Enhanced Diffusion in Disordered Systems

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E. Ben-Naim and P.L. Krapivsky, Phys. Rev. Lett. **102**, 190602 (2009)

Talk, paper available from: http://cnls.lanl.gov/~ebn

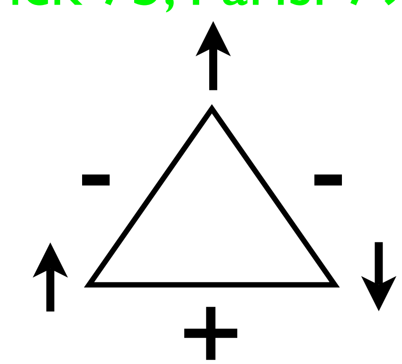
Non-local effects in pattern formation, Haifa, June 18, 2009

Plan

1. Model: diffusion of interacting particles in disordered one-dimensional system
2. Motion of non-interacting particles in disorder
3. Motion on interacting particles in disorder

Disorder

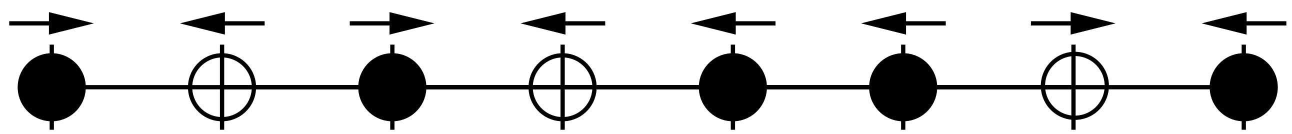
- Disorder underlies many interesting phenomena
  - Localization (Anderson 58)
  - Glassiness & slow relaxation (Sherrington & Kirkpatrick 75, Parisi 79)
  - Frustration (Ramirez 94)



- Influence of disorder:
  - Well understood for non-interacting particles
  - Open question for interacting particles (lee & ramakrishnan 85)
- De-localization of two interacting particles (Shepelyansky 93)

Interplay between disorder and particle interaction

Model System



- Infinite one-dimensional lattice
- Identical particles with concentration c
- Dynamics: particles move left and right with two rules:

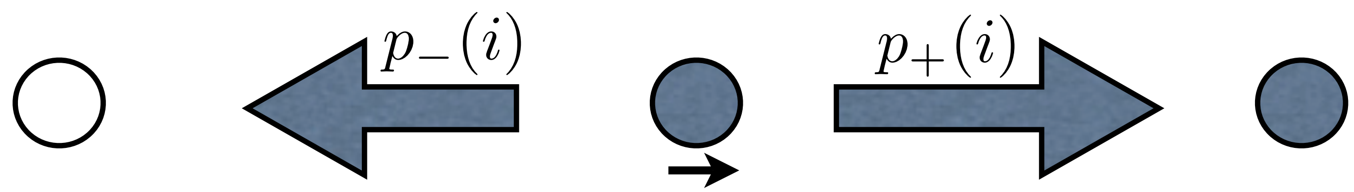
(i) **Disorder**: random, uncorrelated bias at each site

$$p_+ = \begin{cases} \frac{1}{2} + \epsilon & \text{with probability} = \frac{1}{2} \\ \frac{1}{2} - \epsilon & \text{with probability} = \frac{1}{2} \end{cases}$$

(ii) **Interaction**: via exclusion, one particle per site

Minimal model with disorder and interaction

Particle Dynamics



- Pick a particle out of N randomly
- Say particle is located at site i.

(i) Disorder: site dependent, governs motion

- ➡ With probability $p_+(i)$ move to the right by one site
- ➡ With probability $p\_(i)=1-p_+(i)$ move to the left one site

(ii)Interaction: via exclusion

- ➡ Accept the move if new site is vacant
- ➡ Reject the move if new site is occupied

- Augment time by 1/N

Monte Carlo Simulation Procedure

Parameters

- Two parameters: concentration c, disorder strength ϵ
- Generalizes two "seminal" diffusion processes:

    1. Sinai Diffusion: no interaction, $c \to 0$ Sinai 82

    2. Single-File Diffusion: no disorder, $\epsilon \to 0$ Levitt 73

(i) Disorder is small

$$\epsilon \ll 1$$

(ii) Concentration is finite

$$c = \frac{1}{2}$$

One Question

- Displacement of a particle x
- No overall bias, average displacement vanishes

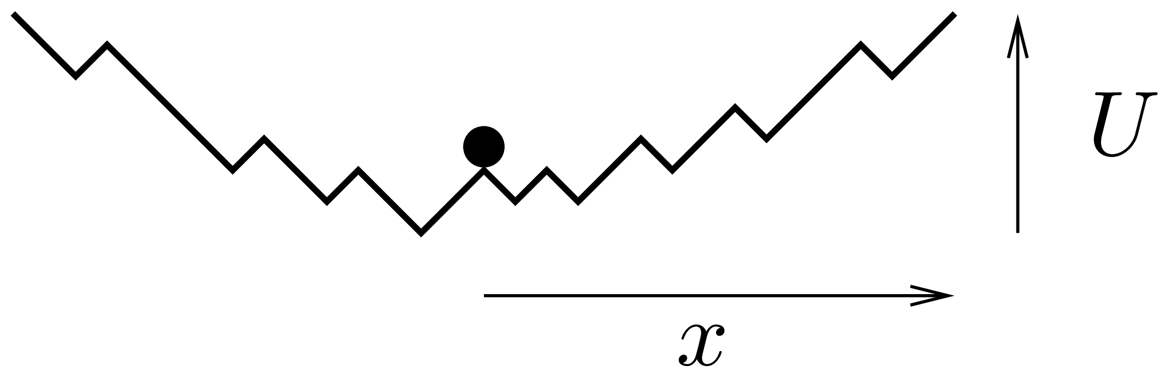
$$\langle x \rangle = 0$$

- How does the variance grow with time?

$$\sigma^2 = \langle x^2 \rangle = ?$$

2. Non-interacting Particles

Non-interacting particles



- Particle is trapped in a stochastic potential well

$$U(x) = \sum_{i=1}^{x} [p_+(i) - p_-(i)]$$

- Potential well is a random walk

$$U \sim \epsilon\sqrt{x}$$

- Escape time is exponential with depth of well

$$t \sim e^{U} \sim e^{\epsilon\sqrt{x}}$$

- Logarithmically slow displacement

$$x \sim \epsilon^{-2}(\ln t)^2$$

Sinai 83

Distribution of Displacements

- Scaled displacement

$$\xi = \frac{x}{(\ln t)^2}$$

- Distribution is exactly known

Golosov 84
Kesten 86

$$F(\xi) = \frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n \left(n+\frac{1}{2}\right)^{-1} \exp\left[-\pi^2|\xi|\left(n+\frac{1}{2}\right)^2\right]$$

