Size of Epidemic Outbreaks

small, medium, or large?

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Complex Systems (T-13)

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Thanks: Aric Hagberg (T-7)

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my talk

Outline

- Introduction: infection processes
- Deterministic versus stochastic description
- Size of outbreaks
- Duration of outbreaks
- Exact results

SIR Infection Processes

- **Total population: N**
- **Susceptibles, Infected, Recovered**

$$N = s + i + r$$

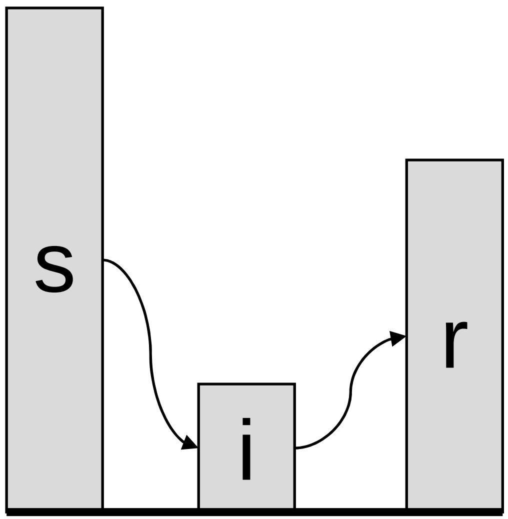
- **Sub-populations change due to:**

(i) infection

$$(s,i,r) \xrightarrow{\alpha s\, i / N} (s-1, i+1, r)$$

(ii) recovery

$$(s,i,r) \xrightarrow{i} (s, i-1, r+1)$$

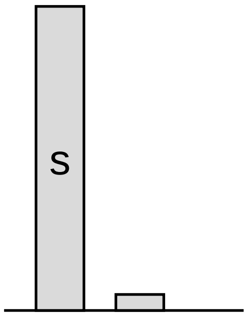


Two <u>dimensionless</u> parameters:
infection rate, population size

The Canonical Problem

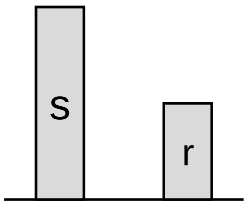
- **Initial state: one infected individual**

$$(s,i,r) = (N-1,1,0) \qquad t = 0$$



- **Final state: no infected individuals**

$$(s,i,r) = (N-n,0,n) \qquad t = t_f$$



What is the size of the epidemic outbreak n?
What is the duration of the outbreak t_f?

Transition probabilities

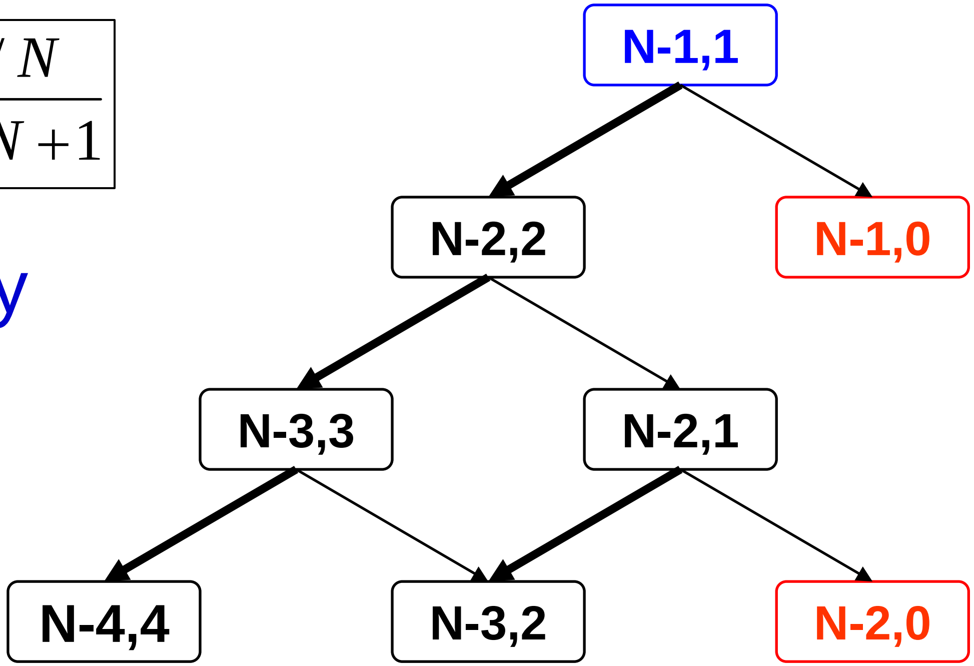
infection probability

$$P_{\text{infect}} = \frac{r_{\text{infect}}}{r_{\text{infect}} + r_{\text{recover}}} = \frac{\alpha s / N}{\alpha s / N + 1}$$

