Maxwell Model of Inelastic Collisions

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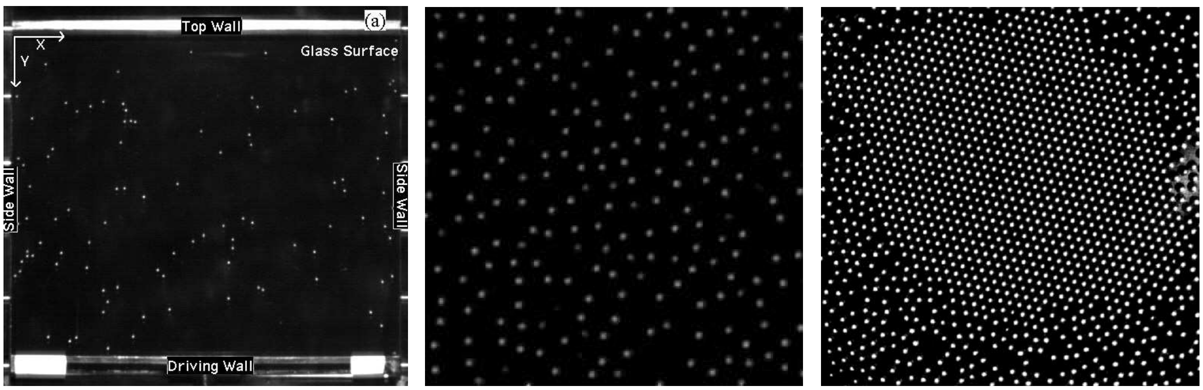
I Motivation
II Freely evolving inelastic gases
III Impurities
IV Social dynamics

with: Paul Krapivsky (Boston)

Experiments: Granular Gases



- Vibration: vertical, horizontal, electrostatic
  Gollub, Swinney, Menon, Aronson, Kudrolli, Urbach

- non-Maxwellian velocity statistics

$$P(v) \sim \exp\left(-|v|^{\alpha}\right) \qquad 1 < \alpha < 1.5$$

- Clustering, density inhomogeneities

- Collective phenomena: phase transitions, pattern formation, shocks

A nonequilibrium gas

Characteristics

- Hard sphere (contact) interactions
- Dissipative (inelastic) collisions

Consequences of energy dissipation

- No energy equipartition
- No ergodicity
- Strong velocity correlations

Challenges

- Hydrodynamics: flow equations
- Kinetic Theory: velocity distributions
- Sharp validity criteria are missing

The Elastic Maxwell Model

J.C. Maxwell, Phil. Tran. Roy. Soc **157**, 49 (1867)

- Infinite particle system
- Binary collisions
- Random collision partners
- Random impact directions $\mathbf{n}$
- Elastic collisions $(\mathbf{g} = \mathbf{v_1} - \mathbf{v_2})$

$$\mathbf{v_1} \to \mathbf{v_1} - \mathbf{g} \cdot \mathbf{n}\, \mathbf{n}$$

- Mean-field collision process
- Purely Maxwellian velocity distributions

$$P(\mathbf{v}) = \frac{1}{(2\pi T)^{d/2}} \exp\left(-\frac{v^2}{2T}\right)$$

What about inelastic, dissipative collisions?

The Inelastic Maxwell Model

- **Inelastic collisions** $r = 1 - 2\epsilon$

$$\mathbf{v}_{1,2} = \mathbf{u}_{1,2} \mp (1-\epsilon)\,(\mathbf{g}\cdot\mathbf{n})\,\mathbf{n}$$

- **Boltzmann equation** $\mathbf{g}\cdot\mathbf{n} \to \langle g \rangle$

$$\frac{\partial P(\mathbf{v},t)}{\partial t} = \int d\mathbf{n} \int d\mathbf{u}_1 \int d\mathbf{u}_2\, \langle g \rangle\, P(\mathbf{u}_1,t)\, P(\mathbf{u}_2,t) \times \Big\{ \delta\left(\mathbf{v}-\mathbf{v}_1\right) - \delta\left(\mathbf{v}-\mathbf{u}_1\right) \Big\}$$

