How to choose a champion

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Talk, papers available from: http://cnls.lanl.gov/~ebn

How to choose a champion

I. Using trees
(tournament = post-season)

II. Using complete graphs
(league = regular season)

III. Using regular random graphs and complete graphs

Randomness in competitions

What is the most competitive sport?

 Soccer

 Baseball

 Hockey

 Basketball

 Football

How to quantify competitiveness?

Parity of a sports league

- Teams ranked by win-loss record
- Win percentage

$$x = \frac{\text{Number of wins}}{\text{Number of games}}$$

- Standard deviation in win-percentage

$$\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

- Cumulative distribution = Fraction of teams with winning percentage < x

$$F(x)$$

Major League Baseball
American League
2005 Season-end Standings

East	W	L	PCT
Boston	95	67	.586
New York	95	67	.586
Toronto	80	82	.494
Baltimore	74	88	.457
Tampa Bay	67	95	.414
Central	**W**	**L**	**PCT**
Chicago	99	63	.611
Cleveland	93	69	.574
Minnesota	83	79	.512
Detroit	71	91	.438
Kansas City	56	106	.346
West	**W**	**L**	**PCT**
Los Angeles	95	67	.586
Oakland	88	74	.543
Texas	79	83	.488
Seattle	69	93	.426

In baseball

$$0.400 < x < 0.600$$

$$\sigma = 0.08$$

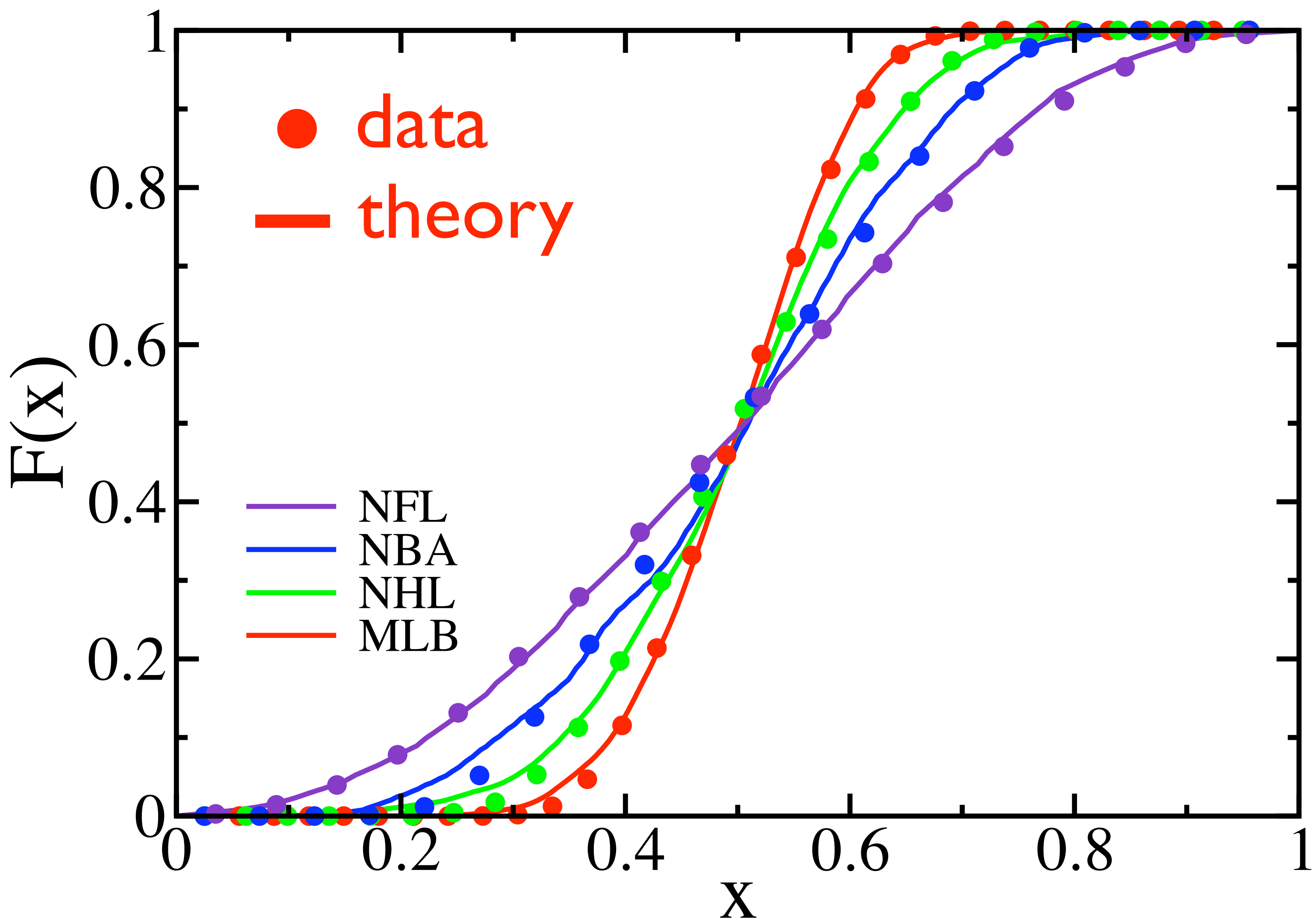
Data

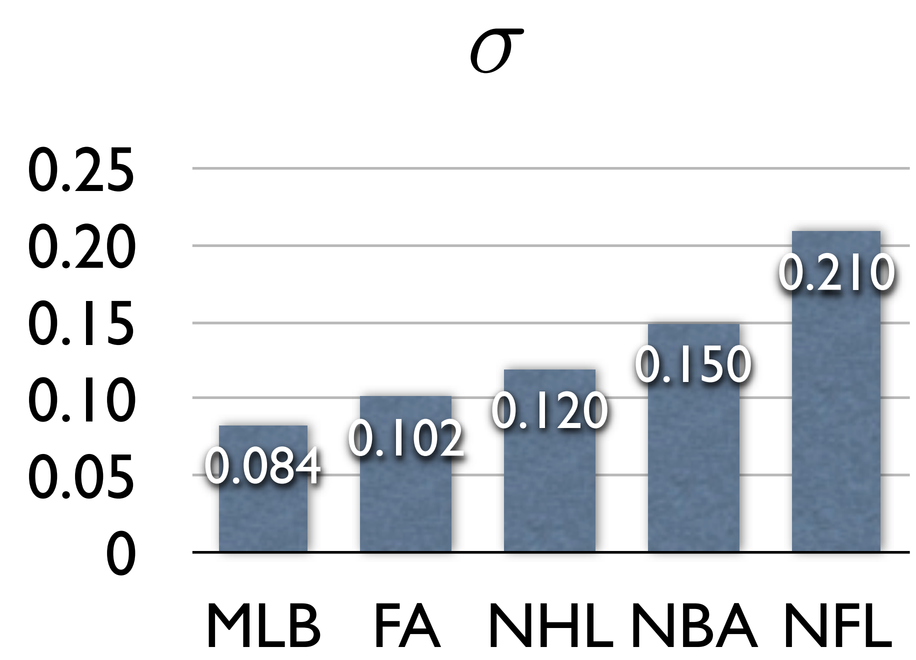
