Nonlinear Integral Equation Formulation of Orthogonal Polynomials

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Working with Carl

- Data: 50 email messages received during collaboration
- AM: 25%, PM: 75%
- “Midnight singularity”
Orthogonal Polynomials 101

• **Definition:** A set of polynomials $P_n(x)$ is orthogonal with respect to the measure $g(x)$ on the interval $[\alpha : \beta]$ if
\[
\int_{\alpha}^{\beta} dx \, g(x) P_n(x) P_m(x) = 0 \quad \text{for all } m \neq n
\]

• **Properties:** Orthogonal polynomials have many fascinating and useful properties:
  
  ▶ All roots are real and are inside the interval $[\alpha : \beta]$
  
  ▶ The orthogonal polynomials form a complete basis: any polynomial is a linear combination of orthogonal polynomials of lesser or equal order
Orthogonal Polynomials 101

- **Specification:** Orthogonal polynomials can be specified in multiple ways (typically linear)

  Example: Legendre Polynomials orthogonal on $[-1 : 1]$ w.r.t $g(x) = 1$

1. **Differential Equation:**

   $$(1 - x^2)P_n''(x) - 2xP_n'(x) + n(n + 1)P_n(x) = 0$$

2. **Generating Function:**

   $$\frac{1}{\sqrt{1 - 2tx + t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

3. **Rodriguez Formula:**

   $$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n$$

4. **Recursion Formula:**

   $$(n + 1)P_{n+1}(x) - (2n + 1)xP_n(x) + nP_{n-1}(x) = 0$$
Nonlinear Integral Equation

• Consider the following nonlinear integral equation

\[ P(x) = \int_{\alpha}^{\beta} dy \, w(y) \, P(y) \, P(x + y) \]

• No restriction on integration limits

• Weight function restricted: non-vanishing integral

\[ \int_{\alpha}^{\beta} dx \, w(x) \neq 0 \]

• Motivation: stochastic process involving subtraction

\[ x_1, x_2 \rightarrow |x_1 - x_2| \quad \quad e^{-x} = 2 \int_{0}^{\infty} dy \, e^{-y} e^{-(x+y)} \]

• Reduces to the Wigner function equation when

\[ [\alpha : \beta] = [-\infty : \infty] \quad \quad w(y) = e^{iy} \]
Constant Solution (n=0)

• Consider the constant polynomial

\[ P_0(x) = a \quad a \neq 0 \]

• A constant solves the nonlinear integral equation

\[ P(x) = \int_\alpha^\beta dy \ w(y) \ P(y) \ P(x + y) \]

• When

\[ a = \int_\alpha^\beta dy \ w(y) \ a^2 \]

• Since \( a \neq 0 \) we can divide by \( a \)

\[ 1 = \int_\alpha^\beta dy \ w(y) \ a. \]

A constant solution exists
Linear Solution (n=1)

- Consider the linear polynomial
  \[ P_1(x) = a + bx \quad b \neq 0 \]

- Let us introduce the shorthand notation
  \[ \langle f(x) \rangle \equiv \int_{\alpha}^{\beta} dx \, w(x) f(x) \]

- The nonlinear integral equation \( P(x) = \langle P(y) P(x + y) \rangle \) reads
  \[ a + bx = \langle P_1(y) [a + by + bx] \rangle \]

1. Equate the coefficients of \( x^1 \) and divide by \( b \neq 0 \)
   \[ b = b\langle P_1 \rangle \Rightarrow \langle P_1(x) \rangle = 1 \]

2. Equate the coefficients of \( x^0 \) and divide by \( b \neq 0 \)
   \[ a = a\langle P_1 \rangle + b\langle y P_1(y) \rangle \Rightarrow \langle xP_1(x) \rangle = 0 \]

Linear solution generally exists

Nonlinear equation reduces to 2 linear inhomogeneous equations for \( a, b \)
Quadratic Solution (n=2)

- Consider the quadratic polynomial
  \[ P_2(x) = a + bx + cx^2 \quad c \neq 0 \]

- The nonlinear integral equation becomes
  \[ a + bx + cx^2 = \langle P_2(y) \left[ a + by + cy^2 + bx + 2cxy + cx^2 \right] \rangle \]

- Successively equating coefficients
  \[ c = c\langle P_2(y) \rangle \quad \Rightarrow \quad \langle P_2(x) \rangle = 1 \]
  \[ b = b\langle P_2(y) \rangle + 2c\langle yP_2(y) \rangle \quad \Rightarrow \quad \langle xP_2(x) \rangle = 0 \]
  \[ a = a\langle P_2(y) \rangle + b\langle yP_2(y) \rangle + c\langle y^2P_2(y) \rangle \quad \Rightarrow \quad \langle x^2P_2(x) \rangle = 0 \]

