Shock Dynamics of Granular Gases

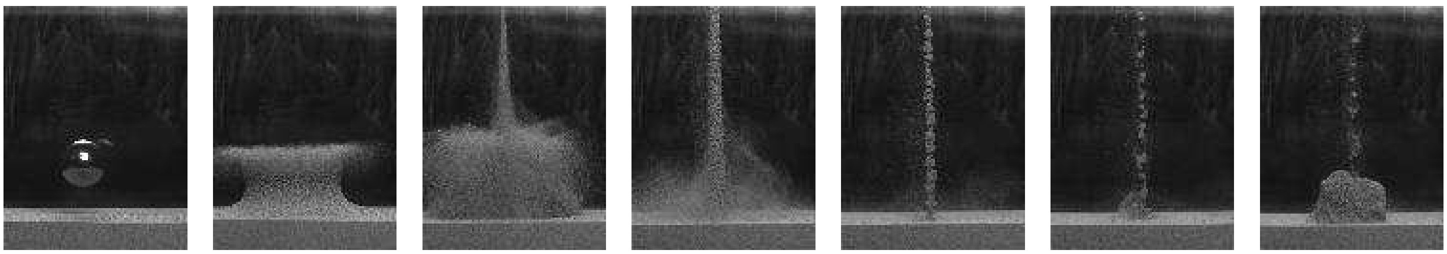
Eli Ben-Naim

Theoretical Division, Los Alamos National Lab

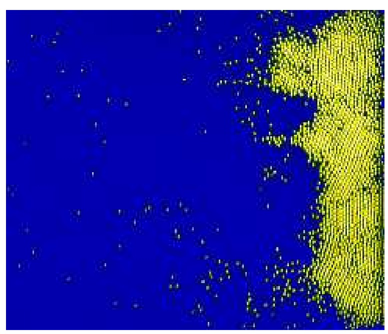
Shiyi Chen (J Hopkins/LANL)
Xiaobo Nie (J Hopkins/LANL)
Gary Doolen (LANL)
Sidney Redner (Boston)

Experiments

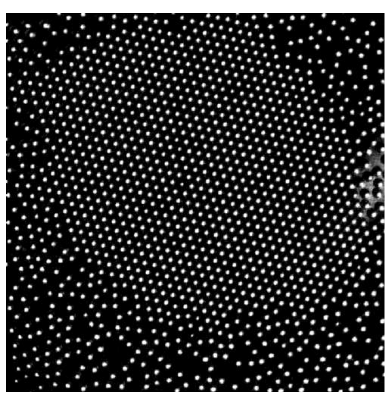
- **Clustering in granular jets** Chen/Lohse 01



- **Density inhomogeneities** Gollub 97



- **Ordered clusters in monolayers** Urbach 97



“A gas of marbles”

- Granular materials: powders, grains
- Geophysics: sand dunes, volcanic flows
- Astrophysics: large scale formation

Characteristics

- Hard sphere interactions
- Dissipative collisions

Challenges

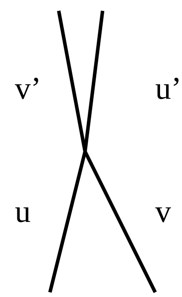
- Hydrodynamics: flow equations
- Kinetic Theory: velocity statistics
- Sharp validity criteria are missing

Inelastic collisions

- Relative velocity reduced by $r = 1 - 2\epsilon$

$$\begin{aligned} \Delta v' &= -r\Delta v \\ v' &= v - \epsilon\Delta v \end{aligned}$$

- Energy dissipation $\Delta E \propto -\epsilon(\Delta v)^2$



Freely cooling gases

- N point particles in 1D ring.
  Random velocity distribution.
  Typical velocity v_0. Typical distance x_0.

- **Dimensionless variables** $x \to x/x_0$, $t \to tv_0/x_0$

— “Temperature” $T(t) = \langle v^2(t)\rangle - \langle v(t)\rangle^2$

— Characteristic time/length scales.

