Nonequilibrium Statistical Physics of Driven Granular Gases

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with
Jon Machta (*Massachusetts)*

talk, papers available: http://cnls.lanl.gov/~ebn

EB and J Machta, PRL **95**, 068001 (2005)
EB, B Machta & J Machta, PRE **72**, 021302 (2005)

June 28, 2006, Isaac Newton Institute, Cambridge, UK
First Passage and Extreme Statistics in Random Processes

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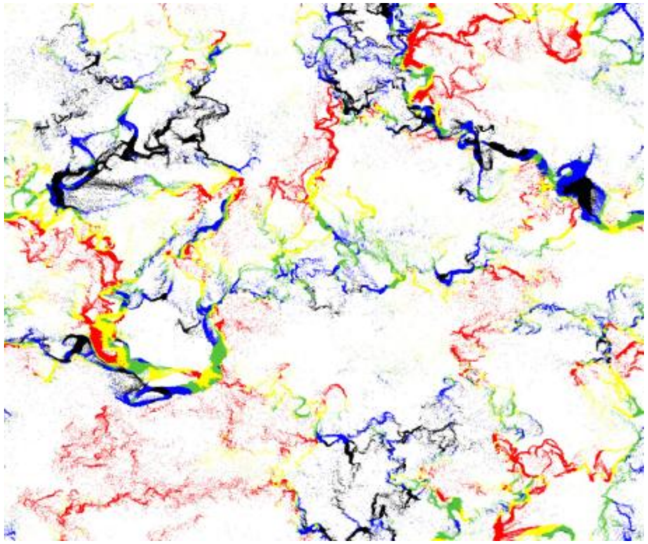
Plan

1. Introduction
2. Kinetic theory of granular gases
3. Freely cooling states
4. Driven steady states I (forcing at large scales)
5. Driven steady states II (forcing at all scales)

Energy dissipation in granular matter

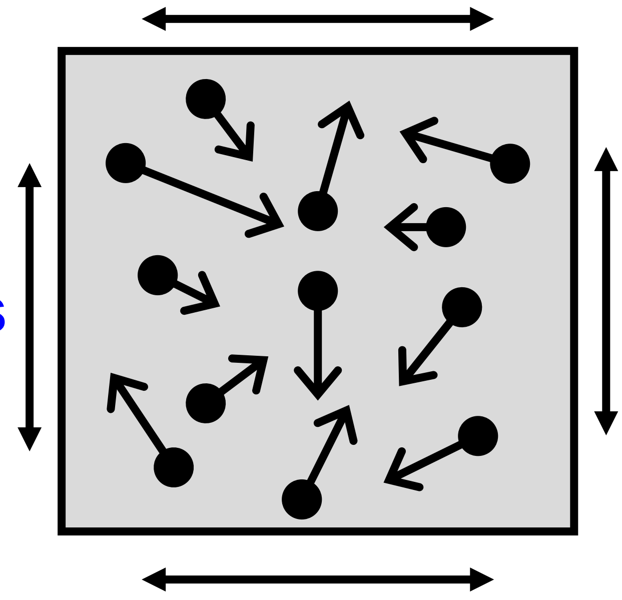


- Responsible for collective phenomena
  - » **Clustering** I Goldhirsch, G Zanetti 93
  - » **Hydrodynamic instabilities** B Meerson 04
  - » **Pattern formation** H Swinney 96
- **Anomalous statistical mechanics**
  - ➢ **No energy equipartition** R Wildman, D Parker 02
  - ➢ **Nonequilibrium energy distributions**

$$P(E) \neq \exp\left(-E/kT\right)$$

Driven Granular Gas

- **Vigorous driving (gravity not relevant)**
- **Spatially uniform system**
- **Particles undergo binary collisions**
- **Velocities change due to:**
  - ☹ **Collisions: lose energy**
  - ☺ **Forcing: gain energy**



- **What is the typical velocity (granular “temperature”)?**

$$T = \langle v^2 \rangle$$

- **How are the velocities distributed?**

$$f(v)$$

Experiments

- **Friction**

  D Blair, A Kudrolli 01
- **Rotation**

  K Feitosa, N Menon 04
- **Driving strength**

  W Losert, J Gollub 98
- **Dimensionality**

  J Urbach & Olafsen 98
- **Boundary**

  J van Zon, H Swinney 04
- **Fluid drag**

  K Kohlstedt, I Aronson, EB 05
- **Long range interactions**

  D Blair, A Kudrolli 01; W Losert 02

  K Kohlstedt, J Olafsen, EB 05
- **Substrate**

  G Baxter, J Olafsen 04

Deviations from equilibrium distribution

Nonequilibrium velocity distributions

A **Mechanically vibrated beads**

F Rouyer & N Menon PRL 00

B **Electrostatically driven powders**

I Aronson, J Olafsen, EB, PRL 05

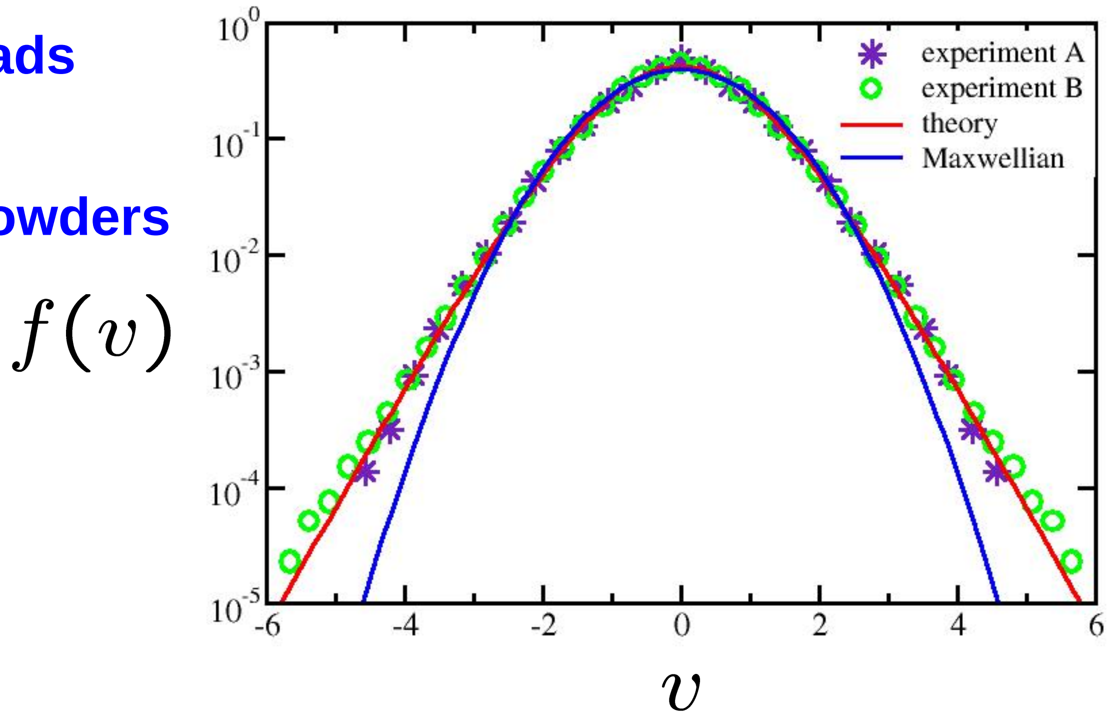
- **Gaussian core**
- **Overpopulated tail**

$$f(v) \sim \exp\left(-|v|^{\delta}\right)$$

$$1 \le \delta \le 3/2$$

- **Fourth Moment (or curtosis)**

$$\frac{\langle v^4 \rangle}{\langle v^2 \rangle^2} = \begin{cases} 3.55 & \text{theory} \\ 3.6 & \text{experiment} \end{cases}$$