recovery probability

$$P_{\text{recover}} = \frac{1}{\alpha s / N + 1}$$

$$(\Delta t)^{-1} = \sum_i r_i$$



Efficient Monte Carlo simulation method

Deterministic Epidemics

- **Evolution of average population, infinite hierarchy**

$$\frac{d\langle s\rangle}{dt} = -\frac{\alpha}{N}\langle si\rangle \qquad \frac{d\langle i\rangle}{dt} = \frac{\alpha}{N}\langle si\rangle - \langle i\rangle$$

- **"Hydrodynamics": ignore correlations**

$$Assume: \quad \langle si\rangle = \langle s\rangle\langle i\rangle \quad Use: \quad S = \langle s\rangle / N, \quad I = \langle i\rangle / N$$

- **SIR equations**

$$\frac{dS}{dt} = -\alpha SI \qquad \frac{dI}{dt} = \alpha SI - I$$

Predicts average behavior for infinite N

Epidemic Outbreaks

616 19. Epidemic Models and the Dynamics of Infectious Diseases

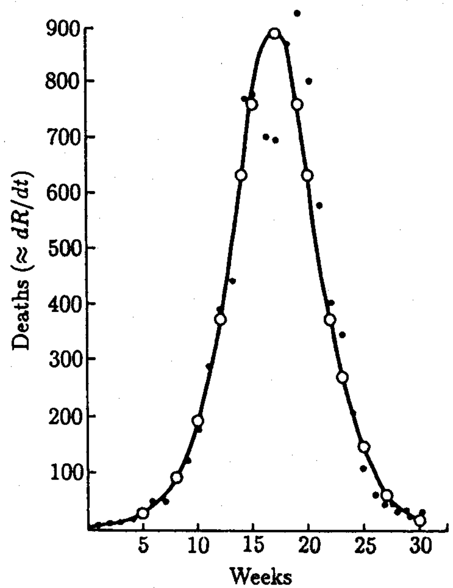
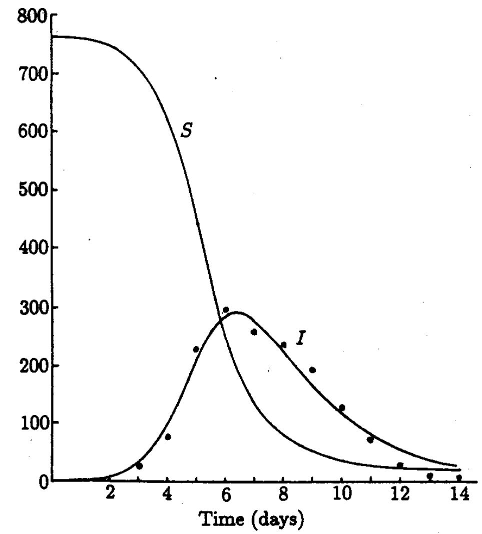


Fig. 19.2. Bombay plague epidemic of 1905–6. Comparison between the data (•) and theory (∘) from the (small) epidemic model and where the number of deaths is approximately dR/dt given by (19.16). (After Kermack and McKendrick 1927)



;. 19.3. Influenza epidemic data (•) for a boys boarding school as reported in British Medical ırnal, 4th March 1978. The continuous curves for the infectives (I) and susceptibles (S) were ;ained from a best fit numerical solution of the SIR system (19.1)–(19.3): parameter values = 763, $S_0 = 762$, $I_0 = 1$, $\rho = 202$, $r = 2.18 \times 10^{-3}$/day. The conditions for an epidemic to ır, namely $S_0 > \rho$ is clearly satisfied and the epidemic is severe since R/ρ is not small.