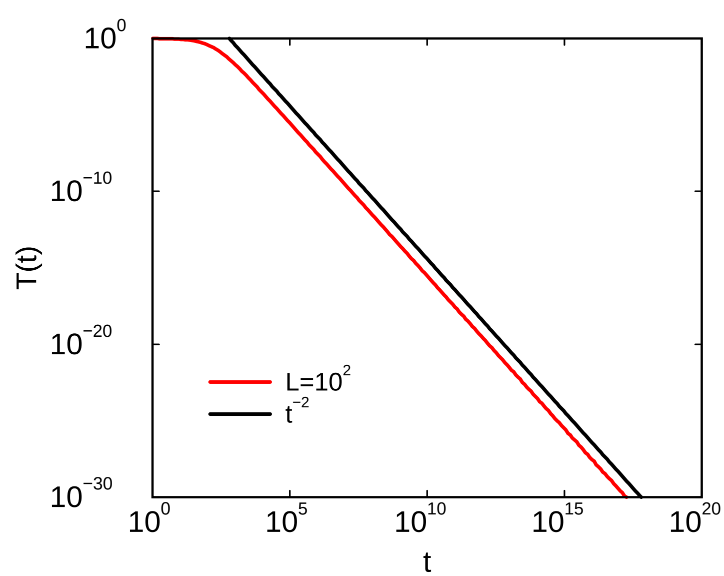
Mean Field Theory

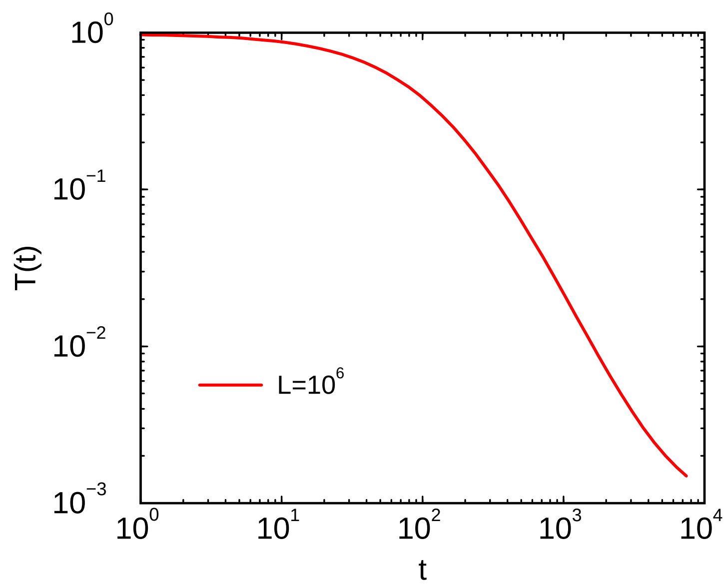
- Energy dissipation $\Delta T \propto -\epsilon(\Delta v)^2$
- Collision frequency $\Delta t \sim \ell/\Delta v \sim (\Delta v)^{-1}$
- Assuming uniform gas $dT/dt \propto -\epsilon T^{3/2}$
- Cooling law

Haff 83

$$T(t) \simeq (1 + A\epsilon t)^{-2} \sim \begin{cases} 1 & t \ll \epsilon^{-1} \\ \epsilon^{-2} t^{-2} & t \gg \epsilon^{-1} \end{cases}$$

- Simulation





Valid only in small systems/early time

The Inelastic Collapse: $N = 3$

1D: Bernu 91, Young 91, 2D: Kadanoff 95

- Deterministic collision sequence $12, 23, 12, \ldots$
- Velocities given by linear combination

$$\begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} = \begin{pmatrix} \epsilon & 1-\epsilon & 0 \\ 1-\epsilon & \epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

- After a pair of collisions (12,23)

$$\mathbf{v}' = \mathbf{M}\mathbf{v} \qquad \mathbf{M} = \mathbf{M}_{12}\mathbf{M}_{23}$$

- $r < 7 - 4\sqrt{3} \Rightarrow$

$$\Delta x, \Delta t \propto \lambda^n \qquad \lambda < 1$$

- Infinite number of collisions in finite time

$\boxed{\textbf{Particles clump}}$

The Sticky Gas ($r = 0$)

Carnavale, Pomeau, Young 90

- Multiparticle aggregate of typical mass m
- **Momentum conservation**

$$P_m = \sum_{i=1}^{m} P_i \quad \Rightarrow \quad P \sim m^{1/2}, \quad v \sim m^{-1/2}$$

- **Mass conservation**

$$\rho = cm = \text{const} \quad \Rightarrow \quad c \sim m^{-1}$$

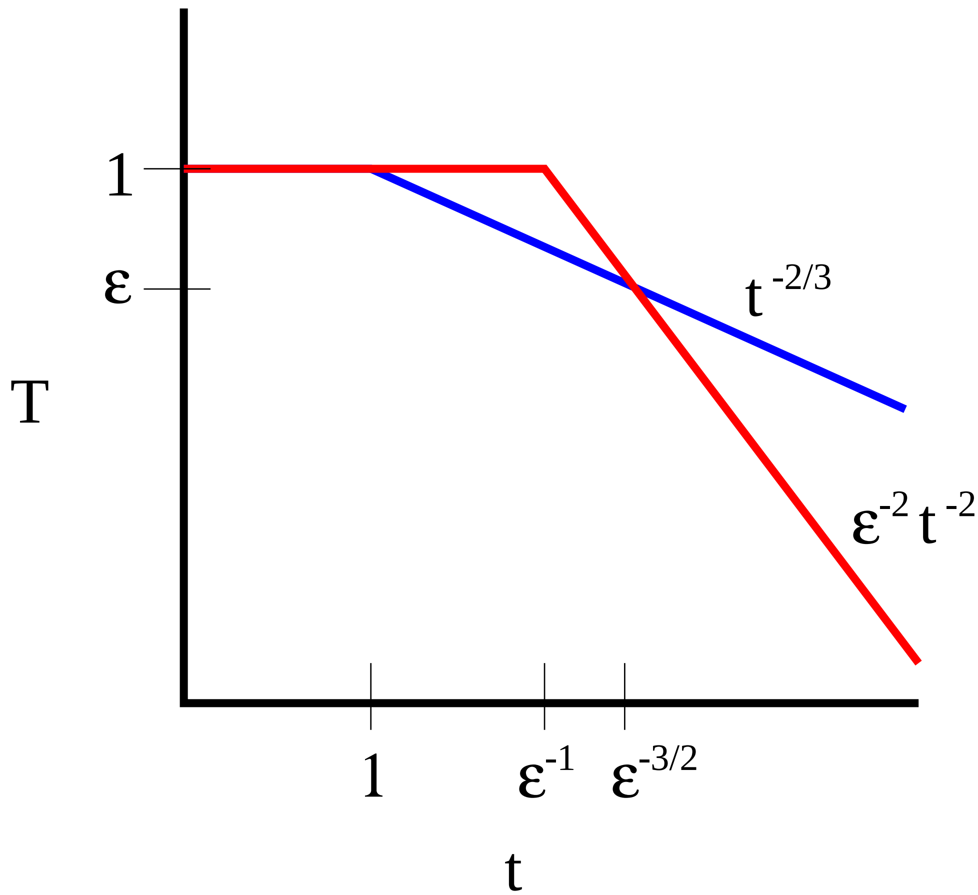
- **Dimensional analysis** $[cv] = [t]^{-1}$