- **Fourier transform** Krupp 1967

$$F(\mathbf{k},t) = \int d\mathbf{v} e^{i\mathbf{k}\cdot\mathbf{v}} P(\mathbf{v},t)$$

- **Closed equations** $\mathbf{q} = (1-\epsilon)\mathbf{k}\cdot\mathbf{n}\,\mathbf{n}$

$$\frac{\partial}{\partial t} F(\mathbf{k},t) + F(\mathbf{k},t) = \int d\mathbf{n}\, F\left[\mathbf{k}-\mathbf{q},t\right] F\left[\mathbf{q},t\right],$$

Theory is analytically tractable

One Dimension

- **Scaling of isotropic velocity distribution**

$$P(\mathbf{v},t) \to \frac{1}{T^{d/2}}\Phi\left(\frac{|\mathbf{v}|}{T^{1/2}}\right) \quad \text{or} \quad F(k,t) \to f\left(k^2T\right)$$

- **Nonlinear and nonlocal equation**

$$-2\epsilon(1-\epsilon)f'(x) + f(x) = f(\epsilon^2 x) f\left((1-\epsilon)^2 x\right)$$

- **Exact solution**

$$f(x) = \left(1+\sqrt{x}\right)e^{-\sqrt{x}} \cong 1 - \frac{1}{2}x + \frac{1}{3}x^{3/2} + \cdots$$

- **Lorentzian^2 velocity distribution**

$$\Phi(v) = \frac{2}{\pi}\frac{1}{(1+v^2)^2}$$

- **Algebraic tail** Baldassari 2001

$$\Phi(v) \sim v^{-4} \qquad w \gg 1$$

Universal scaling function, exponent

Scaling, Nontrivial Exponents

- **Freely cooling case**

$$T = \langle v^2 \rangle \sim t^{-2}$$

- **Governing equation for scaling function**

$$-\lambda\, x\, \Phi'(x) + \Phi(x) = \int d\mathbf{n}\, \Phi(x\xi)\, \Phi(x\eta)$$

$$\lambda = 2\epsilon(1-\epsilon)/d,\ \xi = 1-(1-\epsilon^2)\cos^2\theta,\ \eta = (1-\epsilon)^2\cos^2\theta$$

- **Power-law tails**

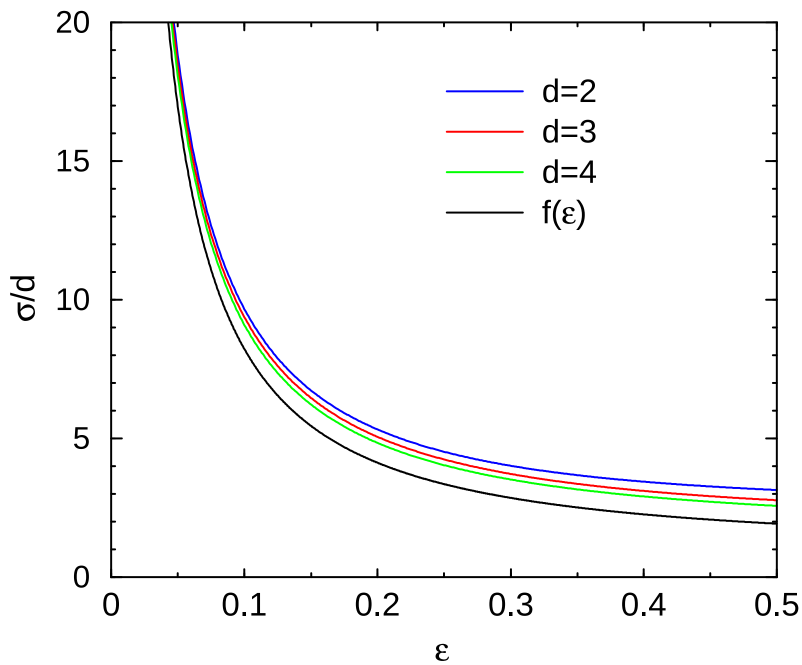
$$\Phi(v) \sim v^{-\sigma}, \qquad v \to \infty.$$