- Non-gaussian statistics

$$F(\xi) \sim \exp\left[\, -\,\text{const.} \times |\xi|\,\right]$$

Early time: random walk

- Ignore biases

$$\epsilon = 0$$

- In each step

$$\begin{aligned} \langle x \rangle &= 0 \\ \langle x^2 \rangle &= 1 \end{aligned}$$

- In t steps: average and variance are additive

$$\begin{aligned} \langle x \rangle &= 0 \\ \langle x^2 \rangle &= t \end{aligned}$$

- Purely diffusive motion

$$\sigma = t^{1/2}$$

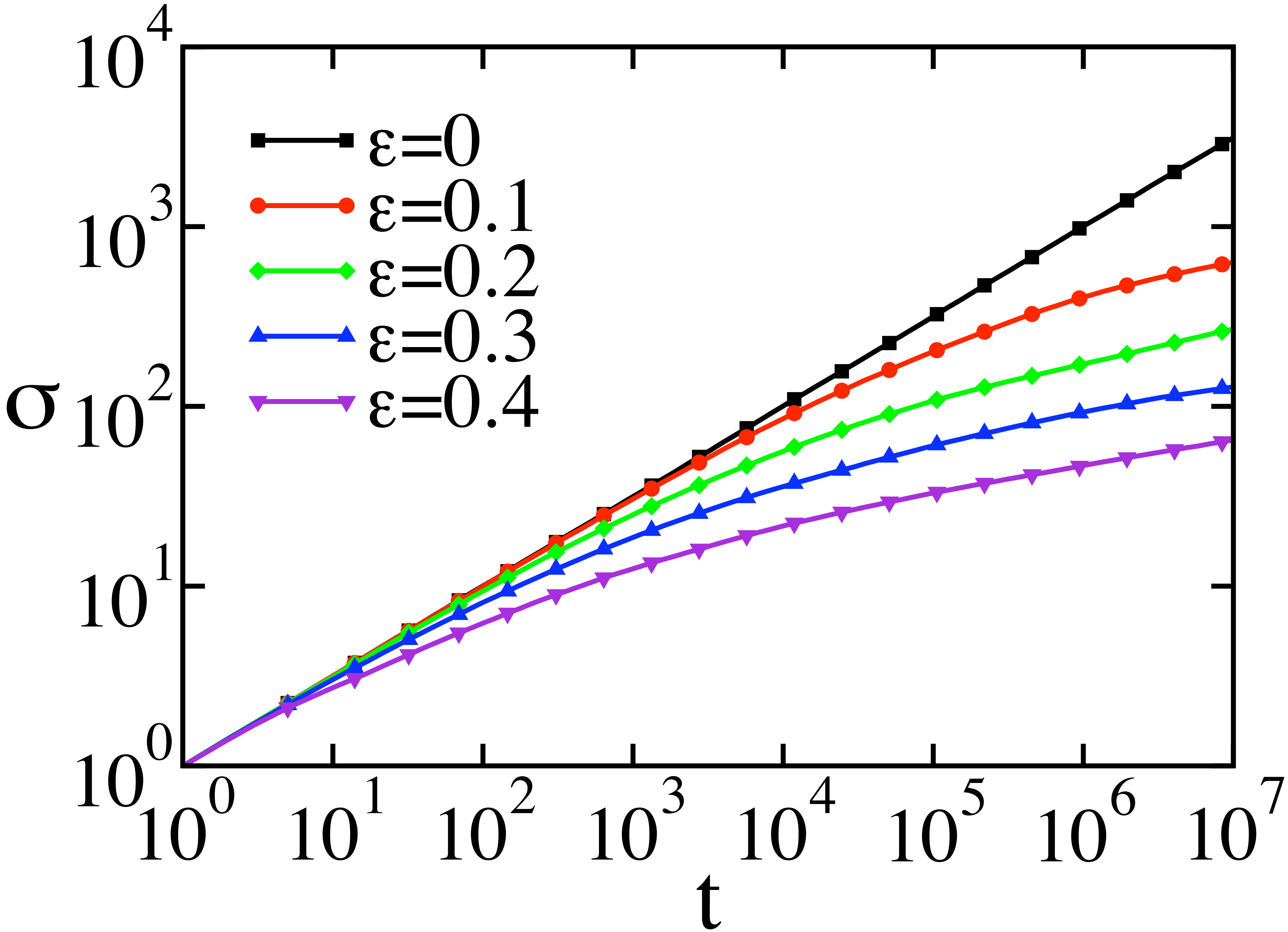
Two time regimes

- When disorder is small, there are two time regimes
- Early times: disorder is irrelevant, simple diffusion
- Late times: disorder is relevant, particle trapped
- Crossover obtained by matching two behaviors

$$\sigma \sim \begin{cases} t^{1/2} & t \ll \epsilon^{-4}, \\ \epsilon^{-2}(\ln t)^2 & t \gg \epsilon^{-4}. \end{cases}$$

Without particle interactions: disorder slows particles down

Numerical Simulations



Monotonic dependence on disorder strength: stronger disorder implies smaller displacement