$$m \sim t^{2/3} \quad v \sim t^{-1/3} \quad T \sim t^{-2/3}$$

- **Final state** 1 aggregate with $m = N$

$$T(t) \sim \begin{cases} 1 & t \ll 1; \\ t^{-2/3} & 1 \ll t \ll N^{3/2}; \\ N^{-1} & N^{3/2} \ll t \end{cases}$$

Monotonicity

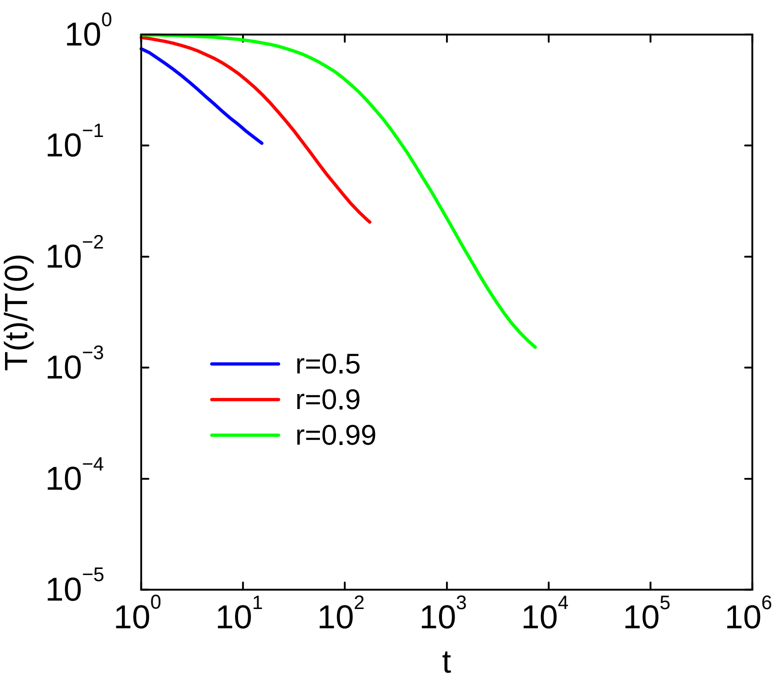


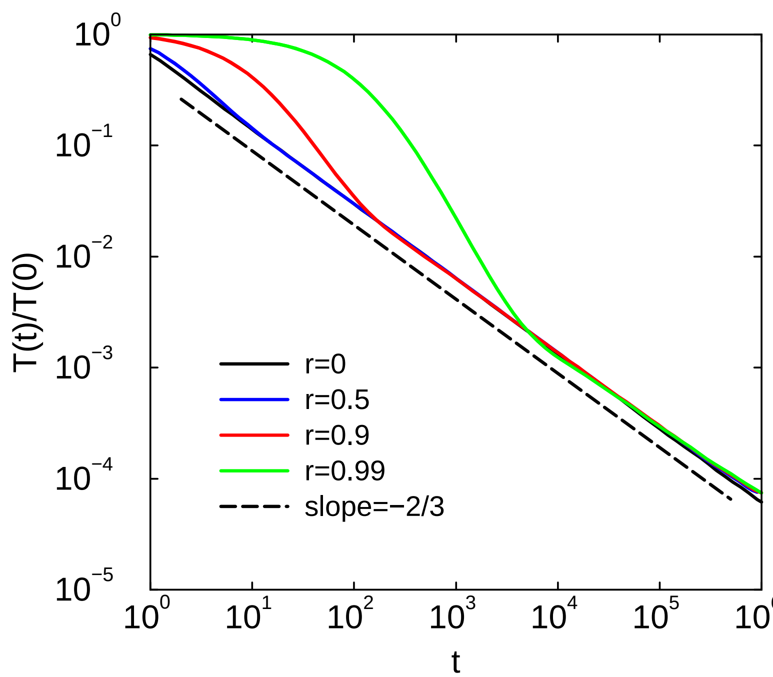
- $T(\epsilon, t)$ decreases monotonically with ϵ, t
- **Sticky gas ($\epsilon = 1/2$) is a lower bound**
- Homogeneous when $N \ll \epsilon^{-1}$ **or** $t \ll \epsilon^{-3/2}$
- Clustering when $N \gg \epsilon^{-1}$ **and** $t \gg \epsilon^{-3/2}$

