Nonlinear Integral Equation Formulation of Orthogonal Polynomials

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December 5, 2006

I. ORTHOGONAL POLYNOMIALS 101

Definition: A set of polynomials $P_n(x)$ is set to be orthogonal with respect to the measure $g(x)$ on the interval $[\alpha : \beta]$ if

$$\int_\alpha^\beta dx\, g(x) P_n(x) P_m(x) = 0, \qquad \text{for all } m \neq n. \tag{1}$$

Specification: There are many ways to specify a set of orthogonal polynomials. For example, take the Legendre Polynomials $P_n(x)$ that are orthogonal on the interval $[-1, 1]$ with respect to the measure $g(x) = 1$. The Legendre polynomials can be uniquely specified in multiple ways:

1. Differential equation
$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0. \tag{2}$$

2. Generating function
$$\frac{1}{\sqrt{1 - 2tx + t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n. \tag{3}$$

3. Rodriguez formula
$$P_n(x) = \frac{1}{2^n\, n!} \frac{d^n}{dx^n} (x^2 - 1)^n. \tag{4}$$

4. Recursion formula
$$(n + 1)P_{n+1}(x) - (2n + 1)xP_n(x) + nP_{n-1}(x) = 0. \tag{5}$$

Properties: Orthogonal polynomials have many fascinating and useful properties. For example, all of their roots are real and inside the interval $[\alpha : \beta]$. They also form a complete basis as any nth order polynomial is a linear combination of the orthogonal polynomials of lesser or equal order.

II. NONLINEAR INTEGRAL EQUATION

In this talk, I will describe how orthogonal polynomials can be specified using nonlinear integral equations. This will be demonstrated using the following integral equation

$$P(x) = \int_\alpha^\beta dy\, w(y)\, P(y)\, P(x+y). \tag{6}$$

The integration limits α and β as well as the function $w(x)$ are arbitrary except for the restriction that $w(x)$ must have a non-vanishing integral, $\int_\alpha^\beta dx\, w(x) \neq 0$.

Note that if we choose $\alpha = -\infty$, $\beta = \infty$, and $w(y) = e^{iy}$, (6) reduces to the heavily studied equation for the Wigner function. The original motivation for considering this nonlinear integral equation is that it describes a stochastic process in which two random variables are subtracted to create a new one. The steady-state probability distribution for this random variable satisfies the integral equation (6) with $\alpha = 0$, $\beta = \infty$, and $w(y) = 2$.

III. INFINITE NUMBER OF SOLUTIONS

This nonlinear equation has an infinite number of solutions. Among these solutions are an infinite set of polynomial solutions. Let us demonstrate this for polynomials of zeroth, first, and second order.

A. n=0

Consider the constant polynomial

$$P_0(x) = a$$

with $a \neq 0$. This is a solution when

$$a = \int_\alpha^\beta dy\, w(y) a^2. \tag{7}$$

Since $a \neq 0$, we can divide by a

$$1 = \int_\alpha^\beta dy\, w(y) a. \tag{8}$$

Since the integral of the function $w(x)$ does not vanish, a constant polynomial solution exists.

B. n=1

Next, we consider, the linear polynomial

$$P_1(x) = a + bx. \tag{9}$$

It is convenient to represent integration with respect to the function $g(x)$ with brackets

$$\langle f(x) \rangle \equiv \int_\alpha^\beta dx\, w(x) f(x). \tag{10}$$

Then the nonlinear integral equation (6) reads

$$a + bx = \langle P_1(y)\, [a + by + bx] \rangle. \tag{11}$$

Here, we have tacitly spelled out only the function $P_1(x+y)$ in the integrand. Solving for the coefficient of x^1 we have

$$b = b\langle P_1 \rangle. \tag{12}$$

Since the polynomial is first order, $b \neq 0$, and therefore,

$$\langle P_1(x) \rangle = 1. \tag{13}$$

Next, we equate the coefficients of x^0 on both sides of the equation

$$a = a\langle P_1 \rangle + b\langle y P_1(y) \rangle. \tag{14}$$

Substituting (13), we have a cancellation and then

$$\langle x P_1(x) \rangle = 1. \tag{15}$$

We see that due to a miraculous cancellation of terms, the nonlinear integral equation reduces to two linear inhomogeneous equations (13) and (15) for the coefficients of the polynomial, a and b. This equation generally has a solution.