- 300,000 Regular season games (all games ever played)
- 5 Major sports leagues in United States & England

sport	league	full name	country	years	games
soccer	FA	Football Association	England	1888-2005	43,350
baseball	MLB	Major League Baseball	US	1901-2005	163,720
hockey	NHL	National Hockey League	US	1917-2005	39,563
basketball	NBA	National Basketball Association	US	1946-2005	43,254
football	NFL	National Football League	US	1922-2004	11,770

source: http://www.shrpsports.com/ http://www.the-english-football-archive.com/

Standard deviation in winning percentage





- Baseball most competitive?
- Football least competitive?

Distribution of winning percentage clearly distinguishes sports

Fort and Quirk, 1995

The competition model

- Two, randomly selected, teams play
- Outcome of game depends on team record
  - Weaker team wins with probability q<1/2 $\longrightarrow \begin{cases} q = 1/2 & \text{random} \\ q = 0 & \text{deterministic} \end{cases}$
  - Stronger team wins with probability p>1/2 $\quad p + q = 1$

$$(i,j) \to \begin{cases} (i+1,j) & \text{probability } p \\ (i,j+1) & \text{probability } 1-p \end{cases} \qquad i > j$$

  - When two equal teams play, winner picked randomly
- Initially, all teams are equal (0 wins, 0 losses)
- Teams play once per unit time $\quad \langle x \rangle = \frac{1}{2}$

