Shock Dynamics of Inelastic Gases

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Motivation: Granular Gases

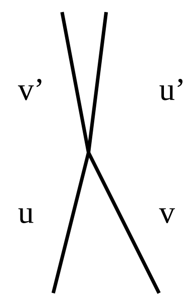
- Applications:
  - — Granular materials: powders, grains.
  - — Geophysical flows.
  - — Large scale formation in universe.
- Characteristics:
  - — Hard sphere interactions.
  - — Dissipative collisions.
- Experimental observations (1D, 2D, 3D):
  - — **Density** inhomogeneities.
  - — **Velocity** correlations, non-Gaussian stat
  - — **Phase transitions:** order-disorder.

Inelastic collisions

- Relative velocity reduced by $r = 1 - 2\epsilon$

$$\begin{aligned} \Delta v' &= -r\Delta v \\ v' &= v - \epsilon \Delta v \end{aligned}$$

- Energy dissipation $\Delta E \propto -\epsilon(\Delta v)^2$



Freely evolving gas

- N point particles in 1D ring.
  Random velocity distribution.
  Typical velocity v_0. Typical distance x_0.

- **Dimensionless variables** $x \to x/x_0$, $t \to tv_0/x_0$

— "Temperature" $T(t) = \langle v^2(t)\rangle - \langle v(t)\rangle^2$

— Characteristic time/length scales.

— Continuum theory?

Mean Field Theory

- Energy dissipation

$$\Delta T \propto -\epsilon(\Delta v)^2$$

- Collision frequency

$$\Delta t \sim \ell/\Delta v \sim (\Delta v)^{-1}$$

- Assuming a uniform gas

$$\begin{aligned} v &\sim \Delta v \sim T^{1/2} \qquad \Delta\ell/\ell \ll 1 \\ \frac{dT}{dt} &\propto \frac{\Delta T}{\Delta t} \propto -\epsilon(\Delta v)^3 \propto -\epsilon\, T^{3/2} \end{aligned}$$

- Cooling law

Haff 83

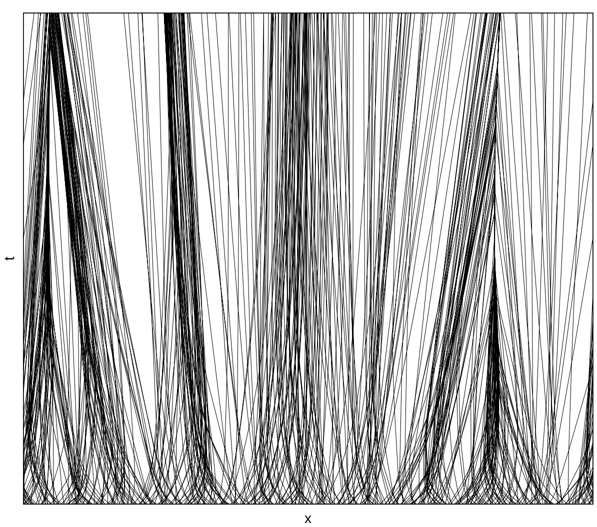
$$T(t) \simeq (1 + A\epsilon t)^{-2} \sim \begin{cases} 1 & t \ll \epsilon^{-1} \\ \epsilon^{-2} t^{-2} & t \gg \epsilon^{-1} \end{cases}$$

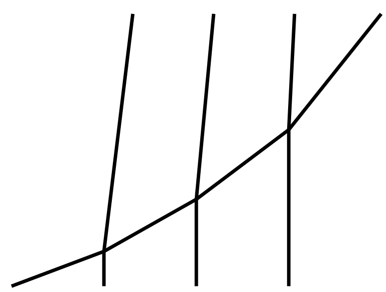
Holds only in early homogeneous phase

The Inelastic Collapse

Bernu 90, Young 91

- 3 particles clump if $r < r_c = 7 - 4\sqrt{3} \cong 0.07$
- Finite time singularity: Cluster formation via infinite collisions when $N > N_c(r)$.
- Estimating the critical mass: $v_N \cong 1 - N\epsilon$ particle passes through if $N < \epsilon^{-1}$
- $N_c \sim \epsilon^{-1} \Rightarrow$ collapse always encountered in the thermodynamic limit $N \to \infty$.