3. Interacting Particles

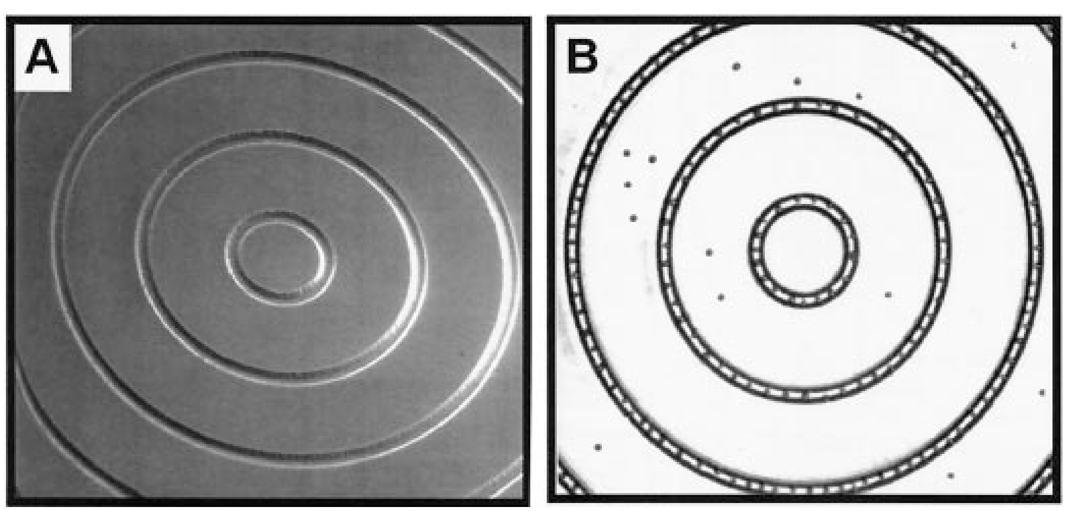
Early times

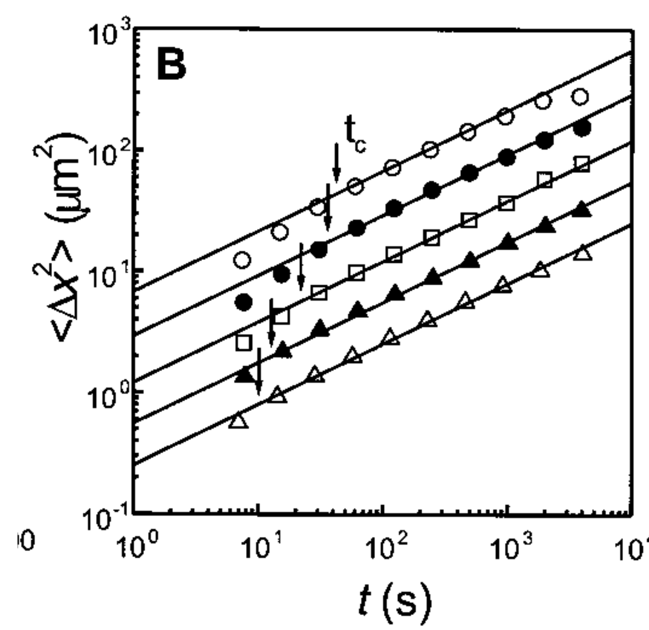
- Disorder is irrelevant, problem reduces to single file diffusion or simple exclusion process
- Particles motion is <u>sub-diffusive</u>

$$\sigma \sim t^{1/4}$$

Harris 63
Levitt 73
Alexander 78
van beijeren 83
Percus 85

- Observed in colloidal rings and biological channels

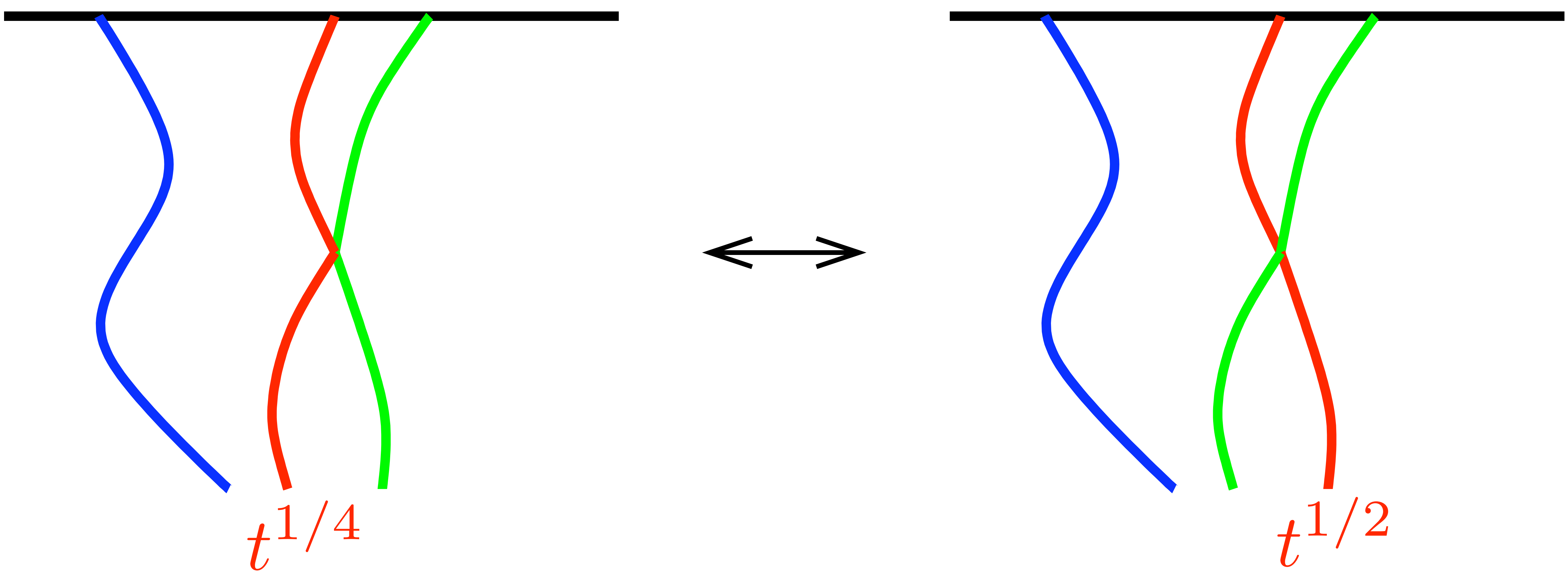




Bechinger 00
Lin 02

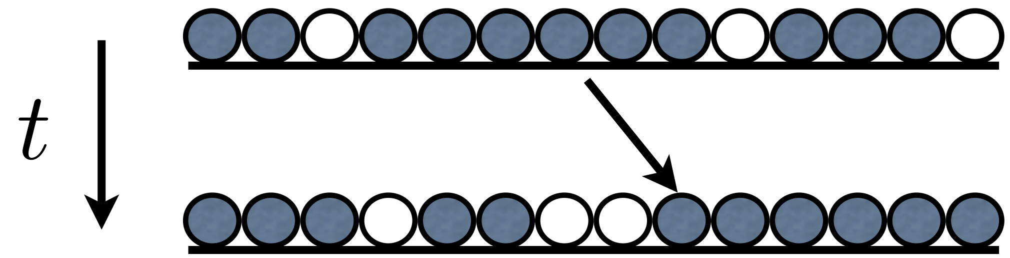
Exclusion hinders motion of particles

interacting particles noninteracting particles



Exchange identities when two particles cross!