Bombay plague epidemic (1905)

Influenza epidemic (1978)

$$N = 763 \quad \alpha \cong 3.7$$

Textbook: JD Murray, Mathematical biology

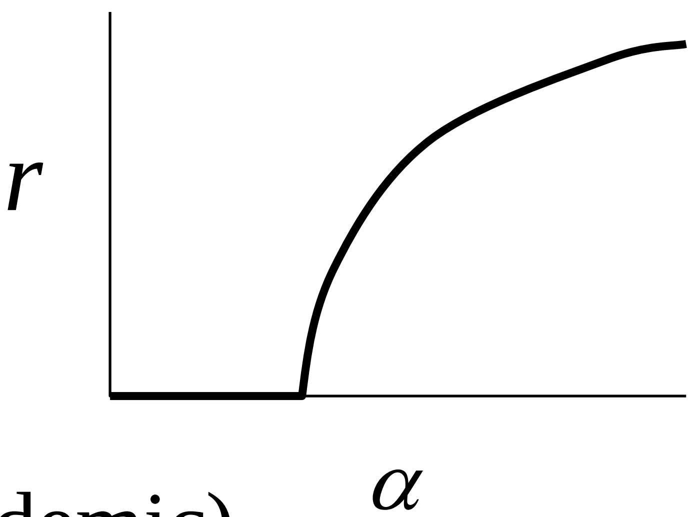
The Epidemic Threshold

- **The average outbreak size**

$$r = \langle n \rangle / N$$

- **From ODE's**

$$r = 1 - S(\infty)$$



Phase transition

$$r = \begin{cases} 0 & \alpha \le 1 \quad \text{(endemic)} \\ 1 - e^{-\alpha r} & \alpha \ge 1 \quad \text{(pandemic)} \end{cases}$$

Deterministic approach predicts r

Behavior at the epidemic threshold

Why worry about the epidemic threshold?

- Evolution (mutation) increases virus lifetime near the epidemic threshold   Antia et al, Nature 2003
- Human efforts (immunization) reduce infection rate

  Anderson & May, 1991
- Actual infection rates can be close to one

  Hethcote, SIAM REV 2000

Size of outbreak at threshold

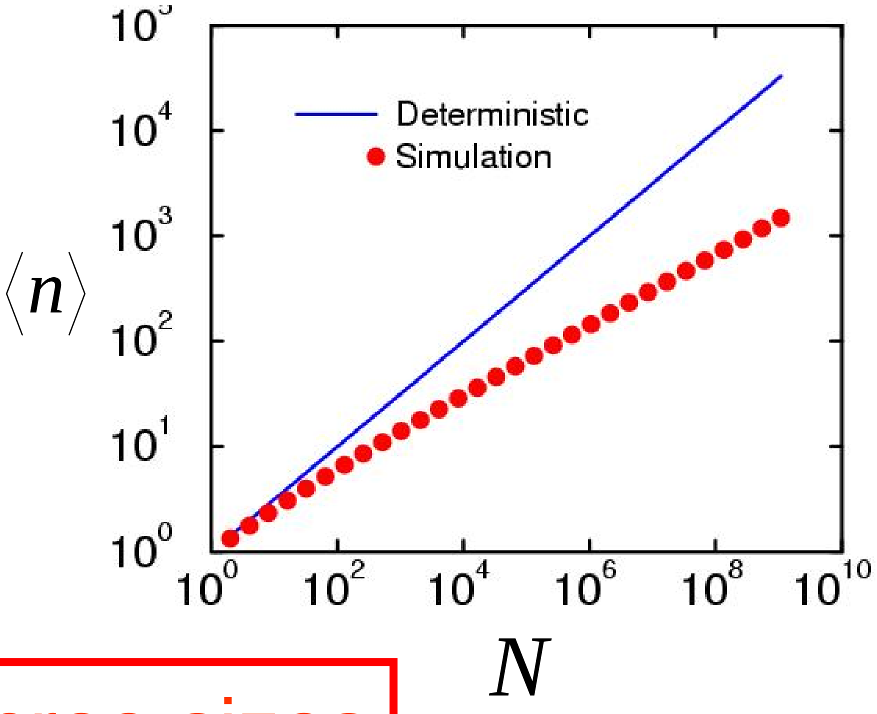
- **Predictions of deterministic approach**

$$r = 1 - \exp[-\alpha(r + 1/N)]$$

$$r = N^{-1/2} \quad \text{for} \quad \alpha = 1$$

- **Average outbreak size**

$$\langle n \rangle = \begin{cases} O(1) & \alpha < 1 \\ N^{1/2} & \alpha = 1 \\ rN & \alpha > 1 \end{cases}$$



