Nonlinear Integral Equation Formulation of Orthogonal Polynomials

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Orthogonal Polynomials 101

- **Definition**: A set of polynomials $P_n(x)$ is orthogonal with respect to the measure $g(x)$ on the interval $[-\alpha : \beta]$ if

$$\int_{\alpha}^{\beta} dx\, g(x) P_n(x) P_m(x) = 0 \qquad \text{for all} \quad m \neq n$$

- **Properties:** Orthogonal polynomials have many fascinating and useful properties:
  - All roots are real and are inside the interval $[-\alpha : \beta]$
  - The orthogonal polynomials form a complete basis: any polynomial is a linear combination of orthogonal polynomials of lesser or equal order

Orthogonal Polynomials 101

- **Specification**: Orthogonal polynomials can be specified in multiple ways (typically linear)

Example: Legendre Polynomials orthogonal on $[-1:1]$ w.r.t $g(x) = 1$

1. Differential Equation:

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$$

2. Generating Function:

$$\frac{1}{\sqrt{1-2tx+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

3. Rodriguez Formula:

$$P_n(x) = \frac{1}{2^n\, n!}\frac{d^n}{dx^n}(x^2-1)^n$$

4. Recursion Formula:

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$$

Nonlinear Integral Equation

- Consider the following nonlinear integral equation

$$P(x) = \int_\alpha^\beta dy\, w(y)\, P(y)\, P(x+y)$$

- No restriction on integration limits
- Weight function restricted: non-vanishing integral

$$\int_\alpha^\beta dx\, w(x) \neq 0$$

- Motivation: stochastic process involving subtraction

$$x_1, x_2 \to |x_1 - x_2|$$

- Reduces to the Wigner function equation when

$$[\alpha : \beta] = [-\infty : \infty] \qquad w(y) = e^{iy}$$

Constant Solution (n=0)

- Consider the constant polynomial

$$P_0(x) = a \qquad\qquad a \neq 0$$

- A constant solves the nonlinear integral equation

$$P(x) = \int_\alpha^\beta dy\, w(y)\, P(y)\, P(x+y)$$

- When

$$a = \int_\alpha^\beta dy\, w(y) a^2$$

- Since $a \neq 0$ we can divide by a

$$1 = \int_\alpha^\beta dy\, w(y) a.$$

A constant solution exists

Linear Solution (n=1)

- Consider the linear polynomial

$$P_1(x) = a + bx \qquad\qquad b \neq 0$$

- Let us introduce the shorthand notation

$$\langle f(x) \rangle \equiv \int_\alpha^\beta dx\, w(x) f(x)$$

- The nonlinear integral equation $P(x) = \langle P(y)P(x+y)\rangle$ reads

$$a + bx = \langle P_1(y)\,[a + by + bx]\rangle$$

1. Equate the coefficients of x^1 and divide by $b \neq 0$

$$b = b\langle P_1\rangle \quad \Rightarrow \quad \langle P_1(x)\rangle = 1$$

2. Equate the coefficients of x^0 and divide by $b \neq 0$

$$a = a\langle P_1\rangle + b\langle yP_1(y)\rangle \quad \Rightarrow \quad \langle xP_1(x)\rangle = 0$$

Linear solution generally exists

Nonlinear equation reduces to 2 linear inhomogeneous equations for a,b

Quadratic Solution (n=2)

- Consider the quadratic polynomial

$$P_2(x) = a + bx + cx^2 \qquad c \neq 0$$

- The nonlinear integral equation becomes

$$\underline{\underline{\underline{a}}} + \underline{\underline{bx}} + \underline{cx^2} = \langle P_2(y) \left[\underline{\underline{\underline{a + by + bx}}} + \underline{\underline{cy^2 + 2cxy}} + \underline{cx^2} \right] \rangle$$

- Successively equating coefficients

$$c = c\langle P_2(y)\rangle \quad \Rightarrow \quad \langle P_2(x)\rangle = 1$$

$$b = b\langle P_2(y)\rangle + 2c\langle yP_2(y)\rangle \quad \Rightarrow \quad \langle xP_2(x)\rangle = 0$$

$$a = a\langle P_2(y)\rangle + b\langle yP_2(y)\rangle + c\langle y^2 P_2(y)\rangle \quad \Rightarrow \quad \langle x^2 P_2(x)\rangle = 0$$