The crossover picture

- Event driven simulations: $N = 10^7$
- Universal cooling law $T(t) \sim t^{-2/3}$

$$T(t) \sim \begin{cases} 1 & t \ll \epsilon^{-1}; \\ \epsilon^{-2} t^{-2} & \epsilon^{-1} \ll t \ll \epsilon^{-3/2} \\ t^{-2/3} & \epsilon^{-3/2} \ll t \ll N^{3/2} \\ N^{-1} & t \gg N^{3/2} \end{cases}$$





Asymptotic behavior is independent of r

Simulation technique

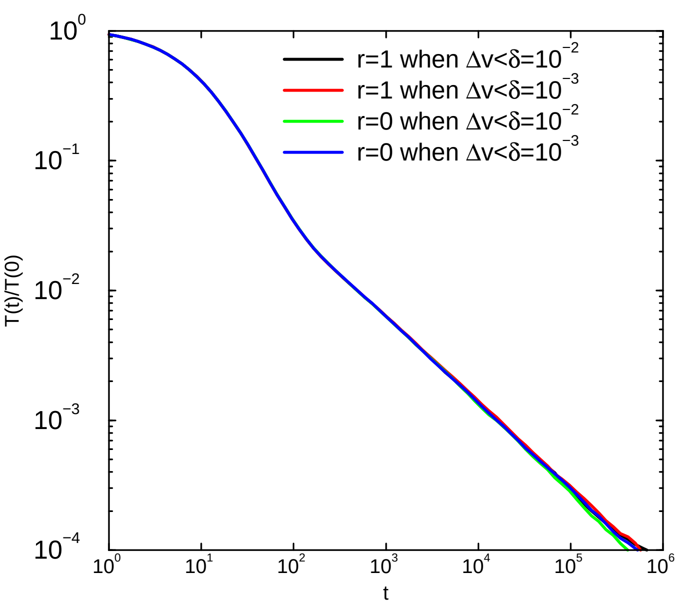
- Relax dissipation below cutoff $\delta \approx 10^{-3}$ (mimic granular particles)

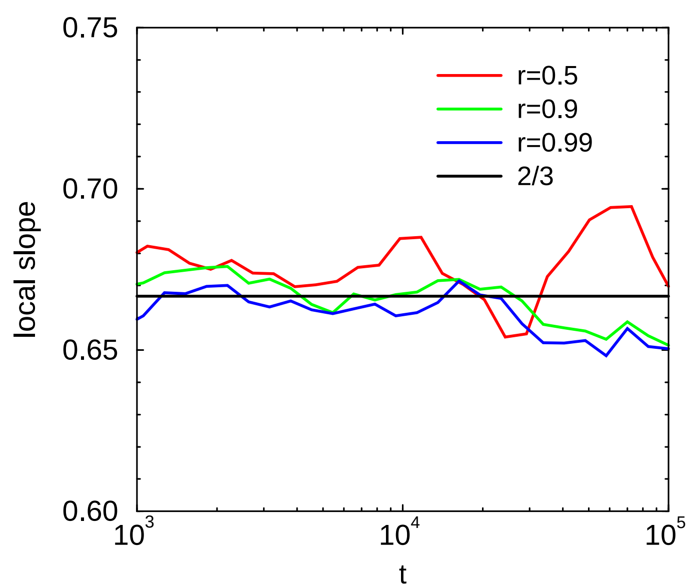
$$r(\Delta v) = \begin{cases} 1 & \Delta v < \delta \\ r & \Delta v > \delta \end{cases}$$

- Results are independent of:

— Threshold value δ

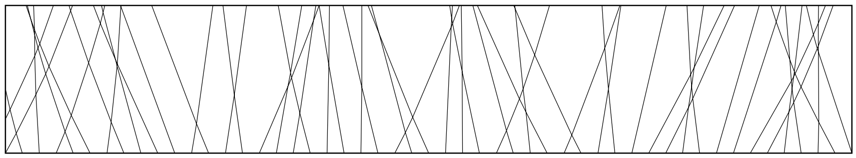
— Subthreshold collision mechanism



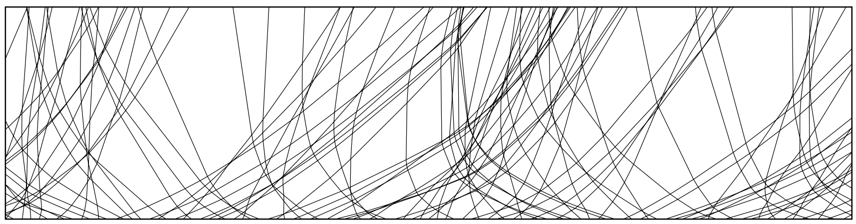


Results valid for $v \ll \delta$, $t \ll \delta^{-3}$

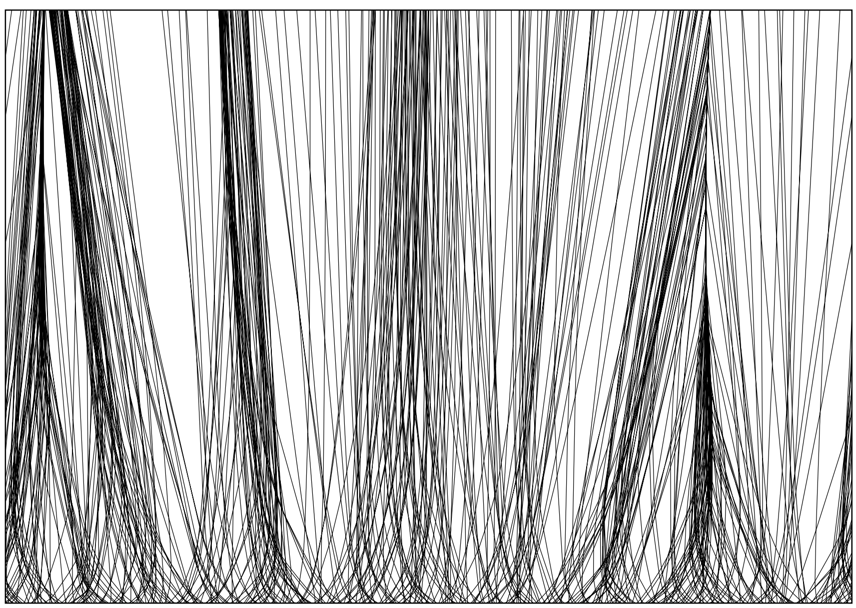
Early = Elastic gas ($r = 1$)



Intermediate = Inelastic gas ($r = 0.9$)



Late = Sticky gas ($r = 0$)



$\boxed{r = 0 \textbf{ is fixed point}}$

The Velocity Distribution

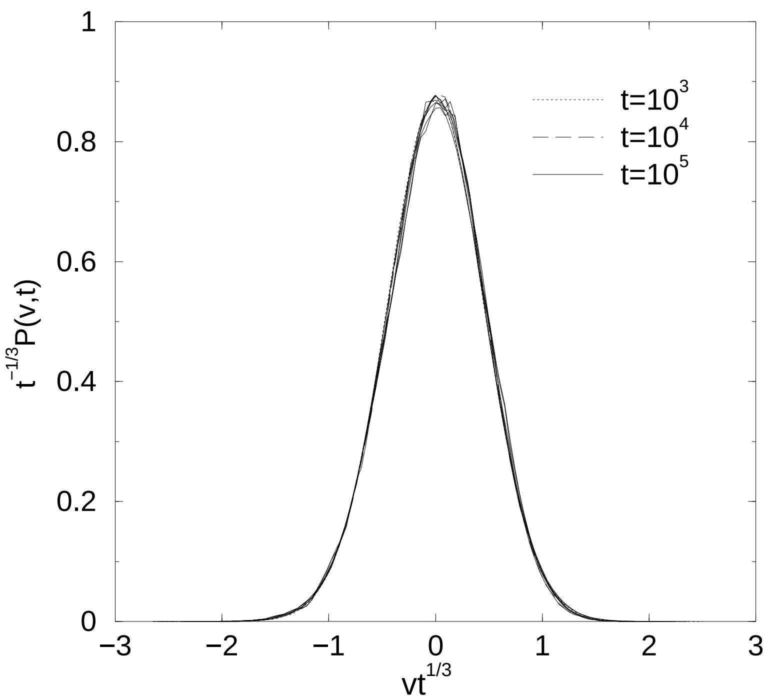
- Self similar distribution

$$P(v,t) \sim t^{1/3}\Phi(vt^{1/3})$$

- Anomalous tail

$$\Phi(z) \sim \exp(-\text{const.} \times z^3) \qquad z \gg 1$$

- Simulation results $r = 0,\ 0.5,\ 0.9$



$$\boxed{P(r,v,t) \textbf{ is function of } z = vt^{1/3} \textbf{ only}}$$

Large velocity tail

- Use scaling behavior

$$P(v,t) \sim t^{\beta}\Phi(z) \qquad z = vt^{\beta}$$

- Assume stretched exponential decay

$$\Phi(z) \sim \exp(-|z|^{\gamma}) \qquad |z| \gg 1$$

- **Lisfhitz/Fisher tail:**

— Focus on fastest $v = 1$ particles
— $t = 0$: Empty interval $L = t$ is empty ahead
— Probability $=\exp(-c_0 L)$

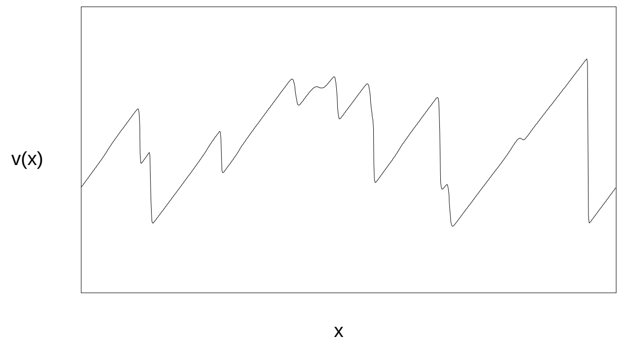
$$P(v = 1, t) \sim \exp(-t)$$

- Equate powers of $t \Rightarrow \beta\gamma = 1$

$$\gamma = \begin{cases} 1 & \text{homogeneous regime} \\ 3 & \text{clustering regime} \end{cases}$$