1. Epidemics come in three sizes
2. Deterministic approach fails

Stochastic Epidemics

- **"Kinetic theory": probability that the system is in a microstate**

$$\frac{d}{dt}P(s,i)=\frac{\alpha}{N}\left[(s+1)(i-1)P(s+1,i-1)-siP(s,i)\right] \\ +\left[(i+1)P(s,i+1)-iP(s,i)\right]$$

- **As a PDE**

$$\partial_t P=\frac{\alpha}{N}\left[\left(\partial_s-\partial_i\right)P+\frac{1}{2}\left(\partial_{i,i}-2\partial_{s,i}+\partial_{s,s}\right)\right]P+\cdots$$

- **Averages follow**

$$\langle s\rangle=\sum sP(s,i) \qquad \langle si\rangle=\sum siP(s,i)$$

Exact, complete description

Textbook: NTJ Bailey, Math. Theo. of Infec. Diseas, 1975.

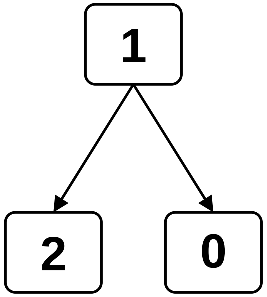
Stochastic epidemics, infinite population

- **Infection probability is fixed (s=N)**

$$P_{\text{infect}} = \frac{\alpha s/N}{\alpha s/N+1} = \frac{\alpha}{\alpha+1}$$

- **Infection process is a branching process**
- **G_n=Probability outbreak size is n, obeys recursion**

$$G_n = \frac{1}{\alpha+1}\delta_{n,1} + \frac{\alpha}{\alpha+1}\sum_{i+j=n} G_i G_j$$



$$G_n = \frac{1}{\alpha+1}\frac{\Gamma(n-1/2)}{\Gamma(n+1)\Gamma(1/2)}\left[1-\left(\frac{1-\alpha}{1+\alpha}\right)^2\right]^{n-1}$$

Outbreak probabilities

- **Probability outbreak has finite size**

$$\sum_n G_n = \begin{cases} 1 & \alpha \le 1 \\ \alpha^{-1} & \alpha \ge 1 \end{cases}$$

- **Distribution of outbreak size**

$$G_n \sim \begin{cases} n^{-3/2} \exp[-n / n_0] & \alpha \ne 1 \\ n^{-3/2} & \alpha = 1 \end{cases}$$

- **Average outbreak size (endemic phase)**

$$\langle n \rangle = (1 - \alpha)^{-1}$$

Size of outbreaks

1. **Assume maximal outbreak size N_***

$$\langle n\rangle = \sum_{n=1}^{N_*} n G_n \sim \sum_{n=1}^{N_*} n^{-1/2} \sim N_*^{1/2}$$

2. **Population depletes, infection rate reduces, epidemic dies out**

$$\alpha_{\text{eff}} N = \alpha(N - N_*) \qquad \Rightarrow \qquad \alpha_{\text{eff}} = 1 - \frac{N_*}{N}$$