Quadratic solution generally exists

Miraculous cancelation of terms

Nonlinear equation reduces to 3 linear inhomogeneous equations for a,b,c

General Properties

- The nonlinear integral equation has two remarkable properties:

1. This equation preserves the order of a polynomial
2. For polynomial solutions, the nonlinear equation reduces to a linear set of equations for the coefficients of the polynomials

- In general, a polynomial of degree n

$$P_n(x) = \sum_{k=0}^{n} a_{n,k}\, x^k$$

- Is a solution of the integral equation if and only if its n+1 coefficients satisfy the following set of n+1 linear inhomogeneous equations

$$\langle x^k P_n(x) \rangle = \delta_{k,0} \qquad (k = 0, 1, \ldots, n)$$

Infinite number of polynomial solutions

The set of polynomial solutions is orthogonal!

- The polynomial solutions are orthogonal w.r.t.

$$g(x) = x\, w(x)$$

- Because

$$\langle x\, P_n P_m \rangle = \sum_{k=0}^{m} a_{m,k} \langle x^{k+1}\, P_n(x) \rangle = \sum_{k=1}^{m+1} a_{m,k-1} \langle x^k\, P_n(x) \rangle = 0$$

- As follows immediately from

$$\langle x^k P_n(x) \rangle = \delta_{k,0}$$

1. The nonlinear integral equation admits an infinite set of polynomial solutions
2. The polynomial solutions forms an orthogonal set

The equations for the coefficients

- The equations for the coefficients

$$\langle x^k P_n(x) \rangle = \delta_{k,0} \qquad (k = 0, 1, \ldots, n)$$

- Can be compactly written as

$$\sum_{j=0}^{n} a_{n,j} m_{k+j} = \delta_{k,0} \qquad (k = 0, 1, \ldots, n)$$

- Or in matrix form

$$\begin{pmatrix} m_0 & m_1 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_n & m_{n+1} & \cdots & m_{2n} \end{pmatrix} \begin{pmatrix} a_{n,0} \\ a_{n,1} \\ \vdots \\ a_{n,n} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

- In terms of the moments of the weight function

$$m_n = \langle x^n \rangle$$

Formula for the polynomials

- Using Cramer's rule, the polynomials can be expressed as a ratio of determinants

$$A_n = \begin{pmatrix} 1 & x & \cdots & x^n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_n & m_{n+1} & \cdots & m_{2n} \end{pmatrix}, \quad B_n = \begin{pmatrix} m_0 & m_1 & \cdots & m_n \\ m_1 & m_2 & \cdots & m_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_n & m_{n+1} & \cdots & m_{2n} \end{pmatrix}$$

$$P_n(x) = \frac{\det A_n}{\det B_n}$$

Examples

- Interval [0:1], weight function w(x)=1

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= 4-6x \\ P_2(x) &= 9-36x+30x^2 \\ P_3(x) &= 16-120x+240x^2-140x^3. \end{aligned}$$

- Jacobi polynomials $P_n(x) \propto G_n(2,2,x)$ orthogonal w.r.t the measure g(x)=x

1. Generalized Laguerre polynomials

$$L_n^{(\gamma)}(x) \qquad \alpha = 0, \quad \beta = \infty, \quad w(x) = x^{\gamma-1}e^{-x}$$

2. Jacoby polynomials

$$G_n(p,q,x) \qquad \alpha = 0, \quad \beta = 1, \quad w(x) = x^{q-2}(1-x)^{p-q}$$

3. Shifted Chebyshev of the second kind polynomials

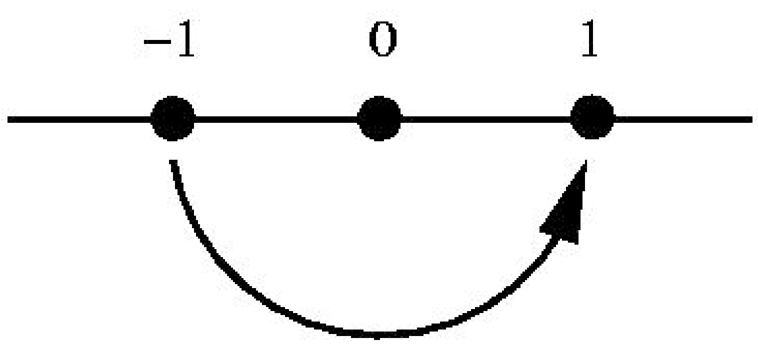
$$U_n^*(x) \qquad \alpha = 0, \quad \beta = 1, \quad w(x) = (1-x)^{1/2}x^{-1/2}$$