Particles coalesce rather than pass through

The Sticky Gas ($r = 0$)

- Multiparticle aggregate of typical mass m
- **Momentum conservation** CPY 90

$$P_m = \sum_{i=1}^{m} P_i \quad \Rightarrow \quad P \sim m^{1/2}, \quad v \sim m^{-1/2}$$

- **Mass conservation**

$$\rho = cm = \text{const.} \quad \Rightarrow \quad c \sim m^{-1}$$

- **Dimensional analysis** $[cv] = [t]^{-1}$

$$m \sim t^{2/3} \quad v \sim t^{-1/3} \quad T \sim t^{-2/3}$$

- **Final state** 1 aggregate with $m = N$

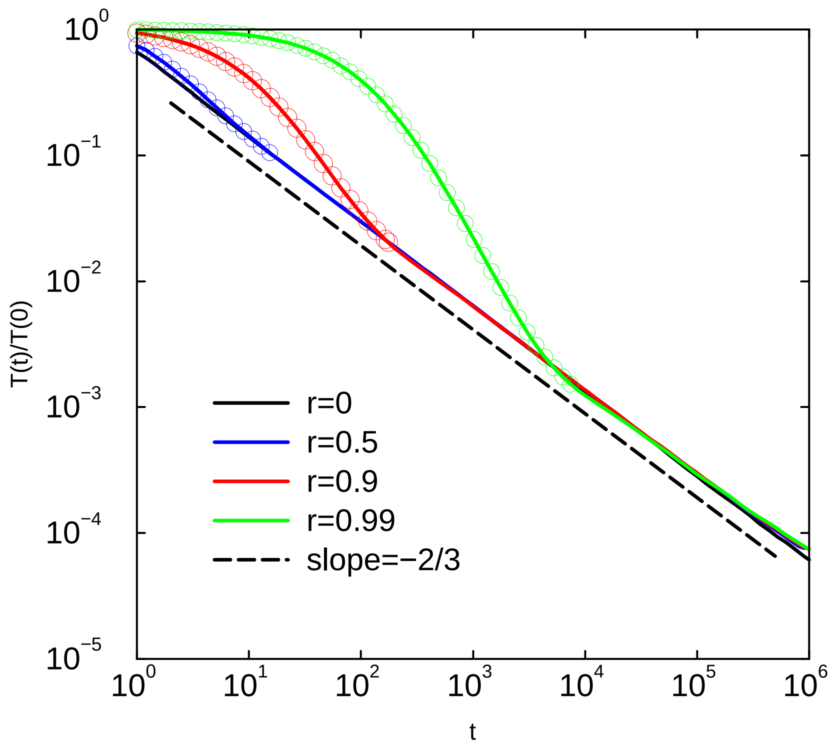
$$T(t) \sim \begin{cases} 1 & t \ll 1; \\ t^{-2/3} & 1 \ll t \ll N^{3/2}; \\ N^{-1} & N^{3/2} \ll t \end{cases}$$

T is monotonic in $r \Rightarrow r = 0$ is a lower bound

Crossover picture

- Large systems $N > \epsilon^{-1}$

$$T(t) \sim \begin{cases} 1 & t \ll \epsilon^{-1}; \\ \epsilon^{-2} t^{-2} & \epsilon^{-1} \ll t \ll \epsilon^{-3/2}; \\ t^{-2/3} & \epsilon^{-3/2} \ll t \ll N^{3/2}; \\ N^{-1} & N^{3/2} \ll t \end{cases}$$