- **Exact solution for the exponent σ**

$$1-\epsilon(1-\epsilon)\,\frac{\sigma-d}{d} = {}_2F_1\left[\tfrac{d-\sigma}{2}, \tfrac{1}{2}; \tfrac{d}{2}; 1-\epsilon^2\right] + (1-\epsilon)^{\sigma-d}\,\frac{\Gamma\left(\frac{\sigma-d+1}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{\sigma}{2}\right)\Gamma\left(\frac{1}{2}\right)}$$

Nonuniversal tails, exponents depend on ϵ, d

The exponent σ



- **Maxwellian distributions:** $d = \infty$, $\epsilon = 0$
- **Diverges in high dimensions**

$$\sigma \cong d\, f(\epsilon)$$

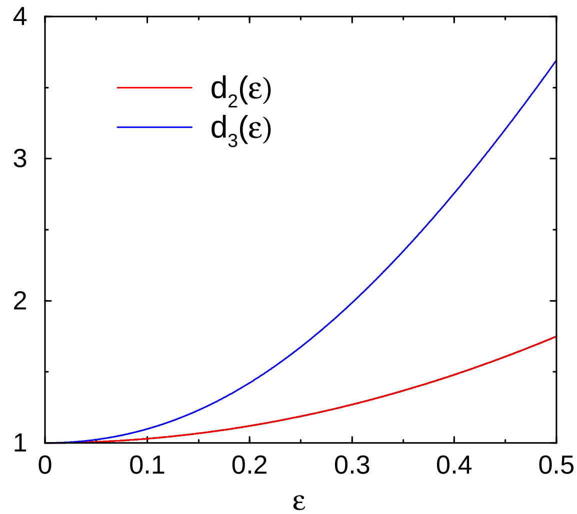
- **Diverges for low dissipation**

$$\sigma \cong d\, \epsilon^{-1}$$

- **In practice, huge**

$$\sigma(d = 3, r = 0.8) \cong 30!$$

Dynamics



- Moments of the velocity distribution

$$M_{2n}(t) = \int d\mathbf{v} |\mathbf{v}|^{2n} P(\mathbf{v}, t)$$

- Multiscaling asymptotic behavior

$$M_{2n} \sim M_2^{\xi_n/2} \qquad \xi_n = \begin{cases} n & d < d_n(\epsilon), \\ \alpha_n(\epsilon) & d > d_n(\epsilon). \end{cases}$$

- Nonlinear multiscaling spectrum (1D):

$$\alpha_n(\epsilon) = \frac{1 - \epsilon^{2n} - (1-\epsilon)^{2n}}{1 - \epsilon^2 - (1-\epsilon)^2}$$

Sufficiently large moments exhibit multiscaling

Velocity Correlations

- Definition (correlation between v_x^2 and v_y^2)

