multiparty system
- Average=5.8, Variance=2.9

Simple model?

The Compromise Model

- Opinion measured by continuum variable

$$-\Delta < x < \Delta$$

- Compromise: via pairwise interactions

$$(x_1, x_2) \to \left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}\right)$$

- Conviction: restricted interaction range

$$|x_1 - x_2| < 1$$

- Initial conditions: uniform distribution

$$P(x, t = 0) = \begin{cases} 1 & |x| < \Delta, \\ 0 & |x| > \Delta. \end{cases}$$

- **Minimal, one parameter model**

- **Mimics competition between compromise and conviction**

Consensus

- Kinetic theory: nonlinear rate equations

$$\frac{\partial P(x,t)}{\partial t} = \int\int_{|x_1-x_2|<1} dx_1 dx_2 P(x_1,t)P(x_2,t)$$
$$\times \left[2\,\delta\left(x-\frac{x_1+x_2}{2}\right) - \delta(x-x_1) - \delta(x-x_2)\right]$$

- Integrable for $\Delta < 1/2$:

$$\langle x^2(t)\rangle = \langle x^2(0)\rangle\, e^{-\Delta t}$$

- Final state: localized

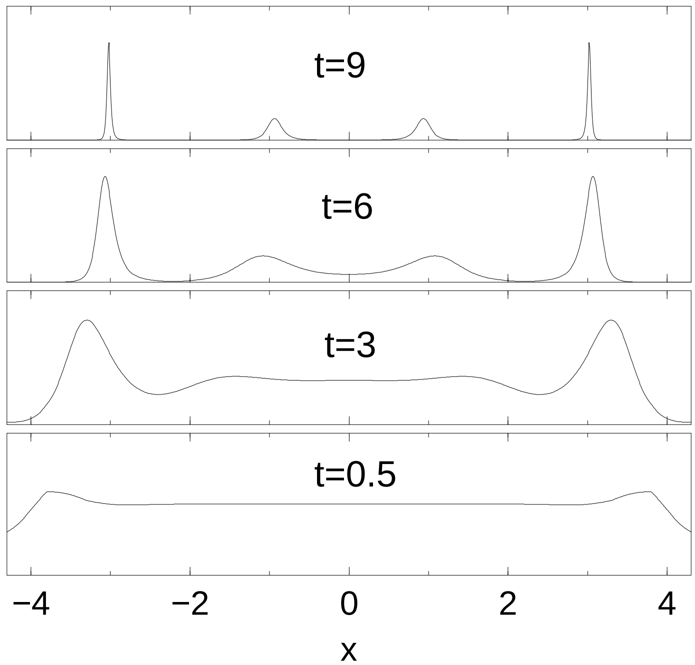
$$P_\infty(x) = 2\Delta\delta(x)$$

- Time dependence: similarity solution

$$\Phi(z) = \frac{2\Delta}{\pi}\frac{1}{(1+z^2)^2} \qquad z = \frac{x}{\langle x^2(t)\rangle^{1/2}}$$

Generally, what is nature of final state?