Excellent agreement between theory and experiment

balance between collisional dissipation, energy injection from walls

Inelastic Collisions (1D)

- **Relative velocity reduced by** $0 < r < 1$

$$v_1 - v_2 = -r(u_1 - u_2)$$

- **Momentum is conserved**

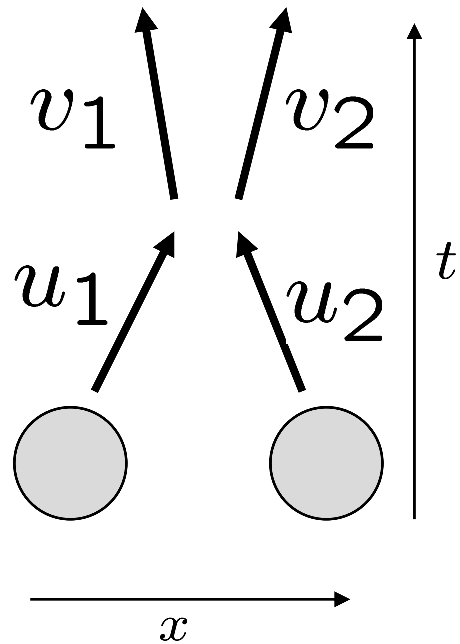
$$v_1 + v_2 = u_1 + u_2$$

- **Energy is dissipated**

$$\Delta E = \frac{1 - r^2}{4}(u_1 - u_2)^2$$

- **Limiting cases**

$$r = \begin{cases} 0 & \text{completely inelastic } (\Delta E = \max) \\ 1 & \text{elastic } (\Delta E = 0) \end{cases}$$



Nonequilibrium: time irreversible process

Inelastic collisions as an averaging process

- **Averaging process**

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} p & q \\ q & p \end{pmatrix} \begin{pmatrix} u_1 \\ v_2 \end{pmatrix}$$

$$r = 1 - 2p \qquad p + q = 1$$

- **Infinite particle system**
  - **Pick two particles with probability proportional to $K(u_1,u_2)$**
  - **Update velocities according to averaging rule**
- **Used in many contexts**
  - **Traffic: headway distance (majumdar, krug)**
  - **Econophysics: assett exchange (krapivsky, redner, slanina)**
  - **Opinion Dynamics (weisbuch, EB)**

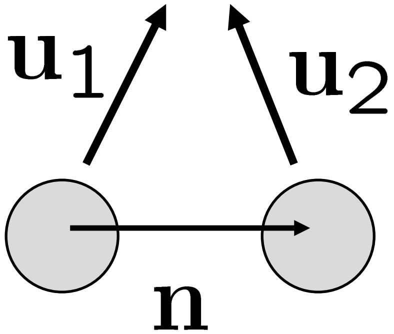
Inelastic Collisions (any D)

- **Normal relative velocity reduced by** $0 < r < 1$

$$(\mathbf{v}_1 - \mathbf{v}_2) \cdot \mathbf{n} = -r(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{n}$$

- **Momentum conservation**

$$\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{u}_1 + \mathbf{u}_2$$



- **Energy loss**

$$\Delta E = \frac{1 - r^2}{4}[(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{n}]^2$$

- **Limiting cases**

$$r = \begin{cases} 0 & \text{completely inelastic } (\Delta E = \max) \\ 1 & \text{elastic } (\Delta E = 0) \end{cases}$$

Non-Maxwellian velocity distributions

JC Maxwell, Phil Trans Roy. Soc. **157** 49 (1867)

1. **Velocity distribution is isotropic**

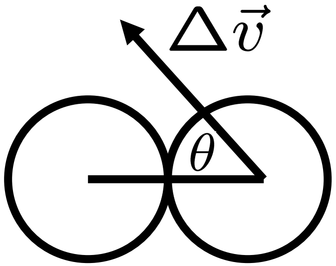
$$f(v_x, v_y, v_z) = f(|v|)$$

2. **No correlations between velocity components**

$$f(v_x, v_y, v_z) \neq f(v_x) f(v_y) f(v_z)$$

Only possibility is Maxwellian

$$f(v_x, v_y, v_z) \neq C \exp\left(-\frac{v_x^2 + v_y^2 + v_z^2}{2T}\right)$$



Granular gases: collisions create correlations

Deviation from Maxwell-Boltzmann

- **Velocity correlations, curtosis:**

$$Q = \frac{\langle v_x^2 v_y^2 \rangle}{\langle v_x^2 \rangle \langle v_y^2 \rangle} \propto \kappa$$

- **Exact expressions**

$$Q = \frac{6(\frac{1-r}{2})^2}{d - \left[1 + 3\left(\frac{1-r}{2}\right)^2\right]}$$

- **Inversely proportional to dimension**

$$Q \sim d^{-1} \qquad d \to \infty$$

- **Vanishes in the elastic limit**

$$Q \sim (1-r)^2 \qquad r \to 1$$

Exact solution of Maxwell's kinetic theory: thermal forcing balances dissipation EB, Krapivsky 02
Experiment: vibrated beads Olafsen 03

The collision rate

- **Collision rate**

$$K(\mathbf{u}_1, \mathbf{u}_2) = |(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{n}|^\lambda$$

- **Collision rate related to interaction potential (elastic)**

$$U(r) \sim r^{-\gamma} \qquad \lambda = 1 - 2\frac{d-1}{\gamma} = \begin{cases} 0 & \text{Maxwell molecules} \\ 1 & \text{Hard spheres} \end{cases}$$

- **Balance kinetic and potential energy**

$$v^2 \sim r^{-\gamma} \quad \Rightarrow \quad r \sim v^{-2/\gamma}$$

- **Collisional cross-section**

$$\sigma \sim v r^{d-1} \quad \Rightarrow \quad \sigma \sim v^{1-\frac{2}{\gamma}(d-1)}$$

The Inelastic Boltzmann equation (1D)

- **Collision rule (linear)** $r = 1 - 2p, \quad p + q = 1$

$$(u_1, u_2) \to (pu_1 + qu_2, pu_2 + qu_1)$$

- **General collision rate**

$$K(v_1, v_2) = |v_1 - v_2|^\lambda \qquad \lambda = \begin{cases} 0 & \text{Maxwell molecules} \\ 1 & \text{Hard spheres} \end{cases}$$

- **Boltzmann equation (nonlinear and nonlocal)**

$$\frac{\partial f(v)}{\partial t} = \iint du_1 du_2 f(u_1) f(u_2) |u_1 - u_2|^\lambda [\delta(v - pu_1 - qu_2) - \delta(v - u_2)]$$

collision rate gain loss

Theory: non-linear, non-local, <u>dissipative</u>

The Inelastic Boltzmann equation

Spatially homogeneous systems
(*StosszahlInsatz*)

$$\frac{\partial f(v)}{\partial t} = \iint du_1 du_2 f(u_1) f(u_2) |u_1 - u_2|^{\lambda} [\delta(v - pu_1 - qu_2) - \delta(v - u_2)]$$

What are the stationary solutions of this equation?
What is the nature of the velocity distribution?