C. n=2

Let us repeat this exercise one more time for the quadratic polynomials

$$P_2(x) = a + bx + cx^2. \tag{16}$$

Now, our nonlinear integral equation (6) is

$$a + bx + cx^2 = \langle P_2(y)\left[a + by + bx + cy^2 + 2cxy + cx^2\right]\rangle. \tag{17}$$

First, we equate the x^2 coefficient and from $c = c\langle P_2(y)\rangle$, we recover (13)

$$\langle P_2(x)\rangle = 1. \tag{18}$$

Next, we equate the x^1 coefficients and

$$b = b\langle P_2\rangle + 2c\langle yP_1(y)\rangle. \tag{19}$$

Again, a cancellation is encountered and the analog of (15) is recovered

$$\langle xP_2(x)\rangle = 0. \tag{20}$$

Last, we equate the x^0 coefficients and then

$$a = a\langle P_2\rangle + b\langle yP_2(y)\rangle + c\langle y^2P_2(y)\rangle. \tag{21}$$

Using the two identities (18) and (20), we now have a third equation

$$\langle x^2P_2(x)\rangle = 0. \tag{22}$$

Therefore, there are three linear inhomogeneous equations (18), (20), and (22) for the three unknown quantities a, b, and c.

IV. GENERAL PROPERTIES

The general pattern is clear. The nonlinear integral (6) equation has two remarkable properties:

1. **It preserves the order of polynomials.**

2. **For polynomial solutions, it reduces to a linear set of equations for the coefficients of the polynomials.**

In general, a polynomial of degree n

$$P_n(x) = \sum_{k=0}^{n} a_{n,k}\, x^k, \tag{23}$$

is a solution of the integral equation if and only if the following set of $n+1$ linear inhomogeneous equations for its $n+1$ coefficients are satisfied:

$$\langle x^k P_n(x)\rangle = \delta_{k,0} \qquad (k=0,1,\ldots,n). \tag{24}$$

Clearly, all our previous linear equations including (8) satisfy this equation.

A. An amazing fact: The set of polynomial solutions is orthogonal!

The linear equations (24) imply that the polynomials $P_n(x)$ are orthogonal with respect to the measure

$$g(x) = x\, w(x). \tag{25}$$

If $P_m(x) = \sum_{k=0}^{m} a_{m,k}\, x^k$ is a polynomial of degree $m < n$, then from (24) we conclude that

$$\langle x\, P_n P_m\rangle = \sum_{k=0}^{m} a_{m,k} \langle x^{k+1}\, P_n(x)\rangle = \sum_{k=1}^{m+1} a_{m,k-1} \langle x^{k}\, P_n(x)\rangle = 0, \tag{26}$$

because every term in the sum equals zero. This is our main result. The nonlinear integral equation (6) admits an infinite set of polynomial solutions. This set is unique; there is one and only one polynomial solution of degree n. These polynomials are orthogonal.

We comment that this integral formulation of orthogonal polynomials is general. The integration limits and the function $w(x)$ are arbitrary apart from the restriction that $w(x)$ have a positive integral $\int_\alpha^\beta dx\, w(x) > 0$.