3. **Outbreak is effectively endemic**

$$\langle n\rangle = (1-\alpha)^{-1} \sim N / N_*$$

4. **Match two estimates: new scaling laws**

$$N_* \sim N^{2/3} \quad \text{and} \quad \langle n\rangle \sim N^{1/3}$$

Distinct outbreak size at the threshold

Behavior extends near threshold

- **Size of near threshold region (scaling window)**

$$(1-\alpha)^{-1} \sim N^{1/3} \quad \Rightarrow \quad 1-\alpha \sim N^{-1/3}$$

- **Outbreak size**

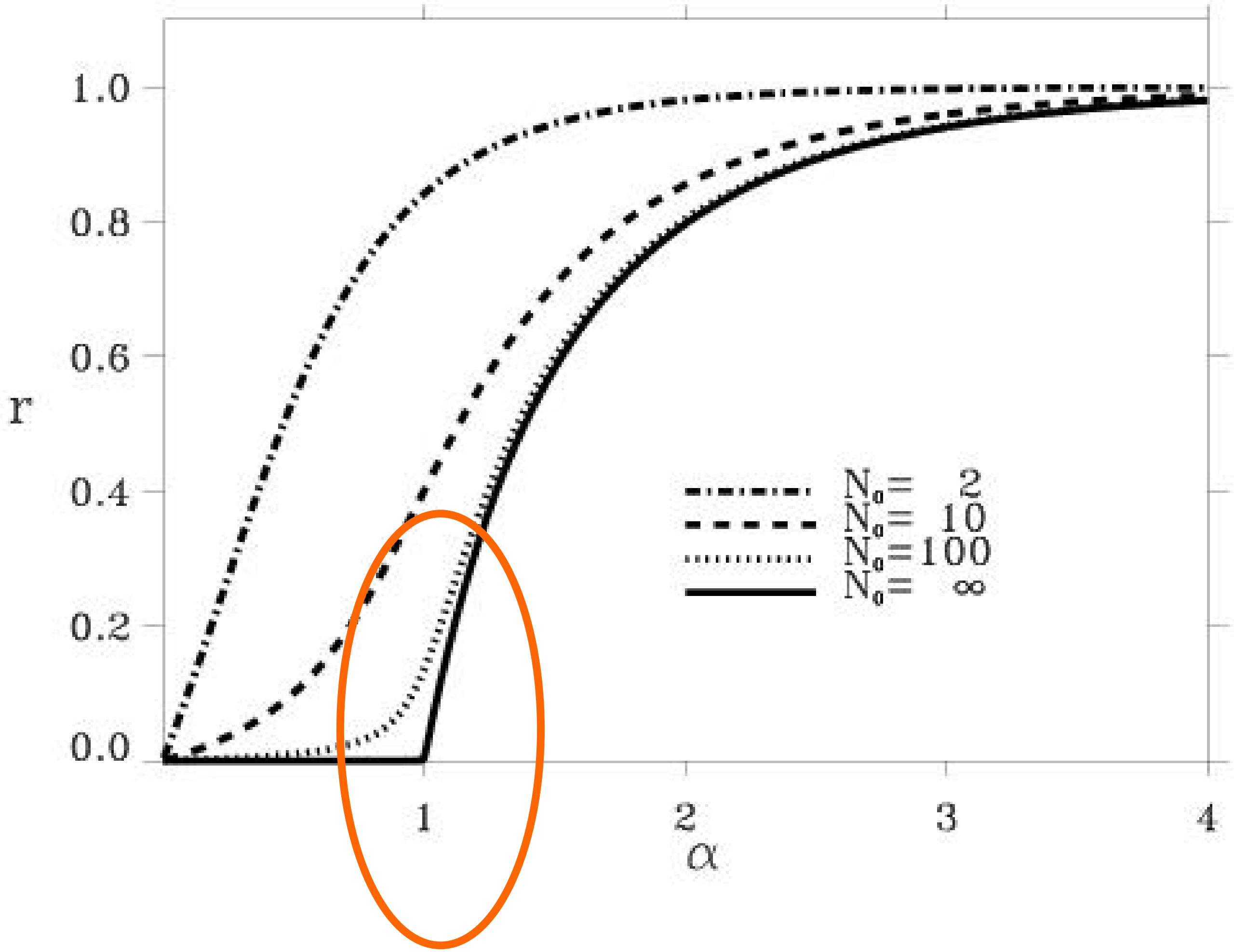
$$\langle n \rangle = \begin{cases} (1-\alpha)^{-1} & 1-\alpha >> N^{-1/3} \\ N^{1/3} & |1-\alpha| << N^{-1/3} \\ rN & \alpha - 1 >> N^{-1/3} \end{cases}$$