Homogeneous cooling state: temperature decay

Haff, JFM 1982

- **Energy loss** $\Delta T \sim (\Delta v)^2$
- **Collision rate** $\Delta t \sim 1/(\Delta v)^\lambda$
- **Energy balance equation**

$$\frac{\Delta T}{\Delta t} \sim -(\Delta v)^{2+\lambda} \quad \Rightarrow \quad \frac{dT}{dt} \sim -T^{1+\lambda/2}$$

- **Temperature decays, system comes to rest**

$$T \sim t^{-2/\lambda} \quad \Rightarrow \quad f(v) \to \delta(v)$$

Trivial stationary solution

Homogeneous cooling states: similarity solutions

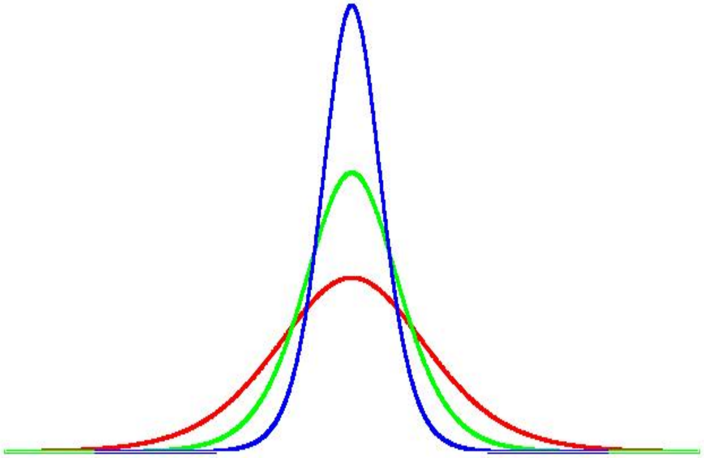
Esipov, Poeschel 97

- **Similarity solution**

$$f(v,t) = t^{1/\lambda}\Phi(vt^{1/\lambda})$$

- **Stretched exponentials (overpopulation)**

$$\Phi(z) \sim \exp\left(-|z|^{\lambda}\right)$$



Are there nontrivial stationary solutions?

- **Stationary Boltzmann equation**

$$0 = \iint du_1 du_2 f(u_1) f(u_2) |u_1 - u_2|^\lambda [\delta(v - pu_1 - qu_2) - \delta(v - u_2)]$$

collision rate gain loss

Naive answer: NO!

- **According to the energy balance equation**

$$\frac{dT}{dt} = -\Gamma$$

- **Dissipation rate is positive**

$$\Gamma \propto \langle |v_1 - v_2|^\lambda (v_1 - v_2)^2 \rangle > 0$$

An exact solution (1D, λ=0)

- **One-dimensional Maxwell molecules**
- **Fourier transform obeys a closed equation** $F(k) = \int dv\, e^{ikv} f(v)$

$$F(k) = F(pk)F(qk)$$

- **Exponential solution**

$$F(k) = \exp(-v_0|k|)$$

- **Lorentzian velocity distribution**

$$f(v) = \frac{1}{\pi}\frac{1}{1+v^2}$$

A nontrivial stationary solution does exist!

Properties of stationary solution

- **Perfect balance between collisional loss and gain**
- **Purely collisional dynamics (no source term)**
- **<u>Family of solutions</u>: scale invariance v→ v/v_0**

$$f(v) = \frac{1}{\pi v_0}\frac{1}{1+(v/v_0)^2}$$

- **Power-law high-energy tail**

$$f(v) \sim v^{-2}$$

- **Infinite energy, infinite dissipation rate!**

Are these stationary solutions physical?

Extreme Statistics (1D)

- **Collision rule: arbitrary velocities**

$$(u_1, u_2) \to (pu_1 + qu_2, pu_2 + qu_1)$$

- **Large velocities: linear but nonlocal process**

$$v \xrightarrow{v^\lambda} (pv, qv)$$

- **High-energies: linear equation**

$$\frac{\partial f(v)}{\partial t} = v^\lambda \left[\frac{1}{p^{1+\lambda}} f\left(\frac{v}{p}\right) + \frac{1}{q^{1+\lambda}} f\left(\frac{v}{q}\right) - f(v)\right]$$

gain gain loss

Linear, nonlocal evolution equation

Stationary solution (1D)

- **High-energies: linear equation**

  May still hold when velocities are correlated

$$f(v) = \frac{1}{p^{1+\lambda}} f\left(\frac{v}{p}\right) + \frac{1}{q^{1+\lambda}} f\left(\frac{v}{q}\right)$$

loss gain gain

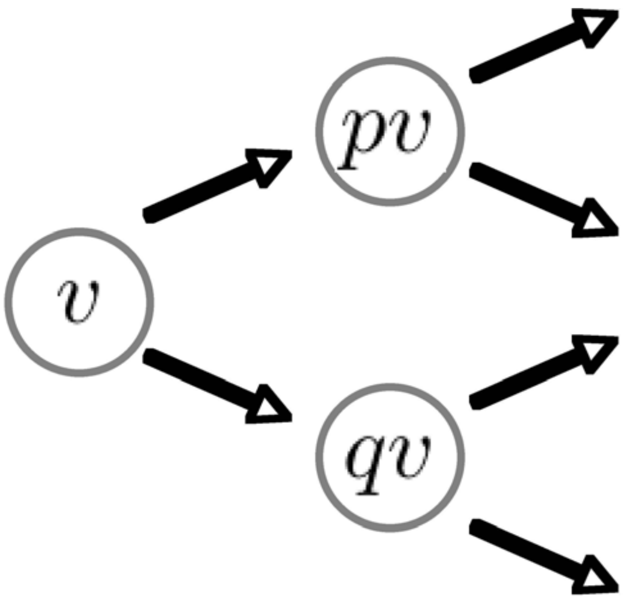
- **Power-law tail**

$$f(v) \sim v^{-2-\lambda}$$

Energy Cascades (1D)