Quadratic solution generally exists
Miraculous cancelation of terms
Nonlinear equation reduces to 3 linear inhomogeneous equations for a, b, c
General Properties

- The nonlinear integral equation has two remarkable properties:
  1. This equation preserves the order of a polynomial
  2. For polynomial solutions, the nonlinear equation reduces to a linear set of equations for the coefficients of the polynomials

- In general, a polynomial of degree $n$

$$P_n(x) = \sum_{k=0}^{n} a_{n,k} x^k$$

- Is a solution of the integral equation if and only if its $n+1$ coefficients satisfy the following set of $n+1$ linear inhomogeneous equations

$$\langle x^k P_n(x) \rangle = \delta_{k,0} \quad (k = 0, 1, \ldots, n)$$

Infinite number of polynomial solutions
The set of polynomial solutions is orthogonal!

• The polynomial solutions are orthogonal w.r.t.

\[ g(x) = x w(x) \]

• Because for \( m < n \)

\[ \langle x P_n P_m \rangle = \sum_{k=0}^{m} a_{m,k} \langle x^{k+1} P_n(x) \rangle = \sum_{k=1}^{m+1} a_{m,k-1} \langle x^k P_n(x) \rangle = 0 \]

• As follows immediately from

\[ \langle x^k P_n(x) \rangle = \delta_{k,0} \]

1. The nonlinear integral equation admits an infinite set of polynomial solutions
2. The polynomial solutions form an orthogonal set
The equations for the coefficients

• The equations for the coefficients
  \[ \langle x^k P_n(x) \rangle = \delta_{k,0} \quad (k = 0, 1, \ldots, n) \]

• Can be compactly written as
  \[ \sum_{j=0}^{n} a_{n,j} m_{k+j} = \delta_{k,0} \quad (k = 0, 1, \ldots, n) \]

• Or in matrix form
  \[
  \begin{pmatrix}
    m_0 & m_1 & \cdots & m_n \\
    m_1 & m_2 & \cdots & m_{n+1} \\
    \vdots & \vdots & \ddots & \vdots \\
    m_n & m_{n+1} & \cdots & m_{2n}
  \end{pmatrix}
  \begin{pmatrix}
    a_{n,0} \\
    a_{n,1} \\
    \vdots \\
    a_{n,n}
  \end{pmatrix}
  =
  \begin{pmatrix}
    1 \\
    0 \\
    \vdots \\
    0
  \end{pmatrix}
  \]

• In terms of the “moments” of the weight function
  \[ m_n = \langle x^n \rangle \]
Formulas for the polynomials

• Using Cramer’s rule, the polynomials can be expressed as a ratio of determinants

\[
A_n = \begin{pmatrix}
1 & x & \cdots & x^n \\
m_1 & m_2 & \cdots & m_{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
m_n & m_{n+1} & \cdots & m_{2n}
\end{pmatrix}, \quad B_n = \begin{pmatrix}
m_0 & m_1 & \cdots & m_n \\
m_1 & m_2 & \cdots & m_{n+1} \\
\vdots & \vdots & \ddots & \vdots \\
m_n & m_{n+1} & \cdots & m_{2n}
\end{pmatrix}
\]

\[
P_n(x) = \frac{\det A_n}{\det B_n}
\]

• Explicit expressions

\[
P_0(x) = 1,
\]

\[
P_1(x) = \frac{m_2 - xm_1}{m_2 - m_1^2},
\]

\[
P_2(x) = \frac{(m_2m_4 - m_3^2) + (m_2m_3 - m_1m_4)x + (m_1m_3 - m_2^2)x^2}{m_4(m_2 - m_1^2) - m_3^2 + 2m_1m_2m_3 - m_2^3}.
\]
Examples

• Interval \([0:1]\), weight function \(w(x)=1\)

\[
P_0(x) = 1 \\
P_1(x) = 4 - 6x \\
P_2(x) = 9 - 36x + 30x^2 \\
P_3(x) = 16 - 120x + 240x^2 - 140x^3.
\]

• Jacobi polynomials \(P_n(x) \propto G_n(2, 2, x)\) orthogonal w.r.t the measure \(g(x)=x\)

1. Generalized Laguerre polynomials

\[
L_n^{(\gamma)}(x) \quad \alpha = 0, \quad \beta = \infty, \quad w(x) = x^{\gamma-1}e^{-x}
\]

2. Jacoby polynomials

\[
G_n(p, q, x) \quad \alpha = 0, \quad \beta = 1, \quad w(x) = x^{q-2}(1 - x)^{p-q}
\]