- **Universal behavior for different N (finite size scaling)**

$$\langle n \rangle / N^{1/3} \to F\left((1-\alpha) N^{1/3}\right)$$

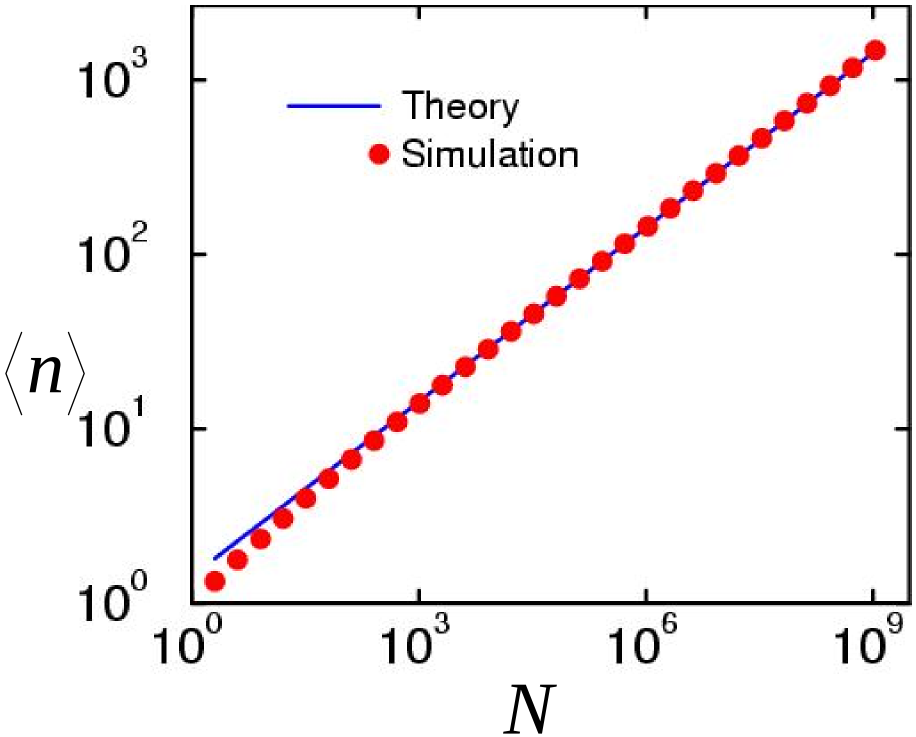
One master curve for all system sizes

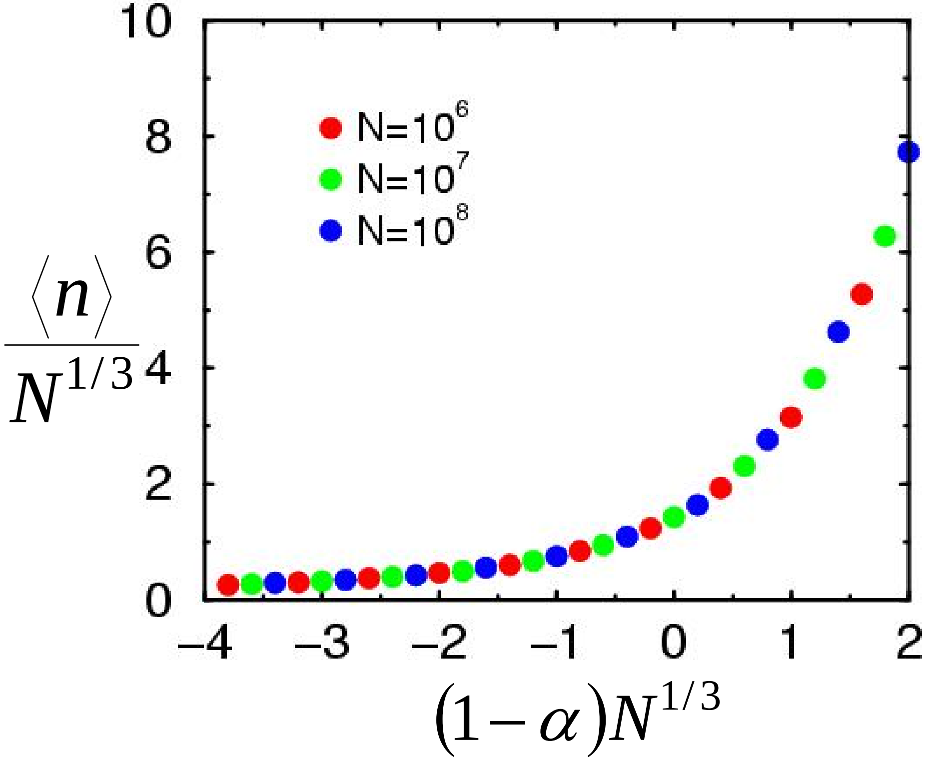
Attack rate versus system size



Numerical simulations

Average over 10^9 independent realizations!