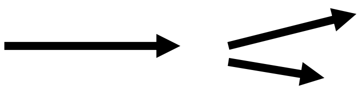
Energetic particles "see" a static medium

$$v \longrightarrow (pv, qv)$$

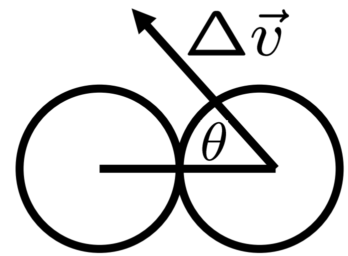


Extreme Statistics (any D)

- **Collision process: large velocities**



$$v \xrightarrow{|v\cos\theta|^\lambda} (\alpha v, \beta v)$$



- **Stretching parameters related to impact angle**

$$\alpha = (1-p)\cos\theta \qquad \beta = \sqrt{1-(1-p^2)\cos^2\theta}$$

- **Energy decreases, velocity magnitude increases**

$$\alpha^2+\beta^2 \le 1 \qquad\qquad \alpha+\beta \ge 1$$

- **Linear equation**

$$\frac{\partial f(v)}{\partial t} = \left\langle (v\cos\theta)^\lambda \left[\frac{1}{\alpha^{d+\lambda}} f\left(\frac{v}{\alpha}\right) + \frac{1}{\beta^{d+\lambda}} f\left(\frac{v}{\beta}\right) - f(v)\right]\right\rangle$$

angular integration $\quad \langle f(\cos\theta)\rangle \propto \int_0^1 d\cos^2\theta\,[\cos^2\theta]^{-1/2}\,[1-\cos^2\theta]^{(d-3)/2} f(\cos\theta)$

Power-laws are generic

- **Velocity distribution always has power-law tail**

$$f(v) \sim v^{-\sigma}$$

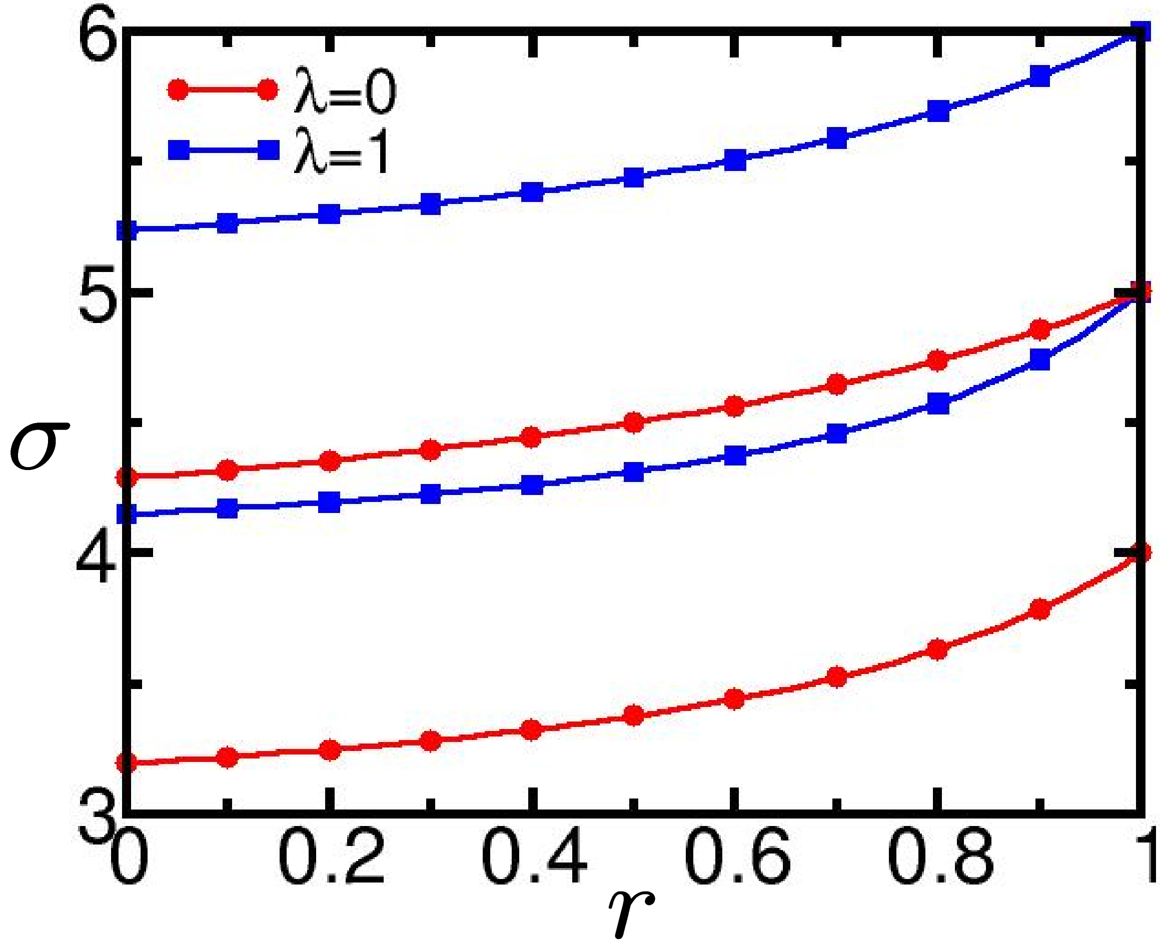
- **Characteristic exponent varies with parameters**

$$\frac{1 - {}_2F_1\left(\frac{d+\lambda-\sigma}{2}, \frac{\lambda+1}{2}, \frac{d+\lambda}{2}, 1-p^2\right)}{(1-p)^{\sigma-d-\lambda}} = \frac{\Gamma(\frac{\sigma-d+1}{2})\Gamma(\frac{d+\lambda}{2})}{\Gamma(\frac{\sigma}{2})\Gamma(\frac{\lambda+1}{2})}$$

- **Tight bounds** $1 \leq \sigma - d - \lambda \leq 2$
- **Elastic limit is singular** $\sigma \to d+2+\lambda$