$\boxed{\textbf{Anomalous velocity statistics}}$

The Inviscid Burgers Equation



- Nonlinear diffusion equation

$$v_t + v v_x = \nu v_{xx} \qquad \nu \to 0$$

- Transform to linear diffusion equation

$$u_t = \nu u_{xx} \qquad \Leftarrow \qquad v = -2\nu (\ln u)_x$$

- Sawtooth (shock) velocity profile

$$v(x,t) = \frac{x - q(x,t)}{t}$$

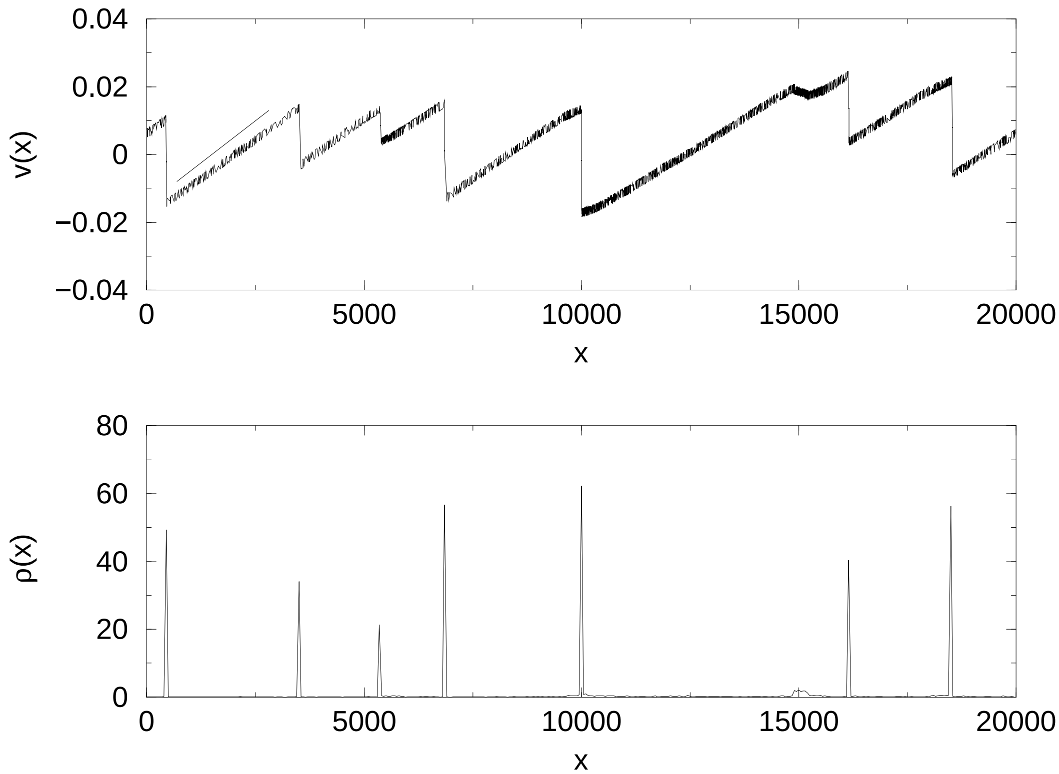
- Shock collisions conserve mass & momentum