Asymptotic behavior is independent of r

The Velocity Distribution

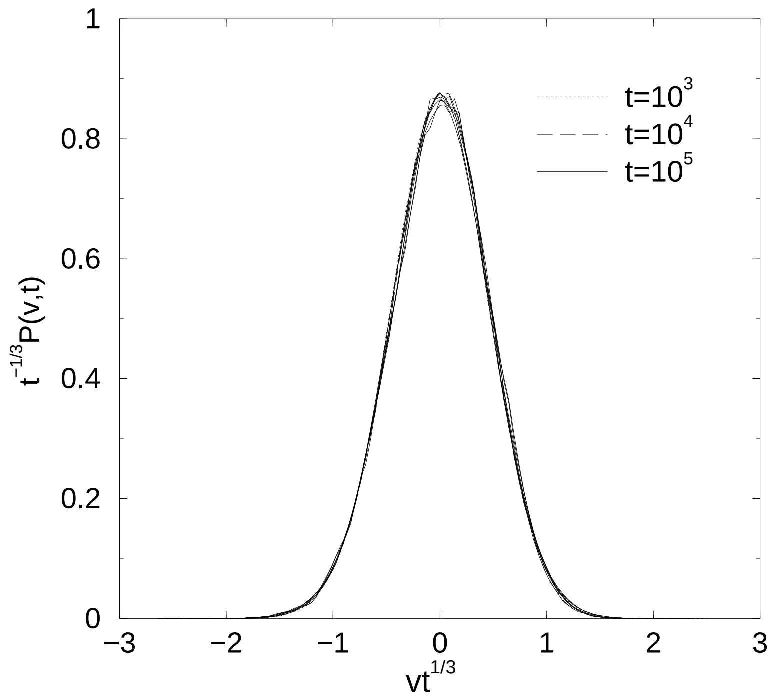
- Self similar distribution

$$P(v,t) \sim t^{1/3}\Phi(vt^{1/3})$$

- Large velocity tail

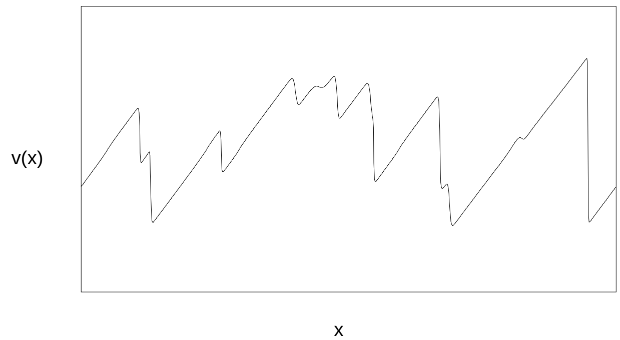
$$\Phi(z) \sim \exp(-\mathrm{const.} \times z^3) \qquad z \gg 1$$

- Simulation results $r = 0,\ 0.5,\ 0.9$



$\boxed{r_{\text{eff}} \equiv 0 \text{ is fixed point}}$

The Inviscid Burgers Equation



- Nonlinear diffusion equation

$$v_t + v v_x = \nu v_{xx} \qquad \nu \to 0$$

- $v = -2\nu(\ln u)_x \Rightarrow u_t = \nu u_{xx}$

$$v(x,t) = \frac{\int dq\, \frac{x-q}{t}\, \exp\left[-\frac{1}{2\nu}\left\{\frac{(x-q)^2}{2t} + \Phi_0(q)\right\}\right]}{\int dq\, \exp\left[-\frac{1}{2\nu}\left\{\frac{(x-q)^2}{2t} + \Phi_0(q)\right\}\right]}$$

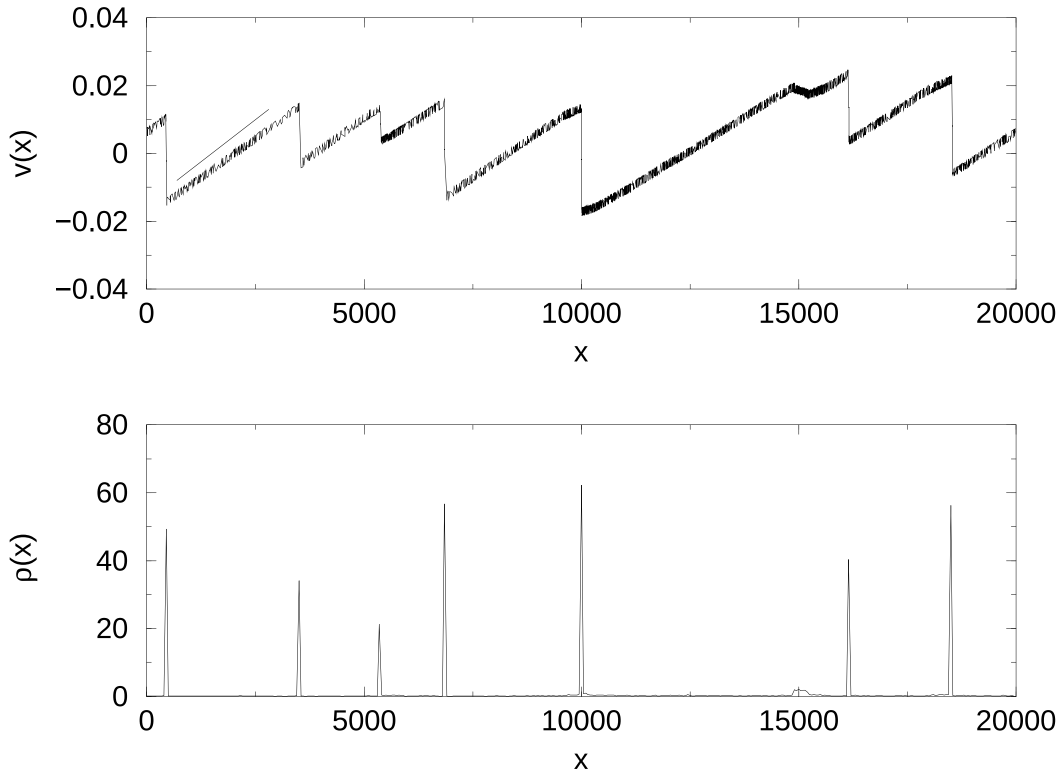
$$\to \frac{x - q(x,t)}{t} \qquad\qquad v_0(x) = \partial_x \Phi_0(x)$$