Dissipation rate always divergent
Energy finite or infinite

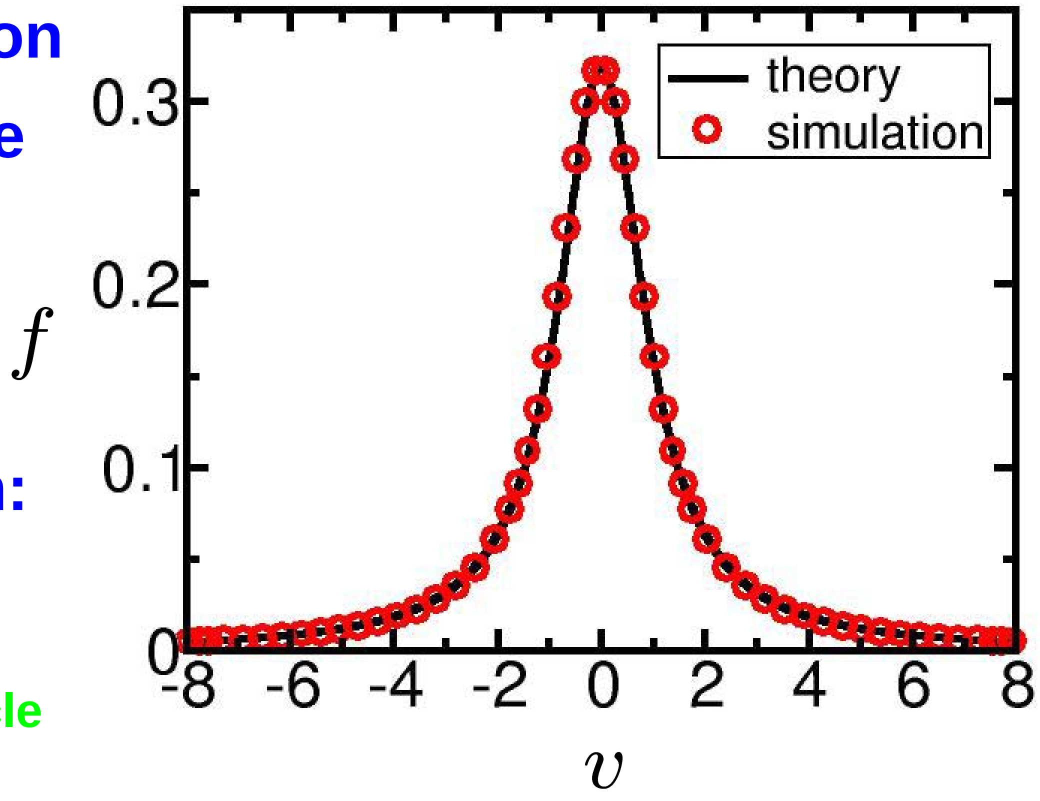
The characteristic exponent σ (d=2,3)



σ varies with spatial dimension, collision rules

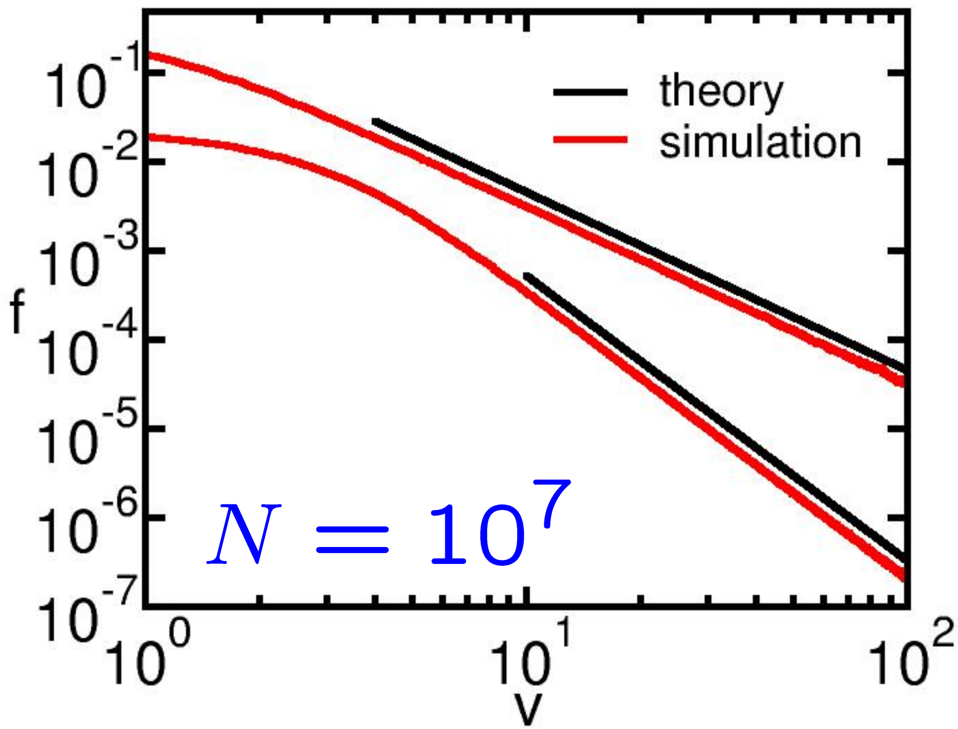
Monte Carlo Simulations: Driven Steady States

- <u>Compact</u> initial distribution
- Inject energy at very large velocity scales only
- Maintain constant total energy
- <u>“Lottery”</u> implementation:
  - Keep track of total energy dissipated, E_T
  - With small rate, boost a particle by E_T



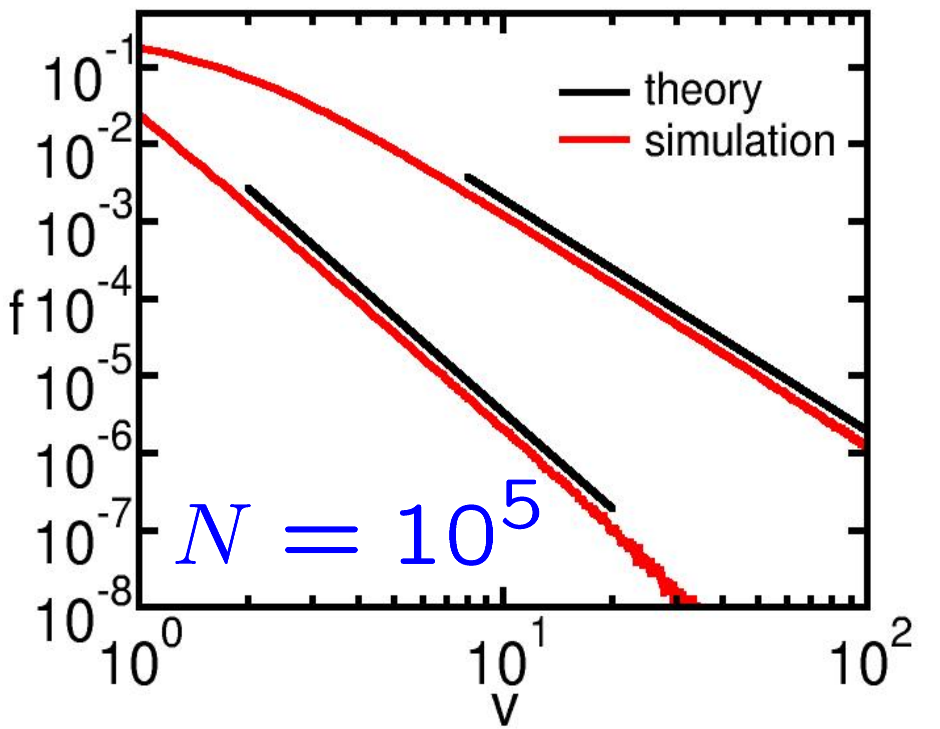
Further confirmation: extremal statistics