Integration in the complex domain

- To specify the Legendre Polynomials

$$P(x) = \int_{-1}^{1} dy\, \frac{1}{y}\, P(y)\, P(x+y)$$

- Perform integration in the complex domain



$$\int_{-1}^{1} \frac{dx}{x} = i\pi$$

- This integration path gives the Legendre Polynomials

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= \frac{i\pi}{2} x, \\ P_2(x) &= 1 - 3x^2, \\ P_3(x) &= \frac{3i\pi}{8}(3x - 5x^3), \end{aligned}$$

Nonlinear integral equation extends to complex domain

Generalization: multiplicative arguments

- Nonlinear equation with multiplicative argument

$$P(x) = \int_\alpha^\beta dy\, w(y)\, P(y)\, P(x\, y)$$

- Infinite set of polynomial solutions when

$$\langle x^k P_n(x) \rangle = 1$$

- These polynomials are orthogonal

$$\langle (1-x) P_n P_m \rangle = \sum_{k=0}^{m} a_{m,k} \left(\langle x^k P_n \rangle - \langle x^{k+1} P_n \rangle \right) = 0 \qquad m < n$$

- Now, the orthogonality measure is

$$g(x) = (1-x) w(x)$$

A series of nonlinear integral formulations

Summary

- A set of orthogonal polynomials w.r.t the measure g(x) can be specified thorugh the nonlinear integral equation

$$P(x) = \int_{\alpha}^{\beta} dy \, \frac{g(y)}{y} \, P(y) \, P(x+y)$$

- For polynomial solutions, this nonlinear equation reduces to a linear set of equations
- Simple, compact, and completely general way to specify orthogonal polynomials
- Natural way to extend theory to complex domain

Outlook

- Non-polynomial solutions
- Multi-dimensional polynomials
- Higher-order nonlinear integral equations
- Polynomials defined on multi-connected domains
- Asymptotic properties of polynomials
- Use nonlinear formulation to derive integral identities

Nonlinear Integral Identities

- Let $P_n(x)$ be the set of orthogonal polynomials specified by the nonlinear integral equation

$$P_n(x) = \int_\alpha^\beta dy\, \frac{g(y)}{y}\, P_n(y)\, P_n(x+y)$$

- Then, any polynomial $Q_m(x)$ of degree $m \le n$ satisfies the nonlinear integral identity

$$Q_m(x) = \int_\alpha^\beta dy\, \frac{g(y)}{y}\, P_n(y)\, Q_m(x+y)$$

Asymptotic Properties & Integral Identities

- Combining the asymptotics of the Laguerre Polynomials

$$\lim_{n\to\infty} n^{-\gamma} L_n^{\gamma}\left(\frac{x}{n}\right) = x^{-\gamma/2} J_{\gamma}(2\sqrt{x})$$

- And the nonlinear integral equation with scaled variables

$$\frac{1}{n^{\gamma}} L_n^{\gamma}\left(\frac{x}{n}\right) = \frac{1}{\Gamma(\gamma)} \int_0^{\infty} dy\, y^{\gamma-1} e^{-y/n} \frac{1}{n^{\gamma}} L_n^{\gamma}\left(\frac{y}{n}\right) \frac{1}{n^{\gamma}} L_n^{\gamma}\left(\frac{x+y}{n}\right)$$

- Gives a standard identity for the Bessel functions

$$2^{\gamma-1} \frac{J_{\gamma}(z)}{2z^{\gamma}} = \frac{1}{\Gamma(\gamma)} \int_0^{\infty} dw\, w^{\gamma-1} J_{\gamma}(w) \frac{J_{\gamma}(\sqrt{w^2+z^2}\,)}{(w^2+z^2)^{\gamma/2}}$$