Heuristic Derivation



$$N_+ = 2$$

$$N_- = 0$$

- Dense limit

$$c \to 1$$

- Particles move by exchanging position with vacancies

$$x = N_+ - N_-$$

- Excess vacancies

$$|N_+ - N_-| \sim N^{1/2}$$

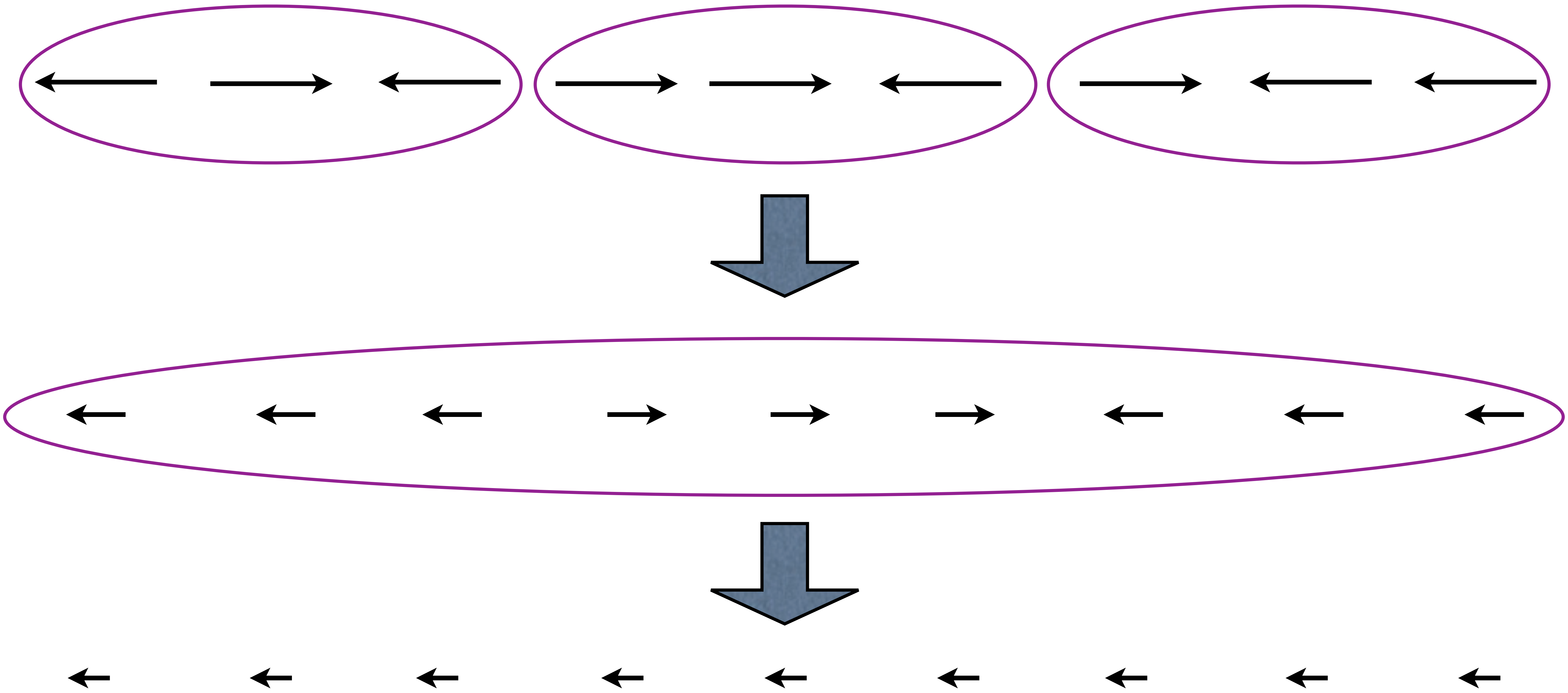
- Total number of vacancies over the diffusive length $t^{1/2}$

$$N \sim (1 - c)t^{1/2}$$

- Displacement

$$x \sim t^{1/4}$$

Random Velocity Field



Magnitude of velocity diminishes with length
Local biases lead to directed motion

bouchaud 90

Intermediate times

- Local biases exist, cause directed motion
- Particle visits $\sigma = n_+ + n_-$ distinct sites in time t
- Disorder is random, so there is a diffusive excess

$$\Delta = |n_+ - n_-| \sim \sigma^{1/2}$$

- Local drift velocity is proportional to excess

$$v \sim \epsilon\,\Delta/\sigma \quad \Rightarrow \quad v \sim \epsilon\,\sigma^{-1/2}$$

- The displacement is super-diffusive

$$\sigma \sim v\,t \sim \epsilon\,t\,\sigma^{-1/2} \quad \Rightarrow \quad \sigma \sim (\epsilon\,t)^{2/3}$$

With particle interactions:
disorder speeds particles up!