Maxwell molecules (1D, 2D)



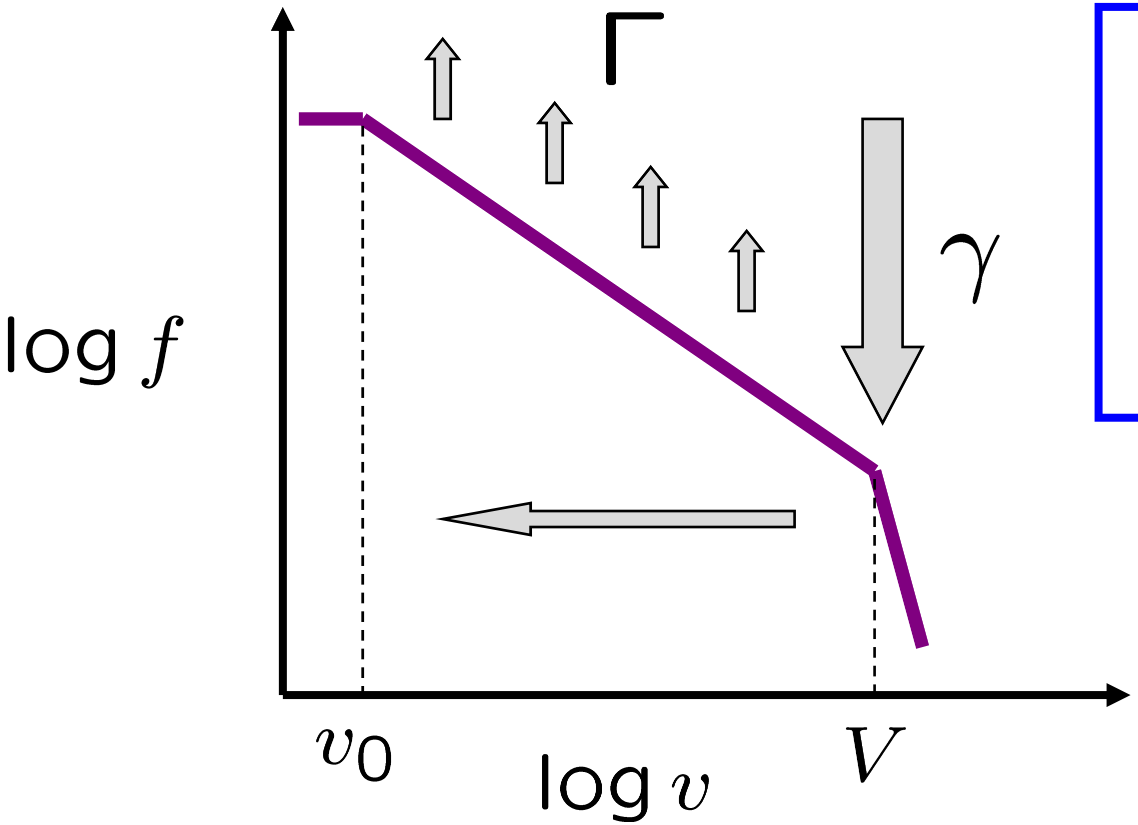
d	theory	simulation
1	2	1.995
2	3.19520	3.19

Hard spheres (1D, 2D)



d	theory	simulation
1	3	2.994
2	4.14922	4.15

Injection, cascade, dissipation



Experimental realization?
Energetic particle "shot" into static medium

Energy balance

$\Gamma \sim \gamma V^2$

- Energy is injected <u>ONLY AT LARGE VELOCITY SCALES!</u>
- Energy cascades from large velocities to small velocities
- Energy dissipated at small velocity scales

Energy balance

- **Energy injection rate** γ
- **Energy injection scale** V
- **Typical velocity scale** v_0
- **Balance between energy injection and dissipation**

$$\Gamma \sim \langle |v|^{2+\lambda} \rangle \quad \rightarrow \quad \gamma \sim V^{\lambda} (V/v_0)^{d-\sigma}$$

- **For "lottery" injection: injection scale diverges with injection rate**

$$V \sim \begin{cases} \gamma^{-1/(2-\lambda)} & \sigma < d+2 \\ \gamma^{-1/(\sigma-d-\lambda)} & \sigma > d+2 \end{cases}$$

Energy injection selects stationary solution → v_0

with Ben Machta (Brown)

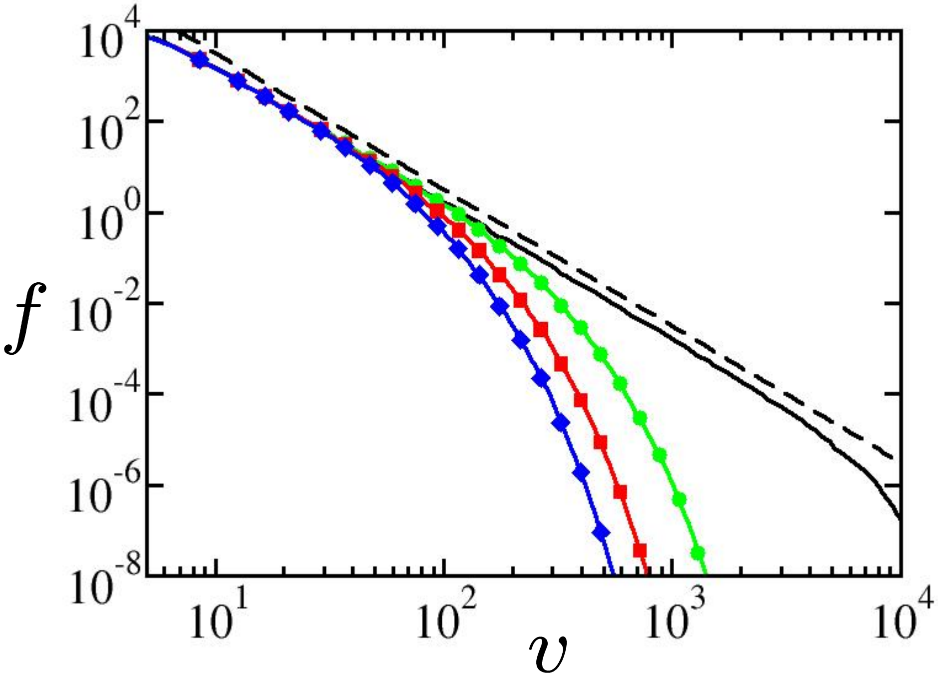
Time dependent solutions (1D, λ>0)

- **Self-similar distribution**

$$f(v,t) \simeq v^{-\sigma}\Phi\left(\frac{v}{V(t)}\right)$$

- **Cutoff velocity decays**

$$V(t) \sim t^{-1/\lambda}$$



- **Scaling function**

$$\Phi(x) = \sum_{n=1}^{\infty} A_n \exp\left[-(2^n x)^\lambda\right] \qquad A_n = \prod_{\substack{k=1 \\ k\neq n}}^{\infty} \frac{1}{1-2^{\lambda(n-k)}}$$