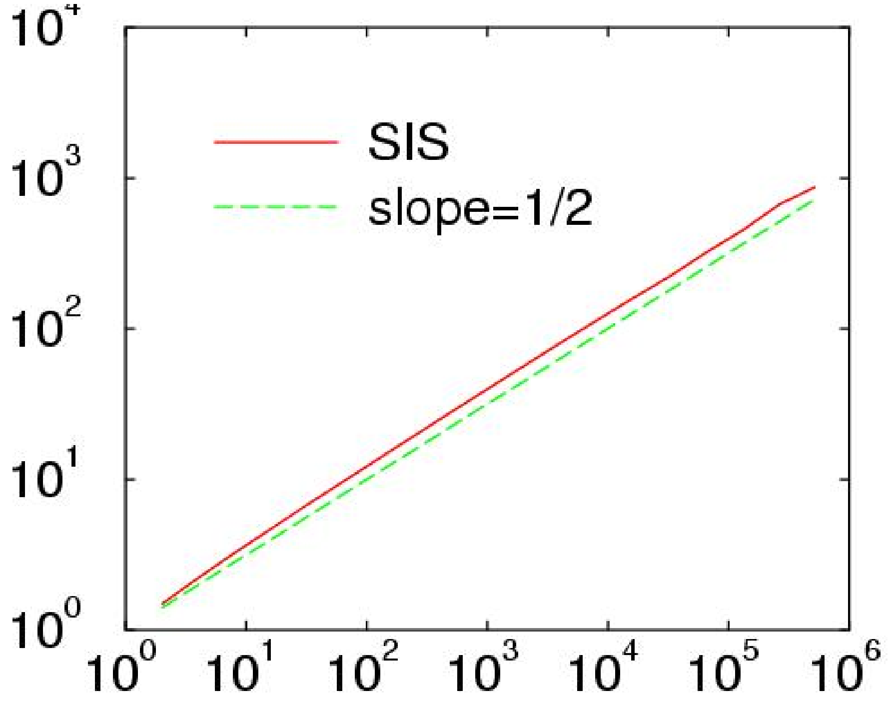


Excellent agreement with theory

SIS process: no depletion effect

SIS model: R immediately becomes S

$$\langle n \rangle \sim N^{1/2} \qquad N_* = N$$



Distribution of duration times

- **Probability of having i infected individuals**

$$\frac{dP_i}{dt} = (i+1)P_{i+1} + (i-1)P_{i-1} - 2P_i$$

- **Exact solution**

$$P_i(t) = t^{i-1}(1+t)^{i+1}$$

- **Survival probability of infection**

$$S(t) = \sum_i P_i = (1+t)^{-1}$$

- **Average number of infected individuals**

$$\langle i \rangle = 1/S = 1+t$$

Duration of outbreaks

- **Number of recovered**

$$dr/dt \approx i \approx t \quad \Rightarrow \quad r \approx t^2$$

- **Maximal duration time**

$$r_* \sim t_*^2 \sim N^{2/3} \quad \Rightarrow \quad t_* \approx N^{1/3}$$

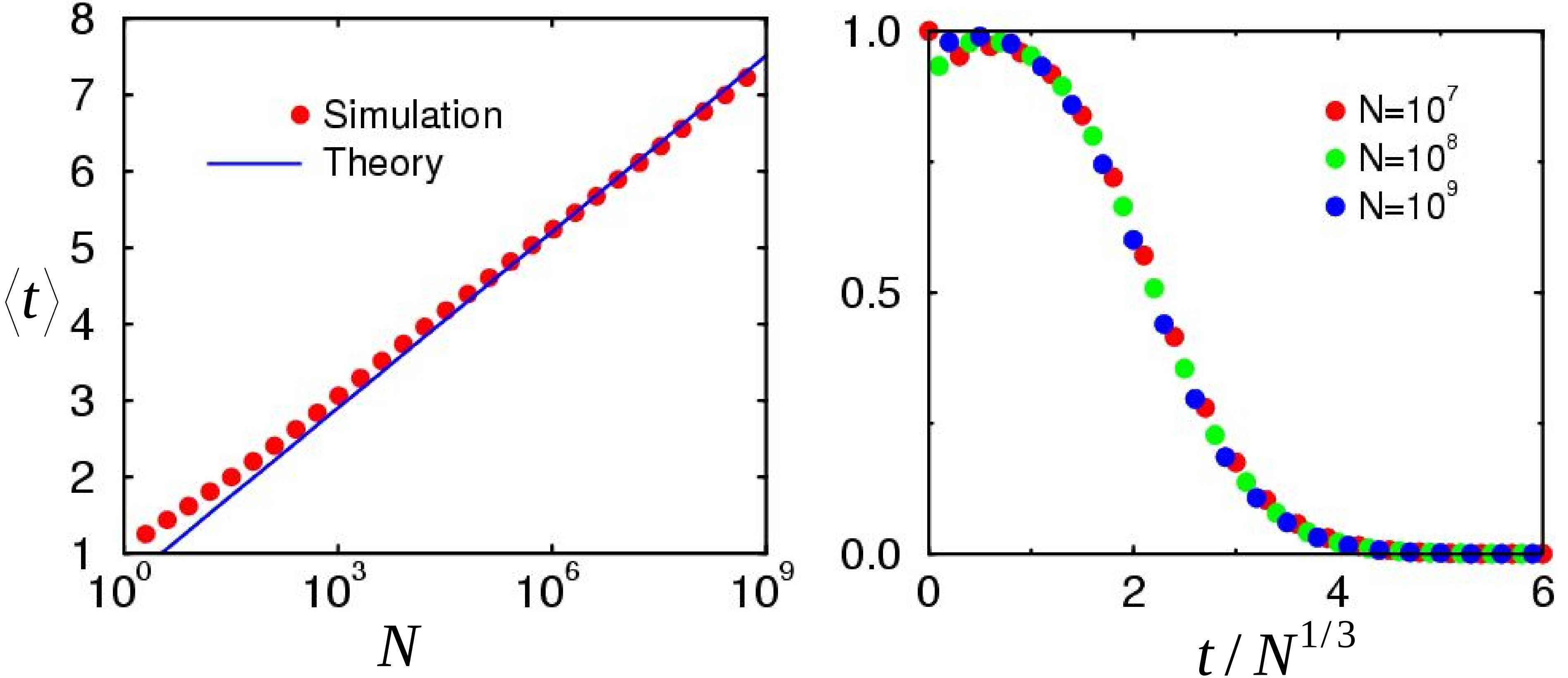
- **Average duration time**

$$\langle t \rangle \sim \int_0^{t_*} t\left(-\frac{dS}{dt}\right) \sim \int_0^{N^{1/3}} t^{-1} \sim \frac{1}{3}\ln N$$

Scaling laws

$$\langle t \rangle \approx \frac{1}{3}\ln N \quad \text{and} \quad t_* \sim N^{1/3}$$

Numerical confirmation



Probability distributions

- **Distributions are self-similar**

$$P(i,t) \to t^{-2}\Phi\left(i/t\right)$$

$$P(r,t) - G_r \to t^{-3}\Psi\left(r/t^2\right)$$

- **Similarity/scaling functions**