- Describes "sticky gas" $r = 0$ Zeldovich RMP 89

Burgers equation $\equiv$ sticky gas $\equiv$ inelastic gas

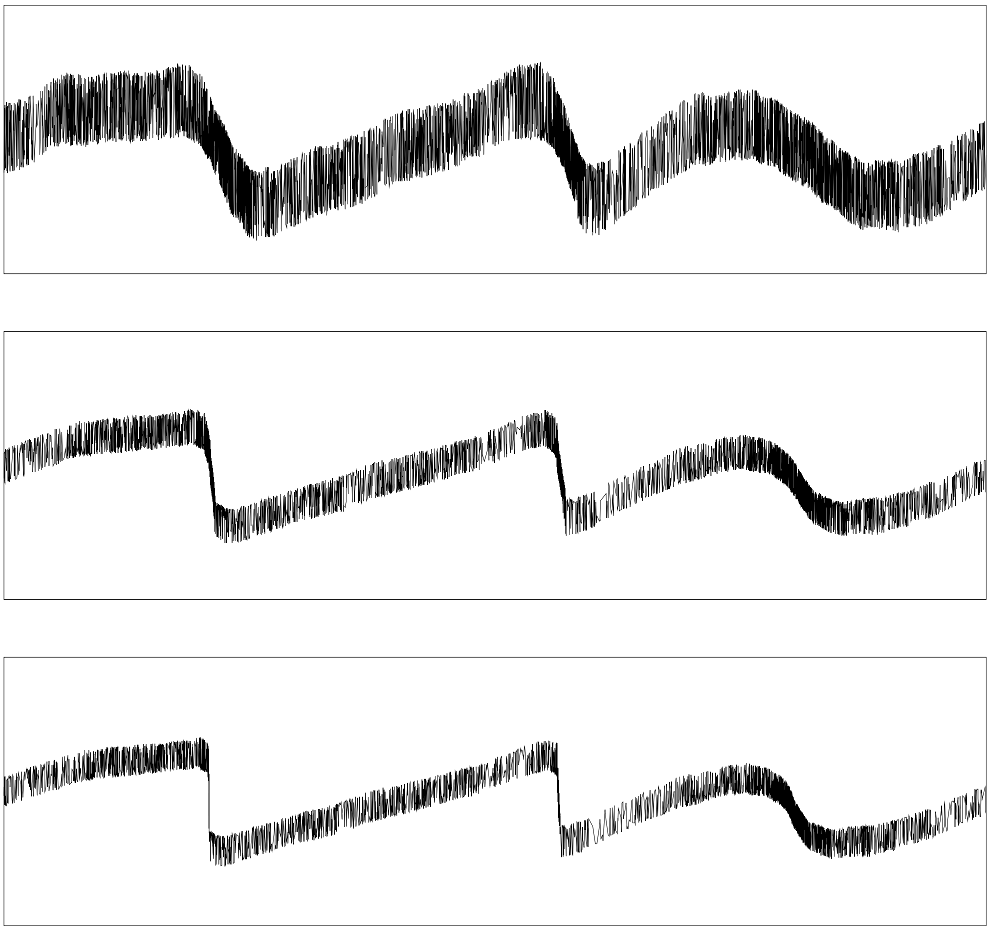
Burgers' eqn Predictions verified in 1D

- Velocity statistics $v \sim t^{-1/3}$
- Discontinuous (shock) velocity profile
- Slope $= t^{-1}$ (simulation with $r = 0.99$)



Collapse $\equiv$ shock formation

Formation of Singularity

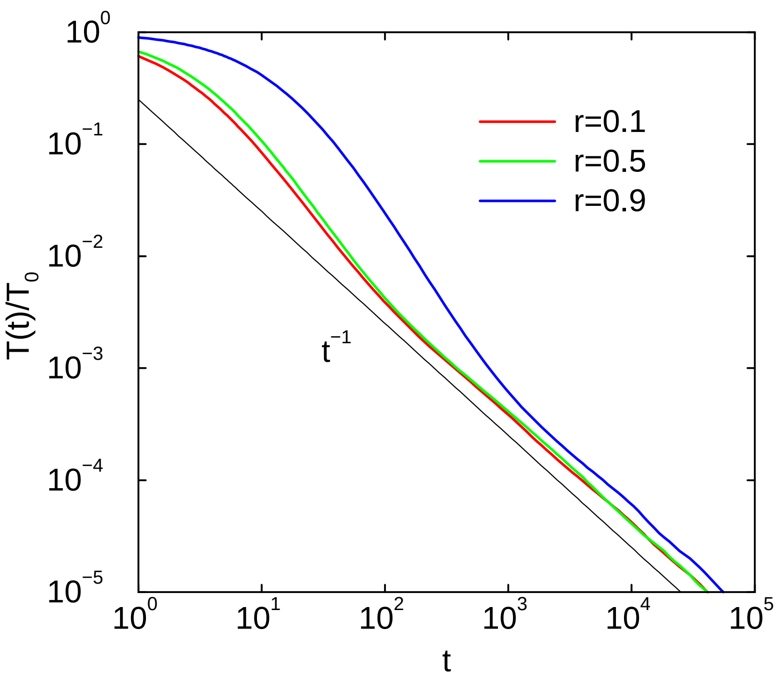


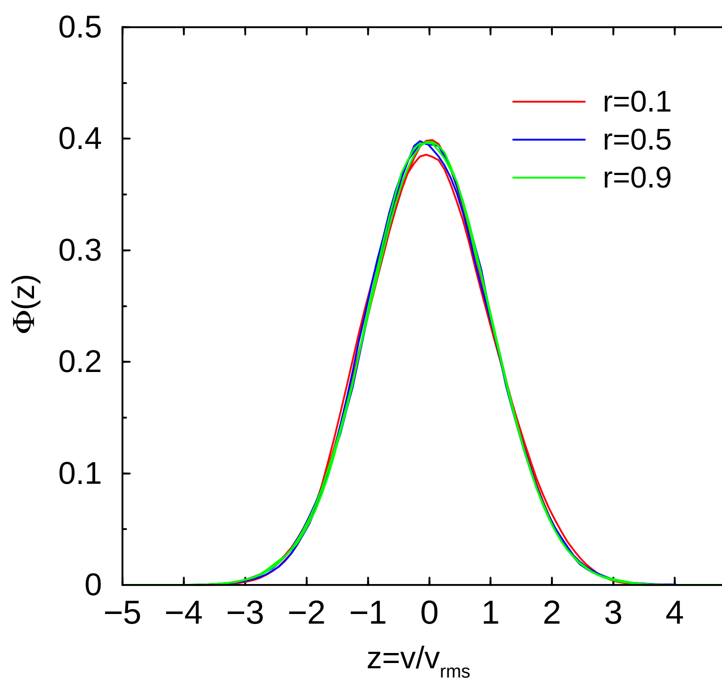
$\boxed{\textbf{Collapse} \equiv \textbf{finite time singularity in } v_t + v v_x = 0}$