Hybrid between steady-state and time dependent state

Solution via Laplace transform

$$p = q = 1/2 \qquad \lambda = 1$$

- **Linear Boltzmann equation**

$$\frac{\partial f(v)}{\partial t} = |v|^\lambda \left[\frac{1}{p^{1+\lambda}} f\left(\frac{v}{p}\right) + \frac{1}{q^{1+\lambda}} f\left(\frac{v}{q}\right) - f(v)\right]$$

- **Scaling solution, cutoff decays with Haff's law**

$$f(v,t) \simeq v^{-\sigma} \Phi\left(\frac{v}{V(t)}\right) \qquad dV/dt = -cV^{1+\lambda}$$

- **Linear, nonlocal equation for scaling function**

$$c\phi'(x) = x^{\lambda-1}\left[p\phi\left(\frac{x}{p}\right) + q\phi\left(\frac{x}{q}\right) - \phi(x)\right]$$

- **Laplace transform equation**

$$(2+s)\tilde{\phi}(s) = 1 + \tilde{\phi}(s/2)$$

- **Infinite product solution**

$$\tilde{\phi}(s) = s^{-1}[1 - g(s)] \longrightarrow g(s) = \frac{1}{1+\frac{s}{2}} g(s)$$

$$g(s) = \prod_{n=1}^{\infty} \frac{1}{1+s/2^n}$$

Extreme statistics

- **Scaling function**

$$\Phi(x) = \sum_{n=1}^{\infty} A_n \exp\left[-(2^n x)^\lambda\right] \qquad A_n = \prod_{\substack{k=1 \\ k\neq n}}^{\infty} \frac{1}{1-2^{\lambda(n-k)}}$$

- **Large velocities: as in free cooling**

$$\Phi(x) \sim \exp(-x^\lambda) \qquad x \to \infty$$

- **Small velocities: non-analytic, log-normal, behavior**

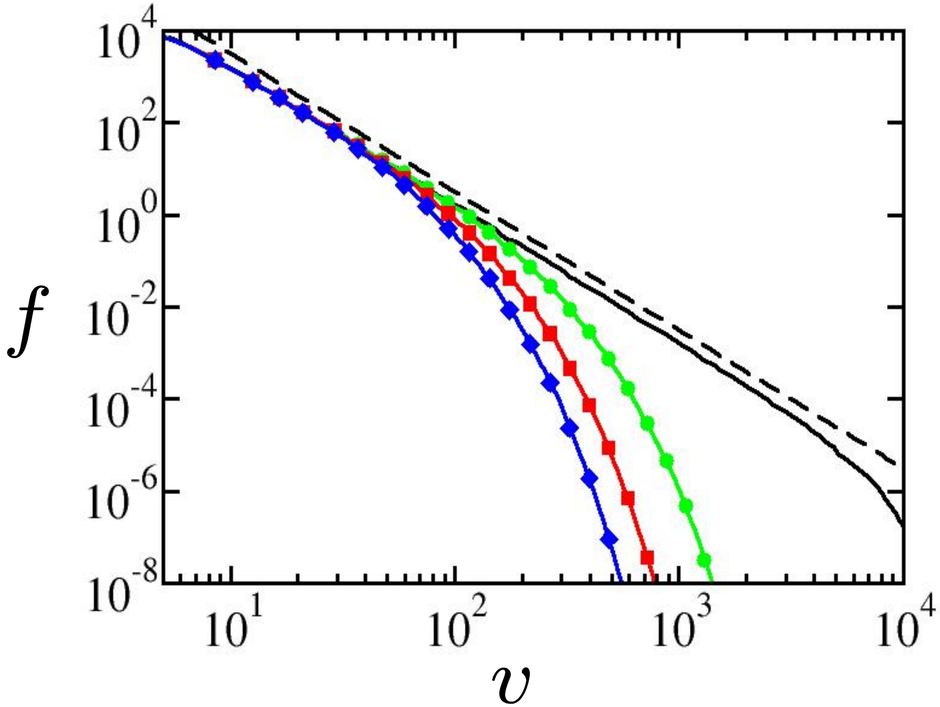
$$1 - \Phi(x) \sim \exp\left[-(\ln x)^2\right] \qquad x \to 0$$

Hybrid between steady-state and time dependent state

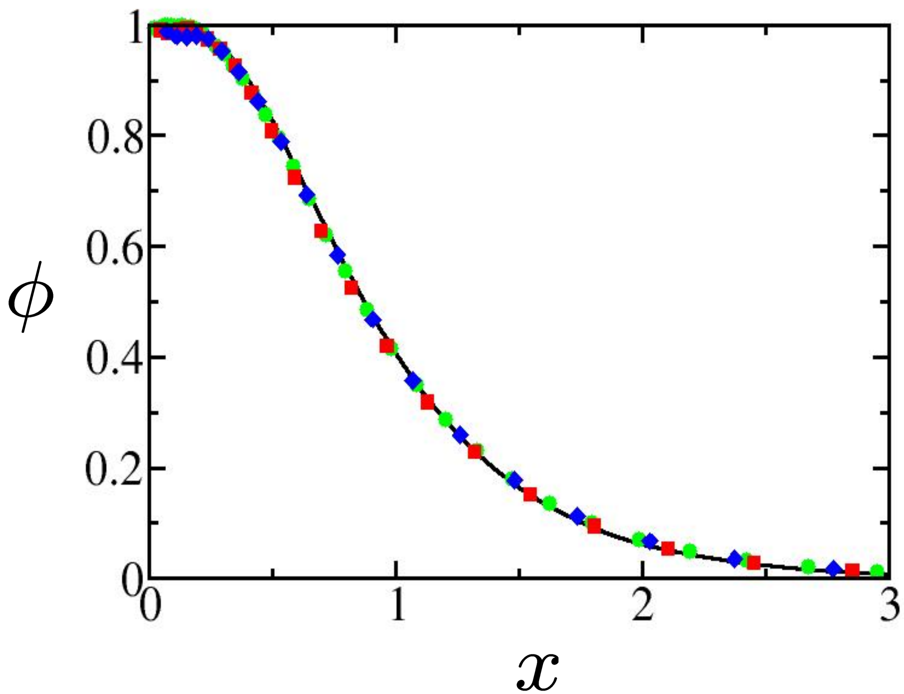
Maxwell Model (λ=0) only unsolved case!

Numerical confirmation

Velocity distribution



Scaling function



A third family of solutions exists

Summary: solutions of kinetic theory

- **Time dependent solution**

$$f(v,t) = t^{1/\lambda}\Psi(vt^{1/\lambda})$$

- **Time independent solution**

$$f_s(v) \sim v^{-\sigma}$$

- **Hybrid solution**

$$f(v,t) = f_s(v)\Phi(vt^{1/\lambda})$$

Are there other types of solutions?