3. Shifted Chebyshev of the second kind polynomials

\[
U_n^*(x) \quad \alpha = 0, \quad \beta = 1, \quad w(x) = (1 - x)^{1/2}x^{-1/2}
\]
Integration in the complex domain

- To specify the Legendre Polynomials

\[ P(x) = \int_{-1}^{1} dy \frac{1}{y} P(y) P(x + y) \]

- Perform integration in the complex domain

\[ \int_{-1}^{1} \frac{dx}{x} = i\pi \quad w(x) = \frac{1}{i\pi x} \]

- This integration path gives the Legendre Polynomials

\[
\begin{align*}
P_0(x) &= 1, \\
P_1(x) &= \frac{i\pi}{2} x, \\
P_2(x) &= 1 - 3x^2, \\
P_3(x) &= \frac{3i\pi}{8}(3x - 5x^3),
\end{align*}
\]

Nonlinear integral equation extends to complex domain
Generalization I: multiplicative arguments

- Nonlinear equation with multiplicative argument
  \[ P(x) = \int_{\alpha}^{\beta} dy \, w(y) \, P(y) \, P(xy) \]

- Infinite set of polynomial solutions when
  \[ \langle x^k P_n(x) \rangle = 1 \]

- These polynomials are orthogonal
  \[ \langle (1 - x) P_n P_m \rangle = \sum_{k=0}^{m} a_{m,k} \left( \langle x^k P_n \rangle - \langle x^{k+1} P_n \rangle \right) = 0 \quad m < n \]

- Now, the orthogonality measure is
  \[ g(x) = (1 - x) w(x) \]

A series of nonlinear integral formulations
Generalization II: iterated integrals

- **Iterate the nonlinear integral equation**
  \[
P(x) = \int_{\alpha}^{\beta} dy \frac{g(y)}{y} P(y) P(x + y)
  \]

- **The double integral equation**
  \[
P(x) = \int_{\alpha}^{\beta} dy \frac{g(y)}{y} \int_{\alpha}^{\beta} dz \frac{g(z)}{z} P(y) P(z) P(x + y + z)
  \]

Similarly specifies orthogonal polynomials
Summary

- A set of orthogonal polynomials w.r.t the measure $g(x)$ can be specified through the nonlinear integral equation

$$P(x) = \int_{\alpha}^{\beta} dy \frac{g(y)}{y} P(y) P(x + y)$$

- For polynomial solutions, this nonlinear equation reduces to a linear set of equations

- Simple, compact, and completely general way to specify orthogonal polynomials

- Natural way to extend theory to complex domain
Outlook

- Non-polynomial solutions
- Multi-dimensional polynomials
- Matrix polynomials
- Polynomials defined on disconnected domains
- Higher-order nonlinear integral equations
- Use nonlinear formulation to derive integral identities
- Asymptotic properties of polynomials
Nonlinear Integral Identities

- Let $P_n(x)$ be the set of orthogonal polynomials specified by the nonlinear integral equation

$$P_n(x) = \int_\alpha^\beta dy \frac{g(y)}{y} P_n(y) P_n(x + y)$$

- Then, any polynomial $Q_m(x)$ of degree $m \leq n$ satisfies the nonlinear integral identity

$$Q_m(x) = \int_\alpha^\beta dy \frac{g(y)}{y} P_n(y) Q_m(x + y)$$
Asymptotic Properties & Integral Identities

• Combining the asymptotics of the Laguerre Polynomials

\[
\lim_{n \to \infty} n^{-\gamma} L_n^\gamma \left( \frac{x}{n} \right) = x^{-\gamma/2} J_\gamma(2 \sqrt{x})
\]

• And the nonlinear integral equation with scaled variables

\[
\frac{1}{n^{\gamma}} L_n^\gamma \left( \frac{x}{n} \right) = \frac{1}{\Gamma(\gamma)} \int_0^\infty dy \ y^{\gamma-1} e^{-y/n} \frac{1}{n^{\gamma}} L_n^\gamma \left( \frac{y}{n} \right) \frac{1}{n^{\gamma}} L_n^\gamma \left( \frac{x+y}{n} \right)
\]

• Gives a standard identity for the Bessel functions

\[
2^{\gamma-1} \frac{J_\gamma(z)}{2z^\gamma} = \frac{1}{\Gamma(\gamma)} \int_0^\infty dw \ w^{\gamma-1} J_\gamma(w) \frac{J_\gamma(\sqrt{w^2 + z^2})}{(w^2 + z^2)^{\gamma/2}}
\]