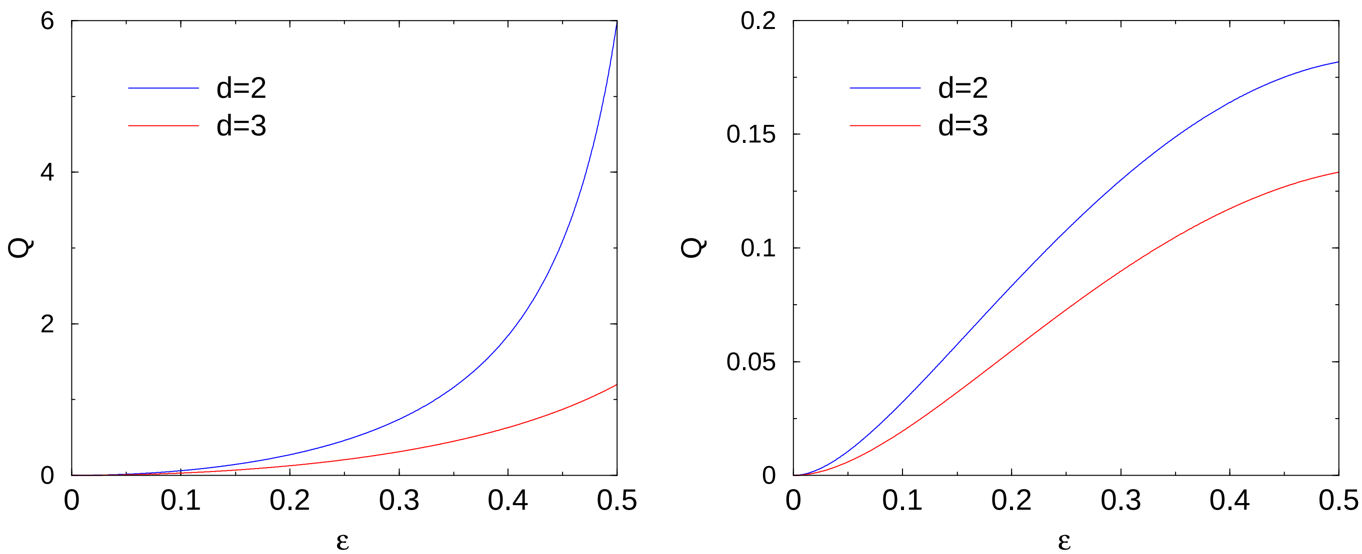
$$Q = \frac{\langle v_x^2 v_y^2 \rangle - \langle v_x^2 \rangle \langle v_y^2 \rangle}{\langle v_x^2 \rangle \langle v_y^2 \rangle}$$

- Unforced case (freely evolving) $P(v) \sim v^{-\sigma}$

$$Q = \frac{6\epsilon^2}{d - (1 + 3\epsilon^2)}$$

- Forced case (white noise) $P(v) \sim e^{-|v|}$

$$Q = \frac{6\epsilon^2(1-\epsilon)}{(d+2)(1+\epsilon) - 3(1-\epsilon)(1+\epsilon^2)}.$$



Correlations diminish with energy input

The "brazil nut" problem

- Fluid background: mass 1
- Impurity: mass m
- Theory: Lorentz-Boltzmann equation
- Series of transition masses

$$1 < m_1 < m_2 < \cdots < m_\infty$$

- Ratio of moments diverges asymptotically

$$\frac{\langle v_I^{2n}\rangle}{\langle v_F^{2n}\rangle} \sim \begin{cases} c_n & m < m_n; \\ \infty & m > m_n. \end{cases}$$

- Light impurity: moderate violation of equipartition, impurity mimics the fluid
- Heavy impurity: extreme violation of equipartition, impurity sees a static fluid

$\boxed{\textbf{series of phase transitions}}$

Conclusions

- non-Maxwellian velocity distributions
- Power-law high energy tails
- non-universal exponents
- Multiscaling of the moments, Temperature insufficient to characterize large moments
- Correlations between velocity components

Ben-Naim and Krapivsky, PRE 61, 5 (00); JPA 35, L147 (02); PRE 66, 011309 (02); EPJE 8, 507 (02).

Ernst and Brito, EL 58, 182 (02); PRE 65, 040301 (02); JSP 109, 407 (02).

Bobylev, Carrido and Gamba, JSP 98, 743 (00).

Baldassarri, Marconi and Puglisi, EL 58, 14 (02); PRE 65, 051301 (02); 66, 011301 (02).

Bobylev and Cercignani, JSP 106, 1039 (02).