Late times

- Interaction is irrelevant, problem reduces to sinai diffusion

- The exponential escape time is dominant

- Imagine particles lines up to exit the cage

$$t \sim e^U \text{ replaced by } t \sim x e^U$$

- Particles motion remains logarithmically slow

$$\sigma \sim \epsilon^{-2} (\ln t)^2$$

Ultimate asymptotic behavior: particles motion is logarithmic slow

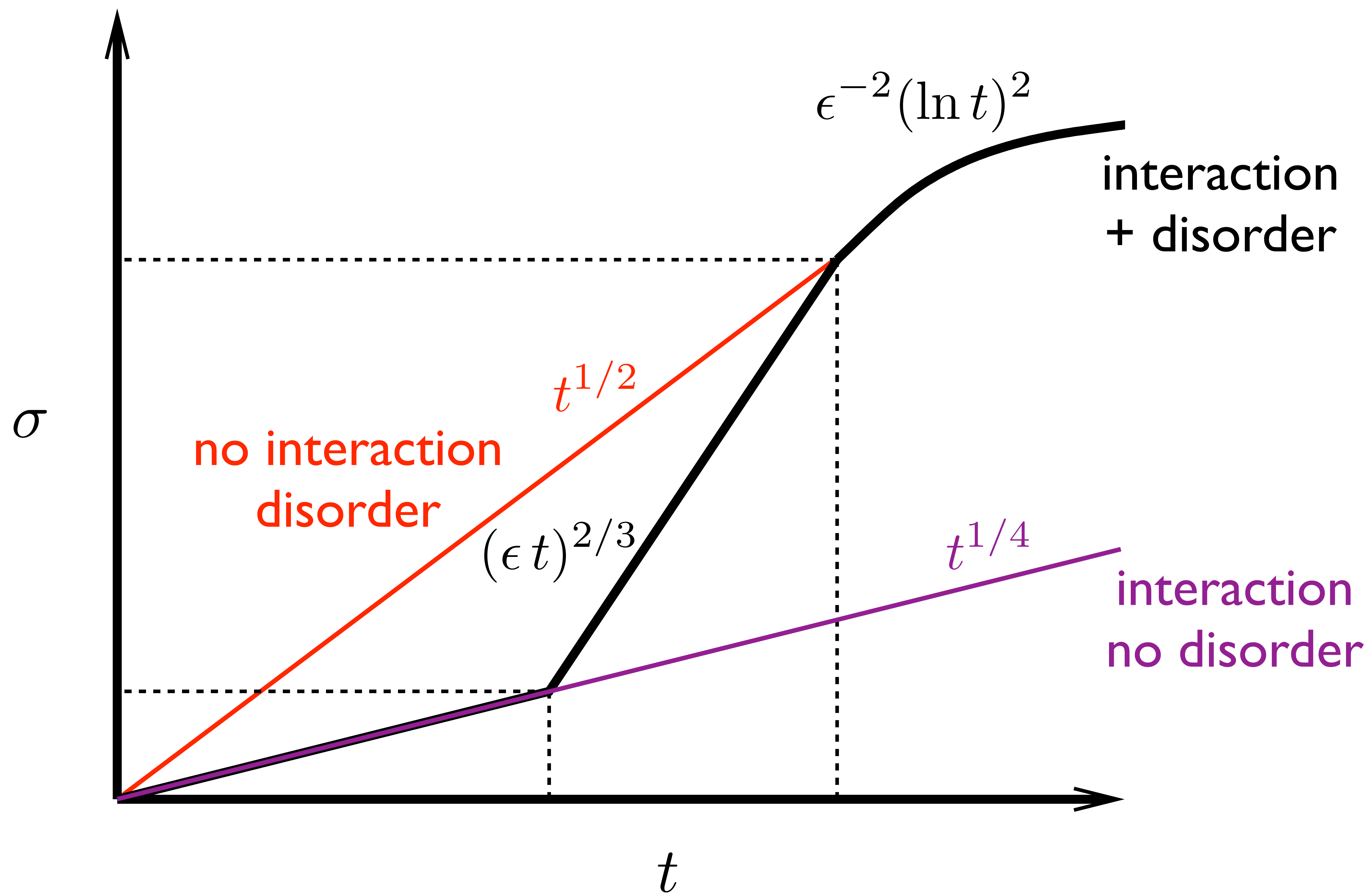
Three time regimes

- Early times: interaction is relevant, sub-diffusion
- Intermediate times: disorder & interaction both relevant, super-diffusion
- Late times: disorder relevant, caging

$$\sigma \sim \begin{cases} t^{1/4} & t \ll \epsilon^{-8/5}, \\ (\epsilon\, t)^{2/3} & \epsilon^{-8/5} \ll t \ll \epsilon^{-4}, \\ \epsilon^{-2}(\ln t)^2 & t \gg \epsilon^{-4}. \end{cases}$$

Small disorder:
mobility is enhanced over a long period

Three time regimes



Can we ignore the cage at intermediate times?

- Of course, the hopping time is of order one
- The escape time is appreciable when

$$t \sim \exp(U) \quad \Rightarrow \quad U \gg 1 \quad \Rightarrow \quad \epsilon\sqrt{x} \gg 1$$

- The cage is relevant only at late times

$$x \gg \epsilon^{-2}$$

Yes, we can (ignore the cage)!

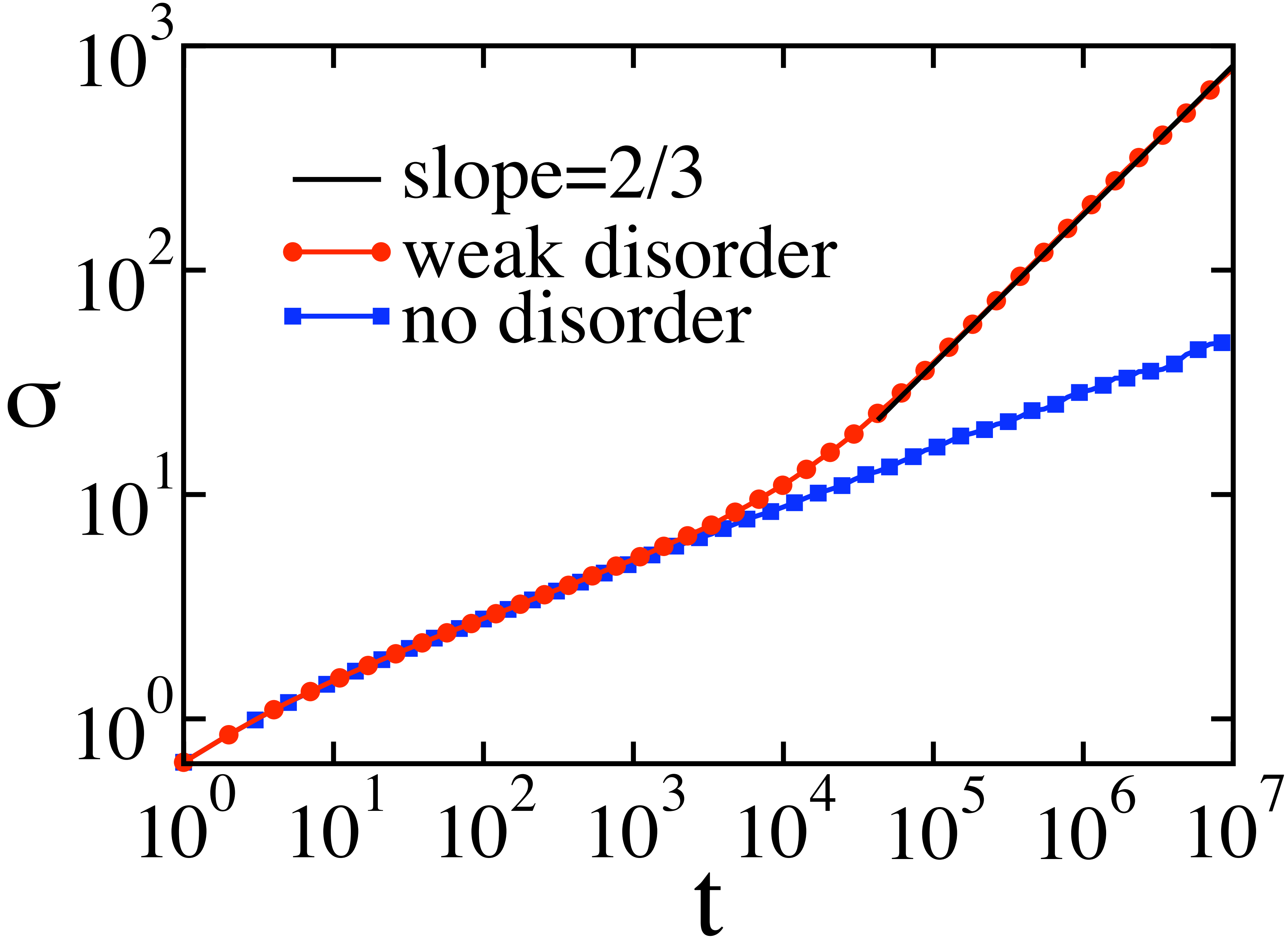
Heuristic argument does not utilize particle interactions!