Diversity

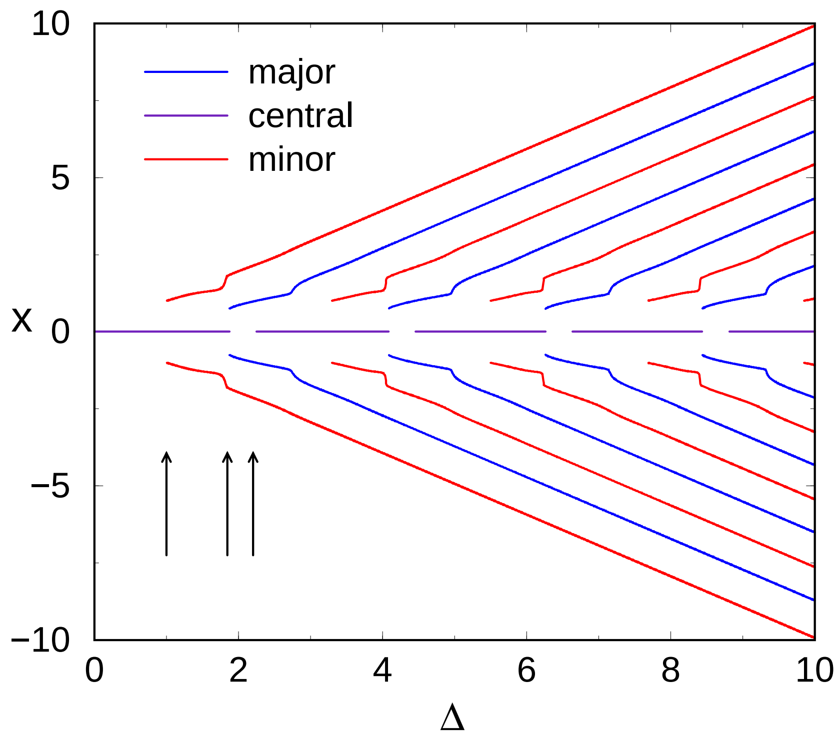


- $\surd$ Numerical integration of rate equations
- Monte Carlo simulation of random process
- Final state:

$$P_\infty(x) = \sum_{i=1}^{N} m_i\, \delta(x - x_i)$$

Multiple localized clusters (parties)

Bifurcations and Patterns



- Periodic sequence of bifurcations

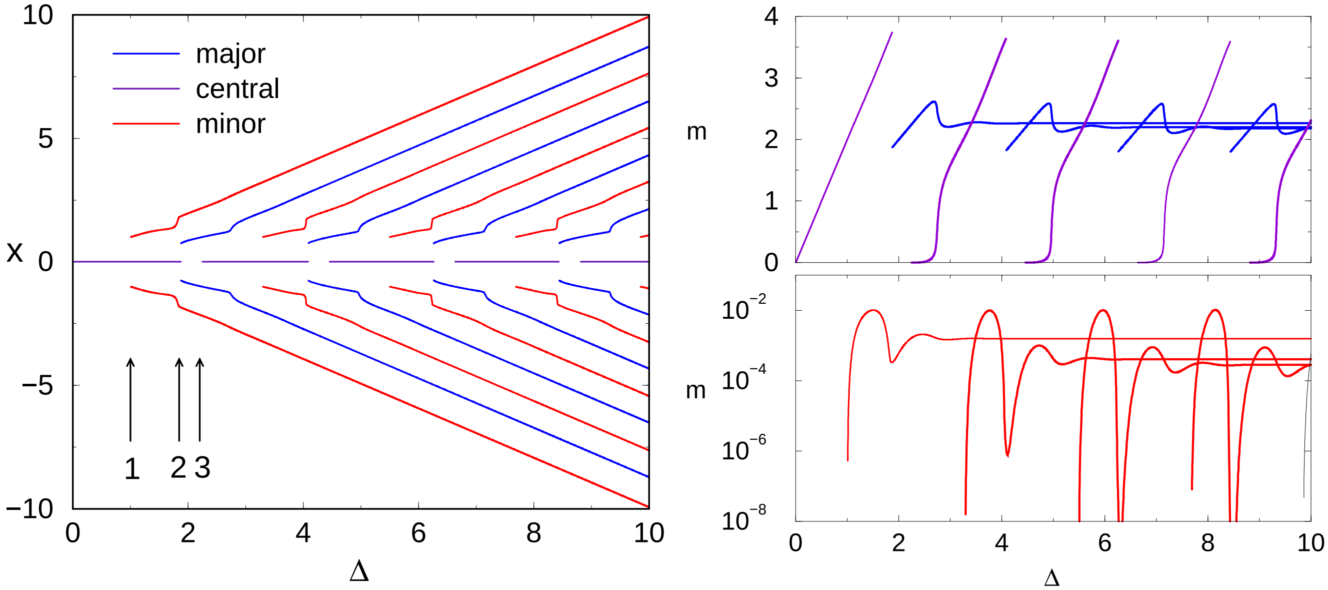
$$x(\Delta) = x(\Delta + L)$$

- Alternating major-minor pattern
- Clusters are equally spaced
- Period $\longrightarrow$ cluster mass, separation