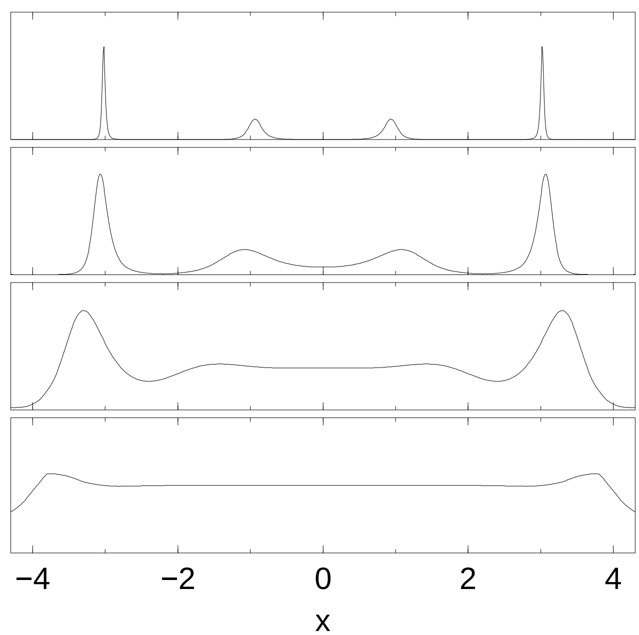
Santos and Dufty, PRL 86, 4823 (01); PRE 64, 051305 (01)

The Compromise Model

- Opinion $-\Delta < x < \Delta$
- Reach compromise in pairs   Weisbuch 2001

$$(x_1, x_2) \to \left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}\right)$$

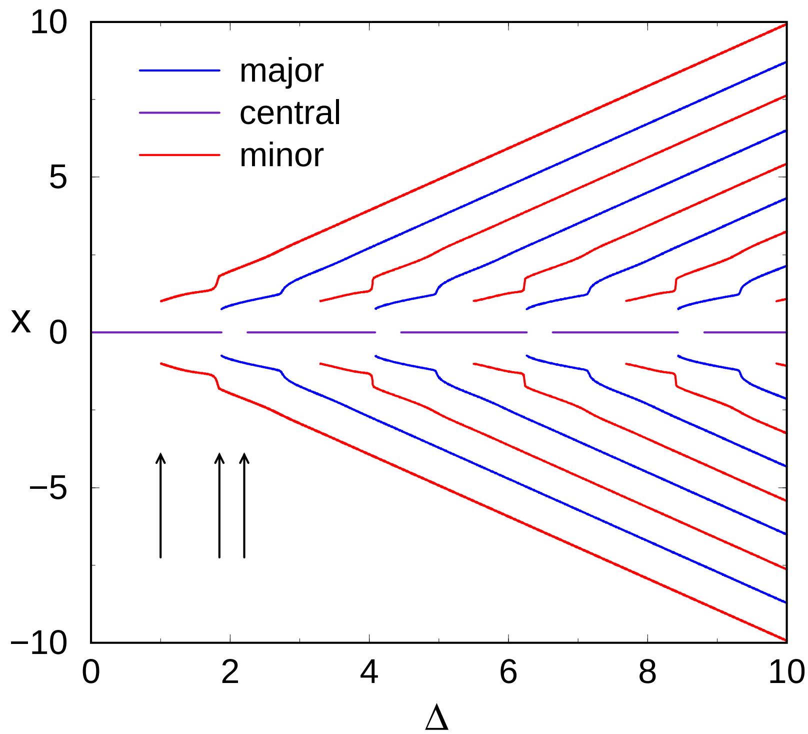
- As long as we are close $|x_1 - x_2| < 1$



$$P_\infty(x) = \sum_i m_i \, \delta(x - x_i)$$

Final State: localized clusters

Bifurcations and Patterns



- Periodic bifurcations

$$x(\Delta) = x(\Delta + L)$$

- Alternating major-minor pattern
- Critical behavior

$$m \sim (\Delta - \Delta_c)^\alpha \qquad \alpha = 3 \text{ or } 4.$$

Self-similar structure