- Therefore, super-diffusive transport must be relevant for non-interacting particles!
- However, the diffusive transport overwhelms the super-diffusive transport

$$t^{1/2} \gg (\epsilon\, t)^{2/3} \qquad \text{for} \qquad t \ll \epsilon^{-4}$$

Small convective correction exists without interactions, but is irrelevant

Early and intermediate time behavior for a weak disorder



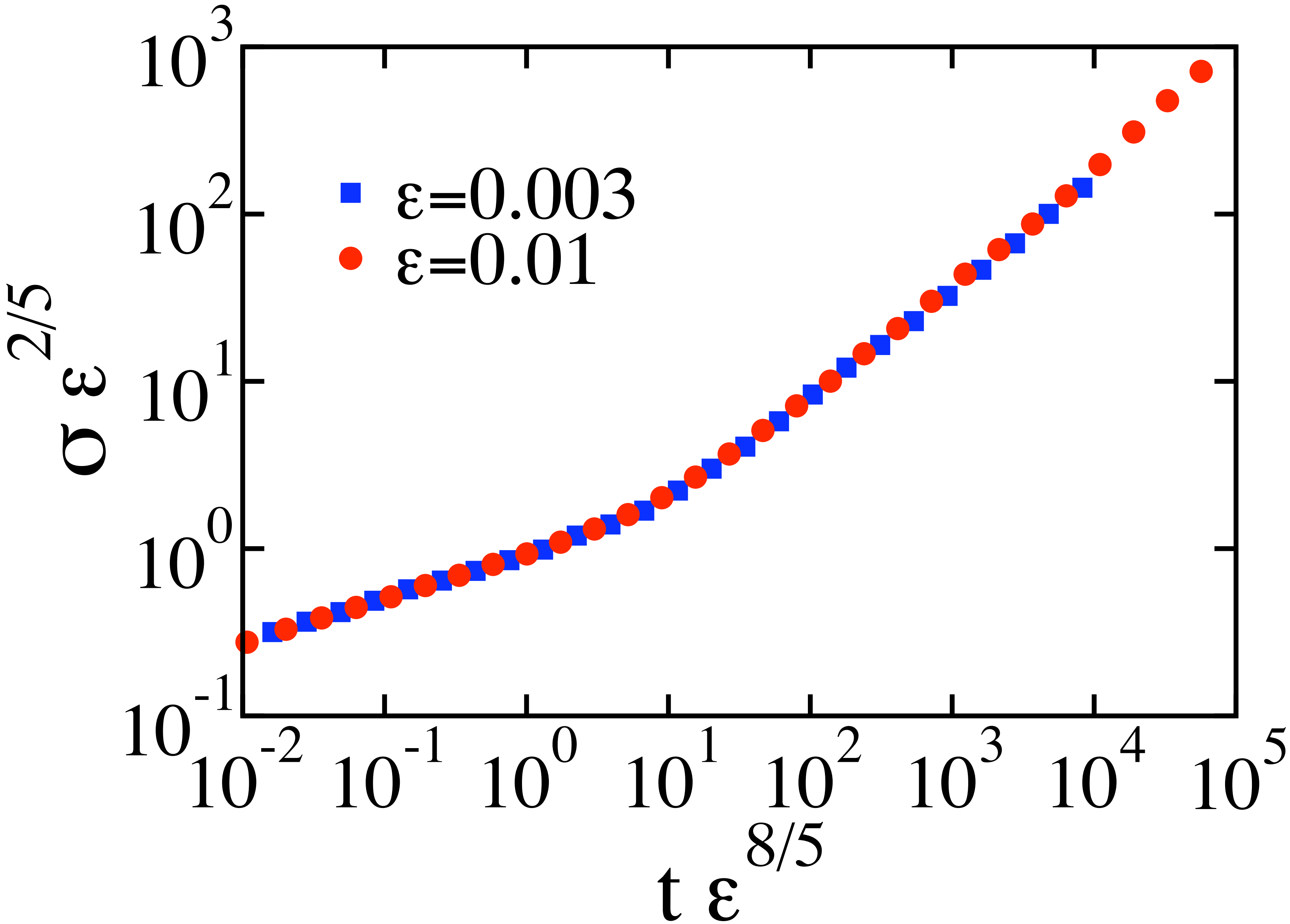
Monte Carlo

number of particles

$N = 4 \times 10^5$

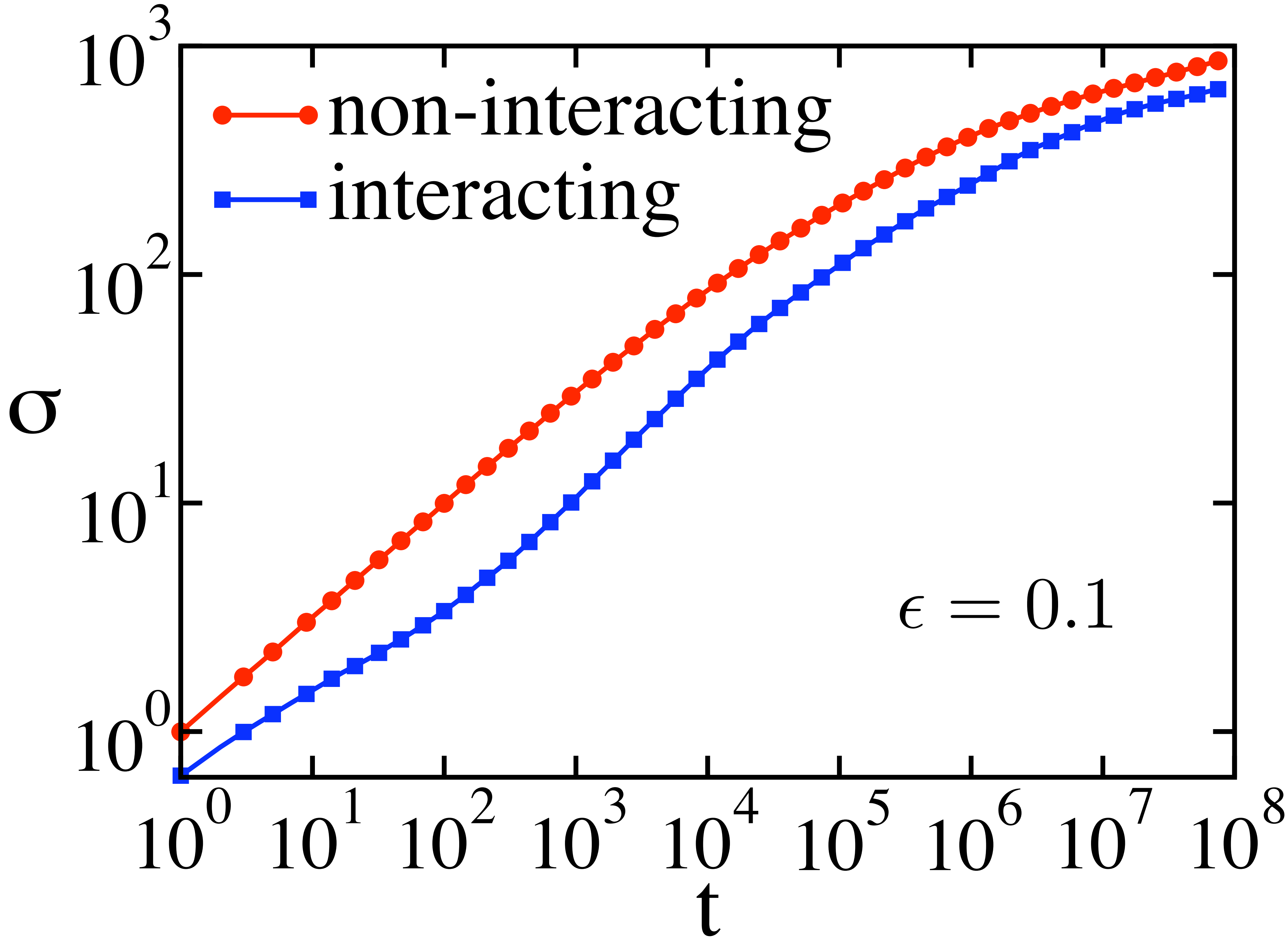
Qualitative and quantitative agreement with scaling theory