Two Dimensions

- Dilute limit $\nu \to 0$ ($\nu = 0.07$)
- Simulations: $N = 10^6$, $\delta = 10^{-5}$
- Universal temperature, velocity distribution

$$T(t) \sim \begin{cases} \epsilon^{-2} t^{-2} & \epsilon^{-1} \ll t \ll \epsilon^{-2}; \\ t^{-1} & \epsilon^{-2} \ll t \ll . \end{cases}$$

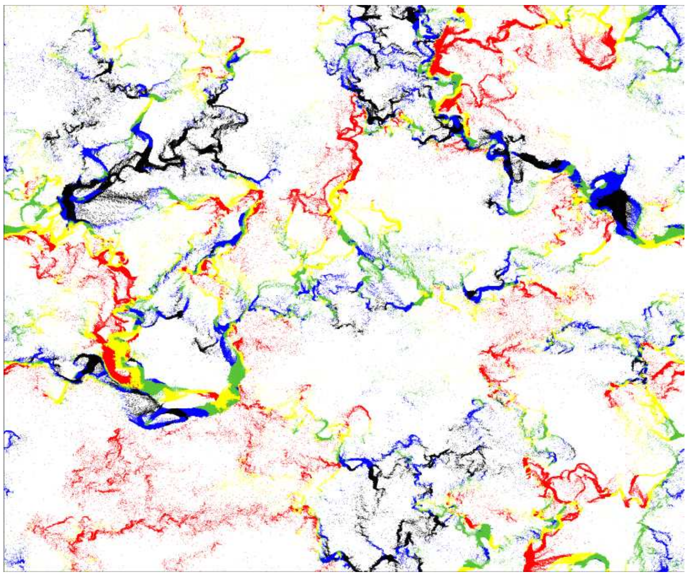
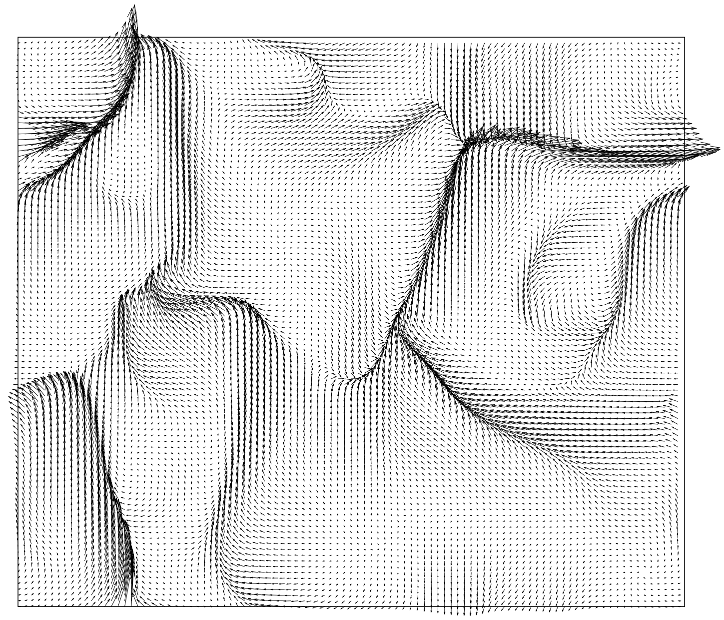




- Gaussian tail $\beta = 1/2 \Rightarrow \gamma = 2$

Preliminary evidence: $r = 0$ remains fixed point

Relation to Burgers equation

- Elongated clusters — Goldhirsch 93
- Well defined local velocity
  $\Rightarrow$ Hydrodynamic description possible

$$\frac{\Delta v}{v} \sim t^{-1/4}$$

- Heuristically: pressure term negligible

$$p \sim T \qquad \nu \sim T^{1/2}$$

Still, open questions remain

Underlying Length Scales

- Burgers equation

$$v_t + v v_x = v_{xx}$$

- Correlation length $= \xi$; Balance terms:

$$\frac{v}{t} \sim \frac{v^2}{\xi} \qquad\qquad \frac{v}{t} \sim \frac{v}{\xi^2}$$

- Momentum conservation $v \sim V^{-1/2} \sim \xi^{-d/2}$

$$\xi \sim \begin{cases} t^{\frac{2}{d+2}} & 0 < d \leq 2 \\ t^{1/2} & 2 \leq d \end{cases}$$

- Temperature $T \sim v^2 \sim \xi^{-d}$

$$T \sim \begin{cases} t^{-2d/(d+2)} & 0 < d \leq 2 \\ t^{-d/2} & 2 \leq d \end{cases}$$

Conclusions

Asymptotic behavior:

- **Governed by cluster-cluster coalescence**
- **Independent of restitution coefficient**
- **Described by inviscid Burgers equation**

Outlook

- **Velocity & spatial correlations**
- **Higher dimensions**

EB, Chen, Doolen, Redner, PRL 83, 4069 (1999)
Nie, EB, Chen, submitted (2002).