The equations (24) can be written compactly as $n+1$ simultaneous linear equations for the $n+1$ coefficients $a_{n,j}$:

$$\sum_{j=0}^{n} a_{n,j} m_{k+j} = \delta_{k,0} \qquad (k=0,1,\ldots,n). \tag{27}$$

Here $m_n = \langle x^n\rangle$ are the moments of $w(x)$. These equations have a solution when the measure $g(x)$ is positive. Another way to write these equation is using matrix notations

$$\begin{pmatrix} m_0 & m_1 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_n & m_{n+1} & \cdots & m_{2n} \end{pmatrix} \begin{pmatrix} a_{n,0} \\ a_{n,1} \\ \vdots \\ a_{n,n} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{28}$$

V. EXAMPLES

Take for example the interval $[0:1]$ and the function $w(x)=1$. Then, the first four polynomials are

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= 4-6x \\ P_2(x) &= 9-36x+30x^2 \\ P_3(x) &= 16-120x+240x^2-140x^3. \end{aligned}$$

These polynomials are proportional to the Jacobi polynomials, $P_n(x) \propto G_n(2,2,x)$ that are orthogonal with respect to the measure $g(x)=x$.

Three classical orthogonal polynomials that can be generated using this approach:

• Generalized Laguerre polynomials $L_n^{(\gamma)}(x)$ using $\alpha=0$, $\beta=\infty$, and $w(x)=x^{\gamma-1}e^{-x}$ for all $\gamma \geq 1$;

• Jacobi polynomials $G_n(p,q,x)$ using $\alpha=0$, $\beta=1$, and $w(x)=x^{q-2}(1-x)^{p-q}$, for $q>1$ and $p-q>-1$;

• Shifted Chebyshev polynomials of the second kind $U_n^*(x)$ using $\alpha=0$, $\beta=1$, and $w(x)=(1-x)^{1/2}x^{-1/2}$.

VI. FORMULAS FOR THE POLYNOMIALS

For completeness, we mention the well-known fact that using Cramer's rule, the polynomial solutions can be written explicitly as ratios of determinants. We define the matrices

$$A_n = \begin{pmatrix} 1 & x & \cdots & x^n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_n & m_{n+1} & \cdots & m_{2n} \end{pmatrix}, \quad B_n = \begin{pmatrix} m_0 & m_1 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_n & m_{n+1} & \cdots & m_{2n} \end{pmatrix},$$

so that $B_n = \langle A_n \rangle$. The polynomials $P_n(x)$ are then given by

$$P_n(x) = \frac{\det A_n}{\det B_n}. \tag{29}$$

The normalization of the polynomials can be given in the form

$$\langle x\, P_n(x)\, P_m(x) \rangle = \delta_{n,m}\, G_n. \tag{30}$$

The normalization factors G_n are ratios of determinants of matrices

$$G_n = \frac{(\det C_n)(\det C_{n+1})}{(\det B_{n+1})^2}, \tag{31}$$

where the matrix C_n is given by

$$C_n = \begin{pmatrix} m_1 & m_2 & \cdots & m_n \\ m_2 & m_3 & \cdots & m_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_n & m_{n+1} & \cdots & m_{2n} \end{pmatrix}.$$

The first three polynomials are

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= \frac{m_2 - x m_1}{m_2 - m_1^2}, \\ P_2(x) &= \frac{(m_2 m_4 - m_3^2) + (m_2 m_3 - m_1 m_4)x + (m_1 m_3 - m_2^2)x^2}{m_4(m_2 - m_1^2) - m_3^2 + 2m_1 m_2 m_3 - m_2^3}. \end{aligned}$$

Also, we give the corresponding normalization factors G_n:

$$\begin{aligned} G_0 &= m_1, \\ G_1 &= m_1 \frac{m_1 m_3 - m_2^2}{(m_2 - m_1^2)^2}, \\ G_2 &= \frac{(m_1 m_3 - m_2^2)(m_1 m_3 m_5 - m_2^2 m_5 - m_1 m_4^2 + 2m_2 m_3 m_4 - m_3^3)}{[m_4(m_2 - m_1^2) - m_3^2 + 2m_1 m_2 m_3 - m_2^3]^2}. \end{aligned}$$

Note that (29–31) are only valid if $\det B_n \neq 0$. This condition holds when the measure $g(x) = x\, w(x)$ is positive on $\alpha \leq x \leq \beta$.