Early and intermediate time behavior for two different weak disorders



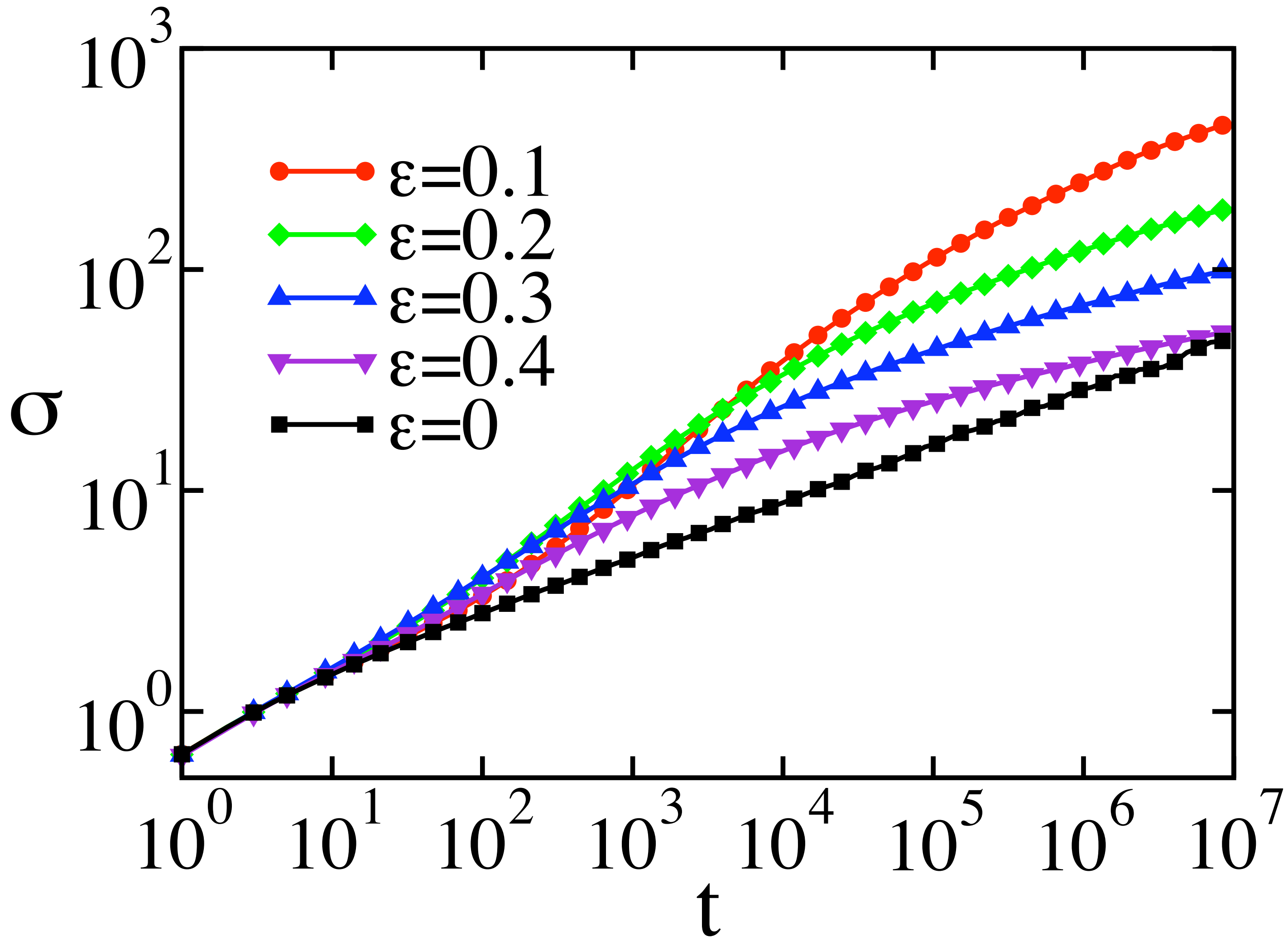
Universal scaling function for the displacement

Late time behavior for a moderate disorder



Suggests that interaction becomes irrelevant

Early and intermediate time behavior for moderate disorders



1. Mobility is enhanced at all disorder strengths
2. Displacement is not monotonic with disorder
3. Eventually, no-disorder catches up
4. But why is the crossover time so large?