Conclusions I

- New class of nonequilibrium steady states
- Energy cascades from large to small velocities
- Power-law high-energy tail
- Energy input at large scales balances dissipation
- Associated similarity solutions exist as well
- Temperature insufficient to characterize velocities
- Experimental realization: requires a different driving mechanism

The Thermally Forced Inelastic Boltzmann equation

- **Energy injection: thermal forcing (at all scales)**

$$dv/dt = \eta$$

- **Energy dissipation: inelastic collision**

$$\frac{\partial f(v)}{\partial t} = D\nabla^2 f(v) + \iint du_1 du_2 f(u_1) f(u_2) |u_1 - u_2|^\lambda [\delta(v - pu_1 - qu_2) - \delta(v - u_2)]$$

- **Steady state equation**

$$0 = D\nabla^2 f(v) + \iint du_1 du_2 f(u_1) f(u_2) |u_1 - u_2|^\lambda [\delta(v - pu_1 - qu_2) - \delta(v - u_2)]$$

Driven Steady States: extremal statistics

T van Noije, M Ernst 97

- **Energy injection: thermal forcing (at all scales)**

$$dv/dt = \eta$$

- **Energy dissipation: inelastic collision**

$$v \longrightarrow (pv, qv)$$

- **Steady state equation**

$$0 = D\frac{d^2 f(v)}{d^2 v} + v^\lambda \left[\cancel{\frac{1}{p^{1+\lambda}} f\left(\frac{v}{p}\right) + \frac{1}{q^{1+\lambda}} f\left(\frac{v}{q}\right)} - f(v)\right]$$

- **Stretched exponentials**

$$f(v) \sim \exp\left(-v^{1+\lambda/2}\right)$$

Nonequilibrium velocity distributions

A **Mechanically vibrated beads**

F Rouyer & N Menon 00

B **Electrostatically driven powders**

I Aronson, J Olafsen, EB

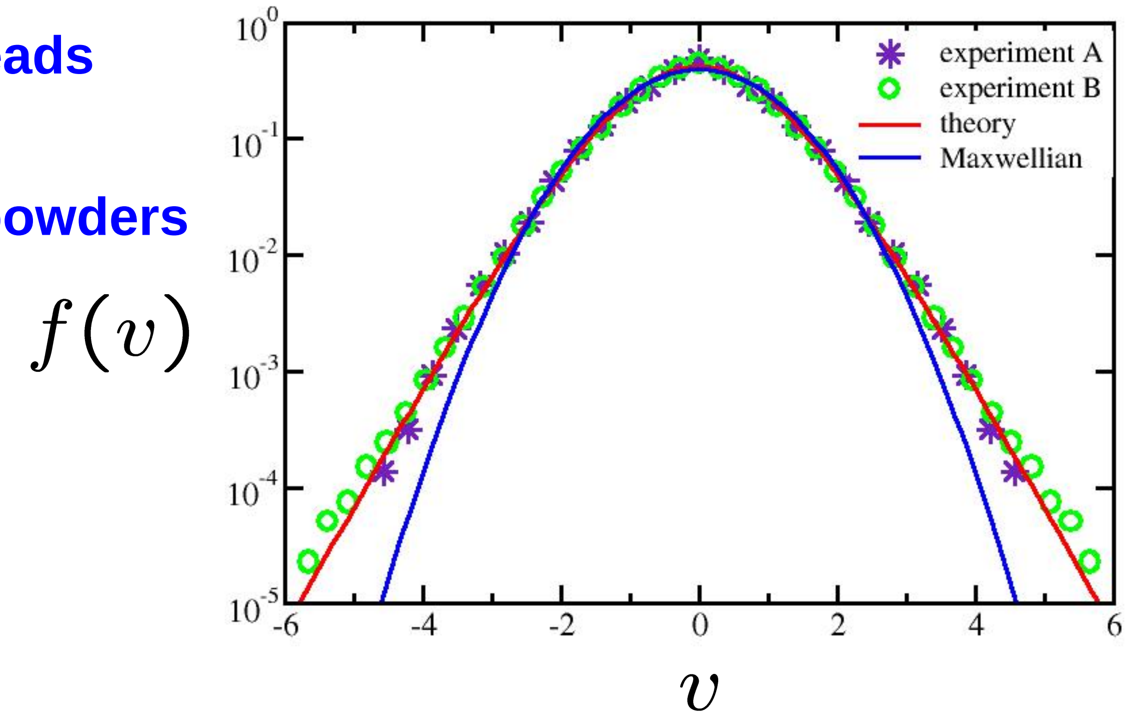
- **Gaussian core**
- **Overpopulated tail**

$$f(v) \sim \exp\left(-|v|^{\delta}\right)$$

$$1 \leq \delta \leq 3/2$$

- **Fourth moment / curtosis**

$$\frac{\langle v^4 \rangle}{\langle v^2 \rangle^2} = \begin{cases} 3.55 & \text{theory} \\ 3.6 & \text{experiment} \end{cases}$$



Excellent agreement between theory and experiment

balance between collisional dissipation, energy injection from walls

Freely cooling states: similarity solutions

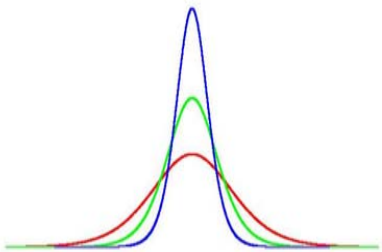
Esipov, Poeschel 97

- **Linearized equation**

$$\frac{\partial f(v)}{\partial t} = v^\lambda \left[\frac{1}{p^{1+\lambda}} f\left(\frac{v}{p}\right) + \frac{1}{q^{1+\lambda}} f\left(\frac{v}{q}\right) - f(v)\right]$$

- **Similarity solution**

$$f(v) \to t^{1/\lambda}\Phi(vt^{1/\lambda})$$



- **Steady state equation**

$$\frac{1}{\lambda}[\Phi(z)+z\frac{d}{dz}\Phi(z)] = z^\lambda \left[\frac{1}{p^{1+\lambda}}\Phi\left(\frac{z}{p}\right) + \frac{1}{q^{1+\lambda}}\Phi\left(\frac{z}{q}\right) - \Phi(z)\right]$$

- **Stretched exponentials (overpopulation)**

$$\Phi(z) \sim \exp\left(-z^\lambda\right)$$

Conclusions II

- Conventional nonequilibrium steady states
- Energy cascades from large to small velocities
- Energy input at ALL scales balances dissipation
- Stretched exponential tails
- Low order moments (temperature, kurtosis) useful
- Excellent agreement between experiments and kinetic theory

Who, then, can calculate the course of a molecule? How do we know that the creation of worlds is not determined by the fall of grains of sand?

Victor Hugo, Les Miserables

I can calculate the motion of heavenly bodies, but not the madness of people.

Isaac Newton

Clustering and Shocks