- Collisions conserve mass & momentum
- Describes "sticky gas" $r = 0$ Zeldovich RMP 89

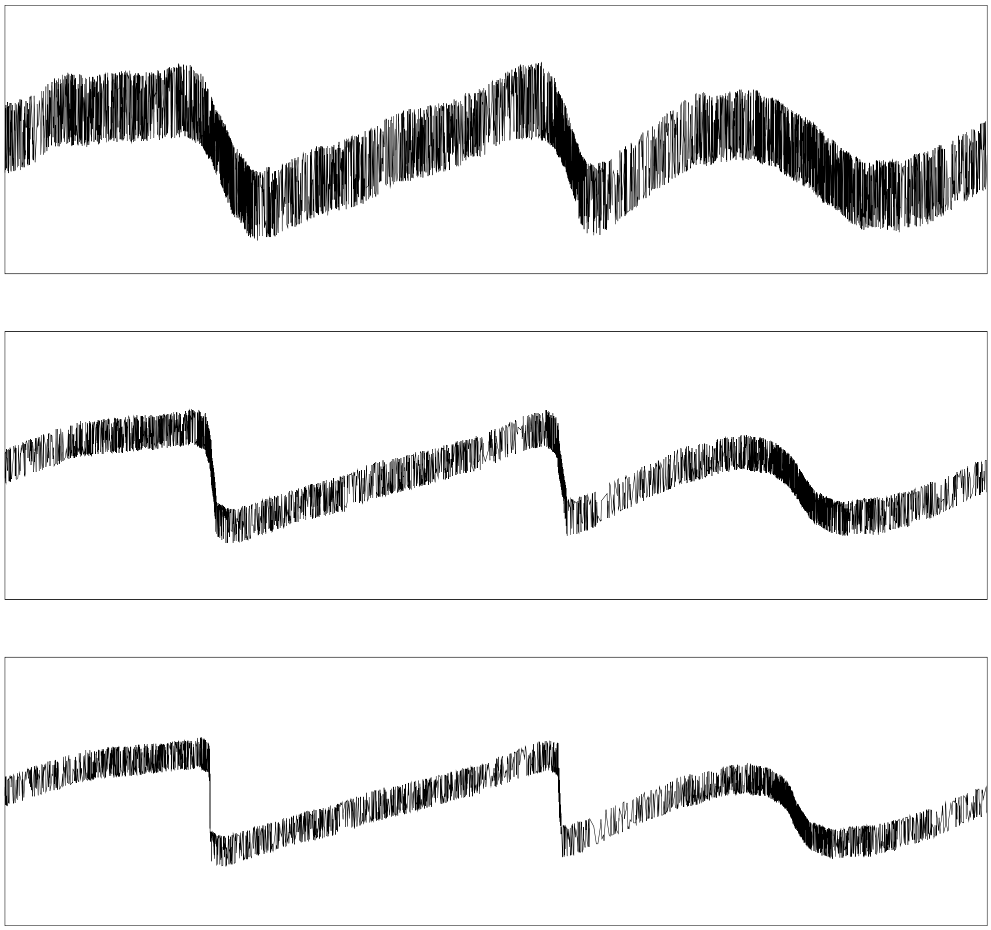
Burgers equation $\equiv$ sticky gas $\equiv$ inelastic gas

Predictions verified in 1D

- Velocity statistics $v \sim t^{-1/3}$
- Discontinuous (shock) velocity profile
- Slope $= t^{-1}$ (simulation with $r = 0.99$)



Formation of Singularity



$\boxed{\textbf{Collapse} \equiv \textbf{finite time singularity in } v_t + v v_x = 0}$