Giant crossover time

- Compare ultimate asymptotic behaviors with interaction
- Without disorder

$$\sigma \sim t^{1/4}$$

- With small disorder

$$\sigma \sim \epsilon^{-2} (\ln t)^2$$

- The crossover time is astronomical

$$t \sim \epsilon^{-8}$$

In practice, small disorder generates stronger transport in an interacting particle system

Generalizations

- ✓ Different concentrations
- ✓ Disorder with variable strengths
- Synchronous dynamics = parallel updates

Qualitative behavior appears to be robust

Summary

- Without interactions: disorder slows particles down
- With interactions: disorder speeds particles up, at least for a very long time
  - Early times: sub-diffusive displacements
  - Intermediate times: super-diffusive displacement
  - Late times: logarithmically slow displacement
- Intricate interplay between interaction and disorder

Outlook

- Beyond scaling theory: a mathematical theory
- Distribution of displacements
- Different types of disorder
- Self-averaging?
- Experiments: colloids, granular biology, microfluids
- Disorder as a mechanism to control transport in matter

Formally

- Particular state of the system

$$|\psi\rangle = |\cdots 001101001 \cdots\rangle$$

- Probabilistic description

$$|\phi(t)\rangle = \sum_{\psi} P(\psi, t)|\psi\rangle$$

- Time evolution

$$\partial_t |\phi\rangle = \mathcal{L}\,|\phi\rangle$$

- Evolution operator

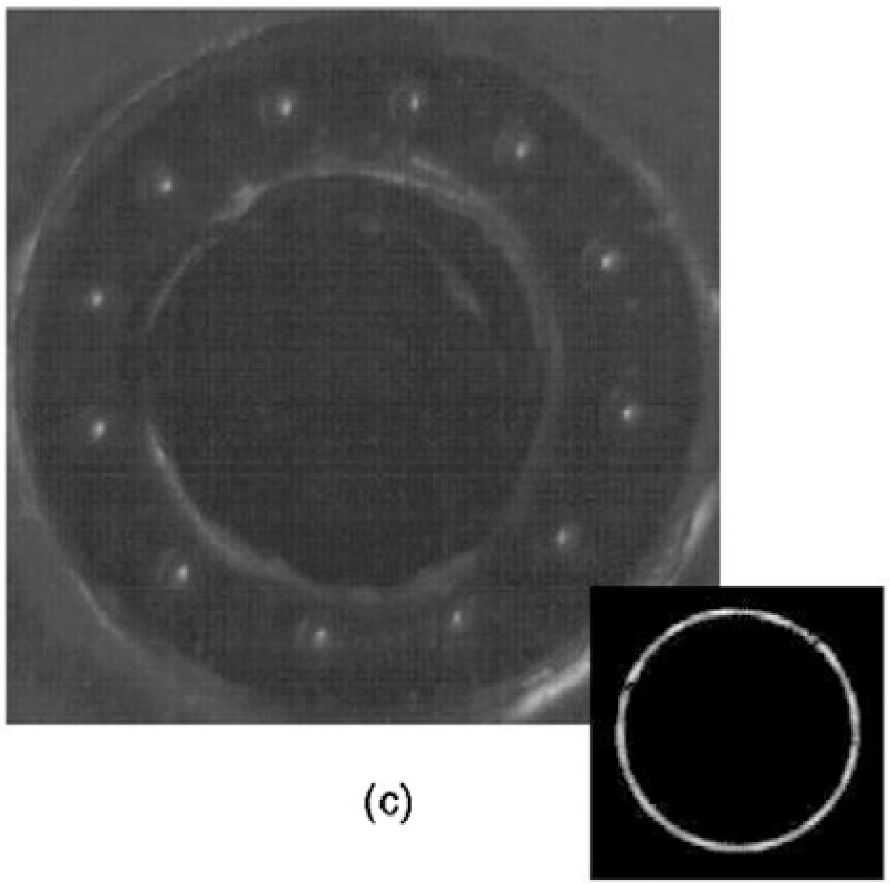
$$\mathcal{L} = \sum_i \left[l_i\, a^\dagger_{i-1} a_i + r_i\, a^\dagger_{i+1} a_i \right]$$

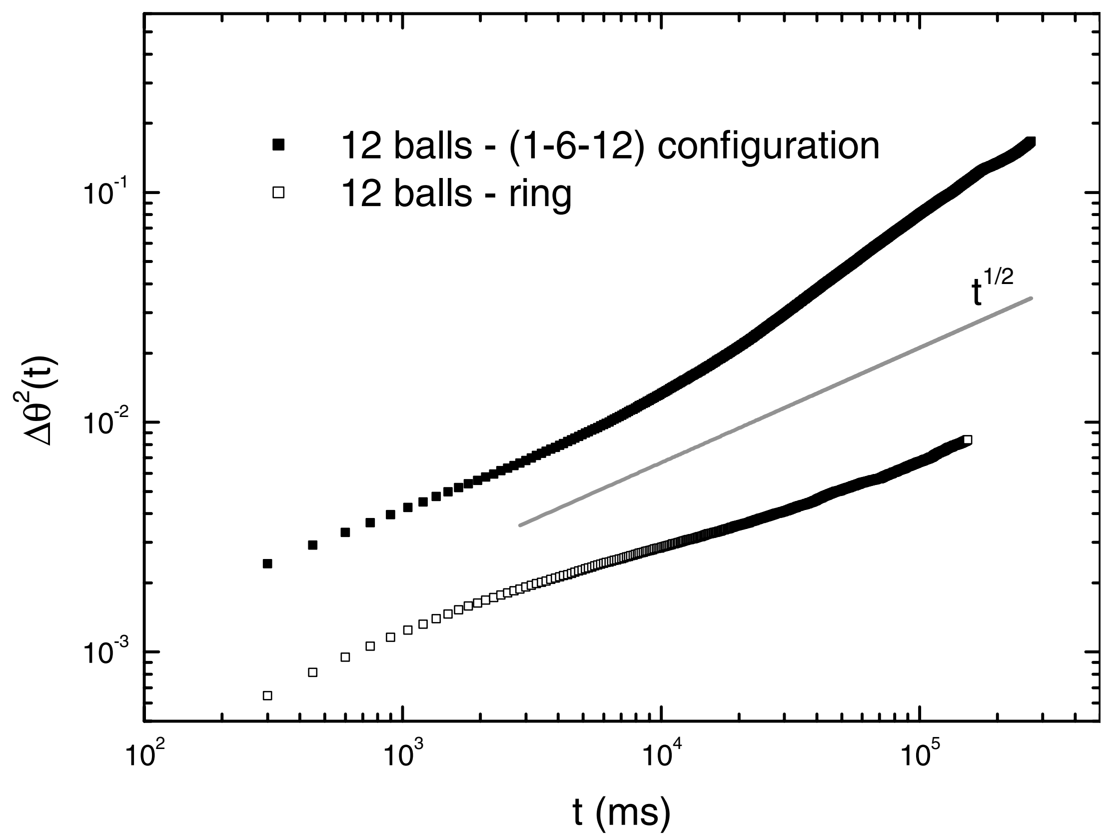
$$a_i|1\rangle = |0\rangle$$
$$a^\dagger_i|0\rangle = |1\rangle$$

- Formal solution

$$|\phi(t)\rangle = e^{\mathcal{L}t}\,|\phi(0)\rangle$$

Experiments: enhanced diffusion





$$\sigma \sim t^{\alpha} \qquad \alpha > 0.3$$

Modulated (irregular) quasi-1D colloidal channel

Coupier, Saint Jean, Guthman 07

“The laws of physics:
it’s what you can and can not do in an experiment”

Talia Ben-Naim, 6