VII. INTEGRATION IN THE COMPLEX DOMAIN

At first, it appears that this formulation is restricted to situations where the function $w(x)$ has a finite integral in the range $[\alpha : \beta]$. For example, to generate the Legendre polynomials, we should consider $w(x) = 1/x$ in $[-1 : 1]$, but the integral of this function is not finite due to the singularity at the origin.

Fortunately, the complex domain comes to our rescue! When the weight function $g(x)$ with respect to which the polynomials are orthogonal does not vanish at $x = 0$ and consequently $w(x) = g(x)/x$ is singular at $x = 0$, the path of integration from α to β may be taken in the complex plane to avoid the singularity at the origin. Using a complex integration path, all steps leading to (26) are valid. Notice that our formulation did not necessarily specify the integration path.

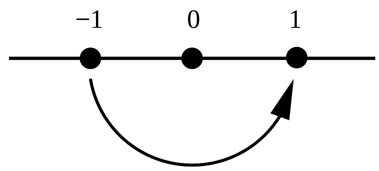

FIG. 1: Integration in the complex domain avoids the singularity at the origin.

For example, consider the Legendre polynomials. There are an infinite number of topologically distinct integration paths that connect -1 to 1. These paths are characterized by their winding numbers. For definiteness, we choose a path that goes from -1 to 1 in the positive (counterclockwise) direction and does not encircle the origin (figure 1). On this path

$$\int_{-1}^{1} \frac{dx}{x} = i\pi. \tag{32}$$

For convenience we chose the the normalization $\int_\alpha^\beta dx\, w(x) = 1$ and consequently, $w(x) = 1/(i\pi x)$. The moments $m_n = \langle x^n \rangle$ are $m_0 = 1$, $m_1 = 2/(i\pi)$, $m_2 = 0$, $m_3 = 2/(3i\pi)$, $\ldots$. From the moment formulas (29), we obtain

$$\begin{aligned}
P_0(x) &= 1, \\
P_1(x) &= \frac{i\pi}{2} x, \\
P_2(x) &= 1 - 3x^2, \\
P_3(x) &= \frac{3i\pi}{8}(3x - 5x^3),
\end{aligned} \tag{33}$$

and so on. These polynomials are the standard Legendre polynomials, except that the odd polynomials have an imaginary multiplicative factor that is determined by the winding number of the integration path. Of course, these polynomials are solutions of the linear equations (24) with the integration path specified above.

To summarize, while the usual theory of orthogonal polynomials is formulated in terms of real integrals, the integral equation (6) provides a simple and natural framework to extend the theory of orthogonal polynomials into the complex domain. In doing so we discover an interesting connection between the polynomial coefficients and the topological winding number of the integration path.

VIII. GENERALIZATIONS

There are many generalizations of the nonlinear integral equation (6).

A. Multiplicative Arguments

If we replace the term $P(x+y)$ in the original nonlinear integral equation (6) by $P(xy)$, we obtain a new class of nonlinear equations:

$$P(x) = \int_\alpha^\beta dy\, w(y)\, P(y)\, P(x\, y). \tag{34}$$

Each of these nonlinear integral equations also has an infinite number of polynomial solutions.

Following the procedure above, the nonlinear integral equation reduces to a set of $n+1$ inhomogeneous equations of the form

$$a_{n,k}\langle x^k P_n(x)\rangle = a_{n,k} \qquad (k = 0, 1, \ldots, n). \tag{35}$$

However, unlike the previous case, the equations are quadratic and there are now 2^{n-1} solutions because each of the coefficients $a_{n,k}$ can be either zero or nonzero for $k = 0, \ldots, n-1$. For each nonzero coefficient the linear equation

$$\langle x^k P_n(x)\rangle = 1 \tag{36}$$

holds for all k for which $a_{n,k} \neq 0$.