$$L = 2.155$$

Self-similar structure, universality

Cluster masses, bifurcation types



- Masses are periodic as well

$$m(\Delta) = m(\Delta + L)$$

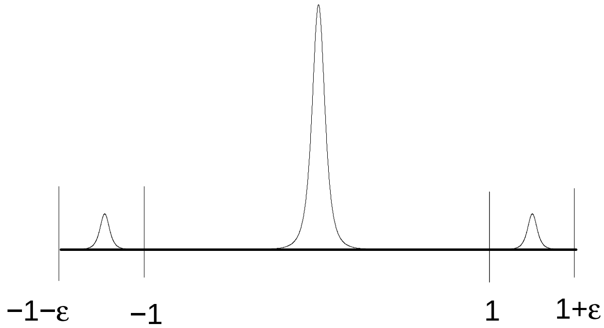
- 3 types of bifurcations:

1. $\emptyset \to \{-x, x\}$ Nucleation of 2 minor branches
2. $\{0\} \to \{-x, x\}$ Nucleation of 2 major branch's
3. $\emptyset \to \{0\}$ Nucleation of central cluster

- Bifurcations occur near origin
- Major: $M \to 2.15$, Minor: $m \to 3 \times 10^{-4}$

Central cluster may or may not exist

Near critical behavior



- Perturbation theory: $\Delta = 1 + \epsilon$
- Central cluster: mass M, $x(\infty) = 0$
- Minor cluster: mass m, $x(\infty) = 1 + \epsilon/2$

$$\frac{dm}{dt} = -mM \quad \to \quad m(t) \sim \epsilon\, e^{-t}$$

- Process stops when $x \sim e^{-t_f/2} \sim \epsilon$
- Final minor cluster mass

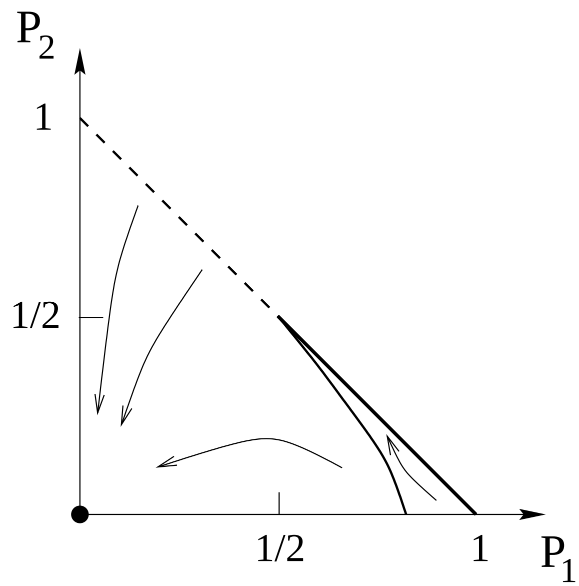
$$m(\infty) \sim m(t_f) \sim \epsilon^3$$

- Argument generalizes to type 3 bifurcations

$$m \sim (\Delta - \Delta_c)^\alpha \qquad \alpha = \begin{cases} 3 & \text{type } 1 \\ 4 & \text{type } 3 \end{cases}$$

$\boxed{\textbf{Masses vanish algebraically near type 1, 3 bif}}$

Discrete Opinions



- Basic process: $(i-1, i+1) \to (i, i)$

- Rate equation:

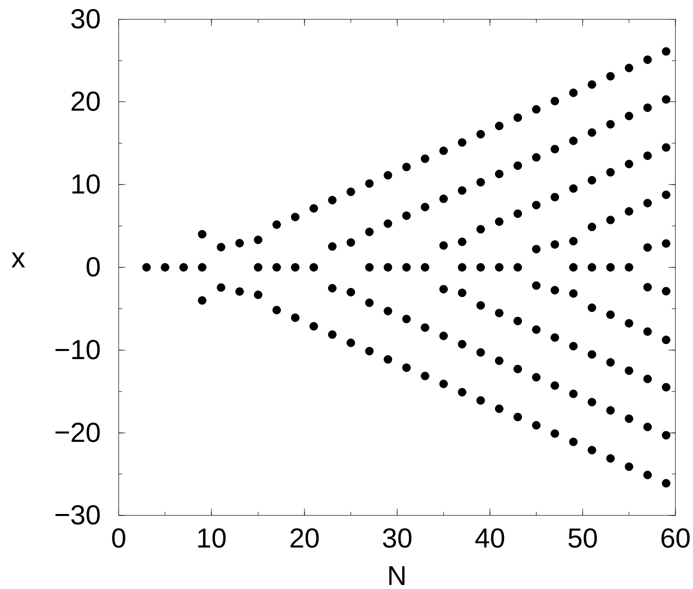
$$\frac{d}{dt}P_i = 2P_{i-1}P_{i-1} - P_i(P_{i-2} + P_{i+2})$$

- Example: 6 states, $P_i = P_{N-i}$

- Initial conditions determine final state

- Isolated fixed points, lines of fixed points

General Features

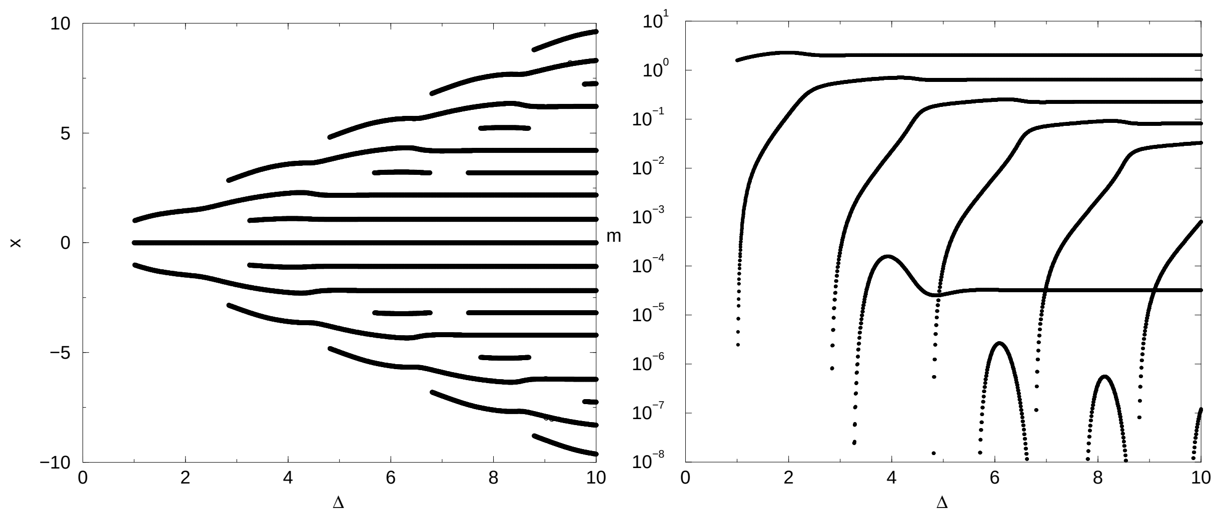


- Dissipative system: volume contracts
- Lyapunov (energy) function exists $\langle x^2 \rangle$
- No cycles or strange attractors
- Uniform state is unstable: $P_i = 1 + \phi_i$

$$\phi_t + \left(\phi + a\phi_{xx} + b\phi^2\right)_{xx} = 0$$

Discrete case yields useful insights

Exponential initial conditions



- Bifurcations induced at the boundary
- Periodic structure
- Two types of bifurcations
  1. Nucleation of major branch
  2. Nucleation of minor branch

Central cluster is stable

Conclusions

- Cluster form via bifurcations
- Periodic structure
- Alternating major-minor pattern
- Central party not always exists
- Power-law behavior near transitions

Outlook

- Role of initial conditions? Classification?
- Role of spatial dimension? Correlations?
- Add disorder, inhomogeneities
- Tiling/Packing in 2D,3D?

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G. Weisbuch, *cond-mat*/0111494.