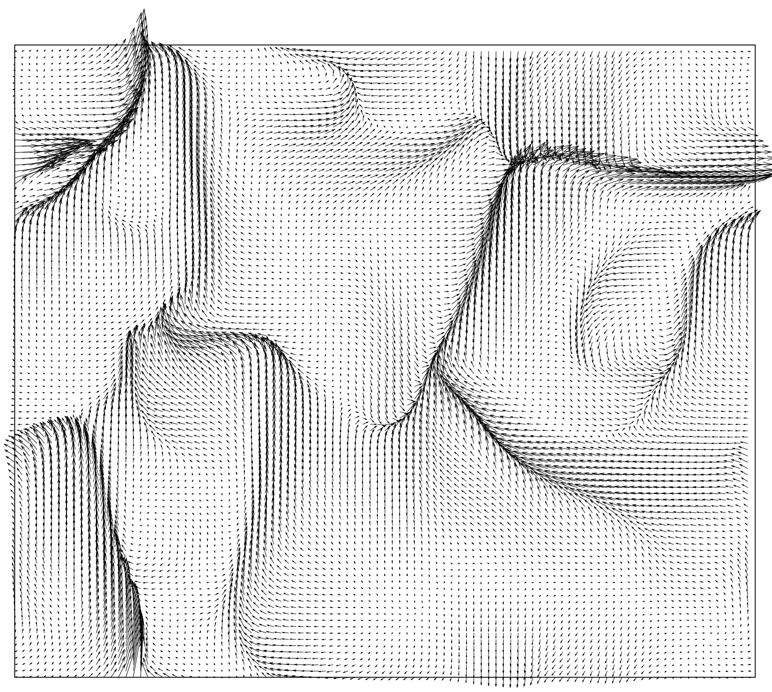
Higher Dimensions

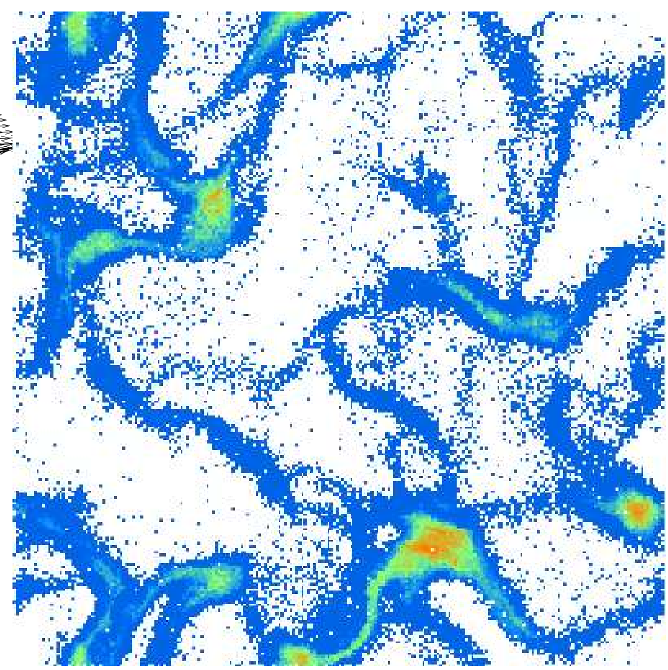
If $r_{\text{eff}} = 0$, then $\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = \nu \nabla^2 \mathbf{v}$ predicts:

- Cooling law for $2 \leq d \leq 4$ Brito & Ernst 98

$$T(t) \sim \begin{cases} 1 & t \ll \epsilon^{-1}; \\ \epsilon^{-2} t^{-2} & \epsilon^{-1} \ll t \ll \epsilon^{-4/(4-d)}; \\ t^{-d/2} & \epsilon^{-4/(4-d)} \ll t \ll N^{2/d}; \\ N^{-1} & N^{2/d} \ll t \end{cases}$$

- Critical size for collapse $N_c(\epsilon) \sim \epsilon^{-2d/(4-d)}$





Luding 99

Underlying Length Scales

- Burgers equation

$$v_t + vv_x = v_{xx}$$

- Correlation length $= \xi$; Balance terms:

$$\frac{v}{t} \sim \frac{v^2}{\xi} \qquad \frac{v}{t} \sim \frac{v}{\xi^2}$$

- Momentum conservation $v \sim V^{-1/2} \sim \xi^{-d/2}$

$$\xi \sim \begin{cases} t^{\frac{2}{d+2}} & 0 < d \leq 2 \\ t^{1/2} & 2 \leq d \end{cases}$$

- Temperature $T \sim v^2 \sim \xi^{-d}$

$$T \sim \begin{cases} t^{-2d/(d+2)} & 0 < d \leq 2 \\ t^{-d/2} & 2 \leq d \end{cases}$$

Conclusions

Asymptotic behavior:

- **Governed by cluster-cluster coalescence**
- **Universal = Independent of degree of dissipation**
- **Described by Burgers equation**

Outlook

- **Velocity & spatial correlations**
- **Predictions in higher dimensions**
- **Use inelastic gases to study shocks**

EB, S. Chen, G. Doolen, S. Redner, PRL **83**, 4069 (1999).