In one special class of solutions all of the coefficients are nonzero and the polynomials are orthogonal with respect to the measure

$$g(x) = (1-x)\, w(x). \tag{37}$$

To verify this, we take a polynomial of degree $m < n$ and observe that

$$\langle (1-x) P_n P_m\rangle = \sum_{k=0}^{m} a_{m,k} \left(\langle x^k P_n\rangle - \langle x^{k+1} P_n\rangle\right) = 0.$$

This equation is valid because all of the terms in the parentheses vanish by virtue of (36). This measure is relevant for the class of polynomials for which $0 \le \alpha < \beta \le 1$.

Here are two classical orthogonal polynomials that can be generated in this way:

• Jacobi polynomials $G_n(p,q,x)$ using $\alpha = 0$, $\beta = 1$, and $w(x) = (1-x)^{p-q-1}x^{q-1}$ with $p-q > 0$ and $q > 0$;

• Shifted Chebyshev polynomials of the second kind $U_n^*(x)$ using $\alpha = 0$, $\beta = 1$, and $w(x) = x^{1/2}(1-x)^{-1/2}$.

B. Linear arguments

For the integral equation with a linearly shifted argument,

$$P(x) = \int_\alpha^\beta dy\, w(y)\, P(y)\, P(x + a + b\, y), \tag{38}$$

where $b \neq 0$ is an arbitrary constant, polynomials of degree n are solutions when $\langle (a + bx)^k\, P_n(x) \rangle = \delta_{k,0}$. This implies that the polynomials are orthogonal with respect to the measure $g(x) = (a + bx)\, w(x)$. Note that when $a = 0$, the polynomials are identical to those generated by the original integral formula (6). Curiously, for $a = 1$ and $b = -1$ we recover the polynomials generated by the integral equation (34).

C. Functional arguments

The integral equation

$$P(x) = \int_\alpha^\beta dy\, w(y)\, P(y)\, P[x + f(y)], \tag{39}$$

where $f(x)$ is a nonconstant function, has polynomial solutions of degree n when $\langle [f(x)]^k\, P_n(x) \rangle = \delta_{k,0}$. These polynomials are not necessarily an orthogonal set. Nevertheless, the polynomial $P_n(x)$ is orthogonal to the function $P_m[f(x)]$ when $m < n$ with respect to the measure $g(x) = f(x)\, w(x)$.

D. Arbitrary functions

The integral equation

$$P(x) = \int_\alpha^\beta dy\, w(y)\, f[P(y)]\, P(x + y), \tag{40}$$

where $f(x)$ is a nonconstant function, has polynomial solutions of degree n when $\langle x^k f[P_n(x)] \rangle = \delta_{k,0}$. This implies that the function $f[P_n(x)]$ is orthogonal to the polynomial $P_m(x)$ with respect to the measure $g(x) = x\, w(x)$.

IX. SUMMARY

1. Any set of orthogonal polynomials with respect to the measure $g(x)$ in the interval $[\alpha : \beta]$ can be formulated via the nonlinear integral equation

$$P(x) = \int_{\alpha}^{\beta} dy \, \frac{g(y)}{y} \, P(y) \, P(x+y). \tag{41}$$

2. For polynomials this nonlinear equation becomes linear.

3. Simple, compact, and completely general way to specify orthogonal polynomials.

4. Natural way to extend theory to complex domain.

X. OUTLOOK

1. Non-polynomial solutions.

2. Multi-dimensional polynomials.

3. Higher-order nonlinear integral equation (by iteration),

4. Other nonlinear integral forms.

5. Use nonlinear equation to generate new orthogonal polynomials.

[1] C.M. Bender and E. Ben-Naim, math-ph/0610071, J. Phys. A, in press.