$$\Phi(x) = \exp(-x)$$

$$\Psi(y) = \frac{\pi^2}{2}\sum_{k=0}^{\infty}(k+1/2)^2 \exp\left[-\pi^2(k+1/2)^2 y\right]$$

- **Asymptotic behaviors**

$$\Psi(y) \sim \begin{cases} 4y^{-1}\exp[-1/y] & y \ll 1 \\ \exp[-\pi^2 y/4] & y \gg 1 \end{cases}$$

- **Laplace transform of joint distribution**

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} db \exp[b\eta]\left[\frac{\sqrt{b}}{\sinh\sqrt{b}}\right]^2 \exp[-\xi\sqrt{b}\coth\sqrt{b}]$$

Master equation for finite systems

- **Alternative derivation of scaling laws**

$$P_t = \left(\partial_{i,i} - \partial_r\right)P + N^{-1}\partial_i(irP)$$

- **Dimensional analysis**

$$ir \sim N \quad + \quad r \sim i^2 \quad \Rightarrow \quad r \sim N^{2/3}$$

- **Reduce equation to (for scaled Laplace transform)**

$$F_\tau = (\beta - \alpha^2)F_\alpha + \alpha F_{\alpha\beta}$$

Conclusion

- Stochastic description needed near threshold
- New scaling laws for the size and duration of outbreaks
- Outbreaks near threshold have distinct size
- Universal behavior near threshold
- Scaling theory useful
- Fluctuations significant even on a complete graph

Outlook

- **Match behavior at threshold-pandemic interface**
- **"Triple-deck" boundary layer?**
- **Form of finite-size scaling functions**

EB, P.L. Krapivsky, q-bio/0402001
Phys. Rev. E, April 2004.