Rate equation approach

- Probability distribution functions

$$g_k = \text{fraction of teams with } k \text{ wins}$$

$$G_k = \sum_{j=0}^{k-1} g_j = \text{fraction of teams with less than } k \text{ wins} \qquad H_k = 1 - G_{k+1} = \sum_{j=k+1}^{\infty} g_j$$

- Evolution of the probability distribution

$$\frac{dg_k}{dt} = (1-q)(g_{k-1}G_{k-1} - g_k G_k) + q(g_{k-1}H_{k-1} - g_k H_k) + \frac{1}{2}\left(g_{k-1}^2 - g_k^2\right)$$

better team wins — worse team wins — equal teams play

- Closed equations for the cumulative distribution

$$\frac{dG_k}{dt} = q(G_{k-1} - G_k) + (1/2 - q)\left(G_{k-1}^2 - G_k^2\right)$$

Boundary Conditions $G_0 = 0$ $\quad G_\infty = 1$ Initial Conditions $G_k(t=0) = 1$

Nonlinear Difference-Differential Equations

Scaling analysis

- Rate equation

$$\frac{dG_k}{dt} = q(G_{k-1} - G_k) + (1/2 - q)\left(G_{k-1}^2 - G_k^2\right)$$

- Treat number of wins as continuous $\quad G_{k+1} - G_k \to \frac{\partial G}{\partial k}$

Inviscid Burgers equation

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial G}{\partial t} + [q + (1-2q)G]\,\frac{\partial G}{\partial k} = 0$$

- Stationary distribution of winning percentage

$$G_k(t) \to F(x) \qquad x = \frac{k}{t}$$

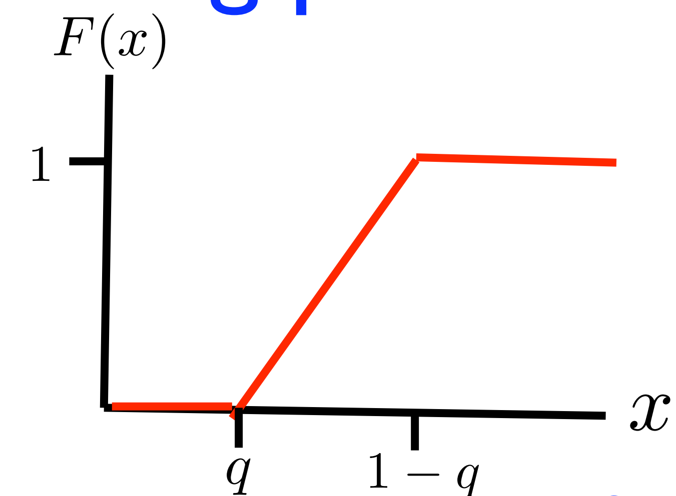
- Scaling equation

$$[(x-q) - (1-2q)F(x)]\,\frac{dF}{dx} = 0$$

Scaling solution

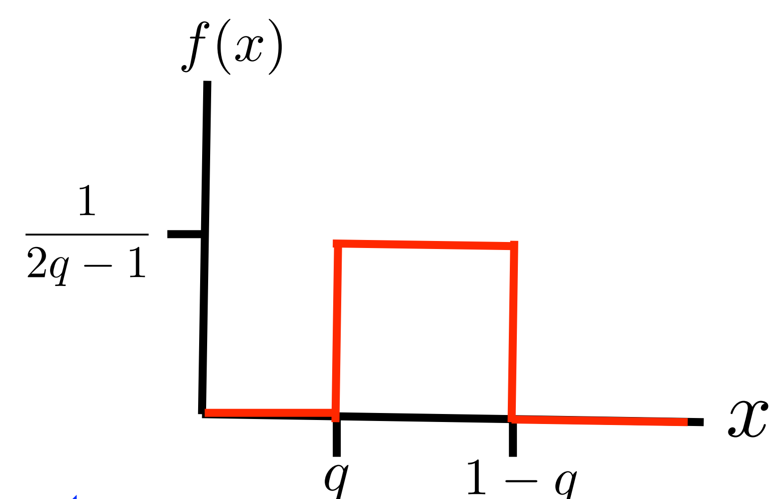
- Stationary distribution of winning percentage

$$F(x) = \begin{cases} 0 & 0 < x < q \\ \dfrac{x-q}{1-2q} & q < x < 1-q \\ 1 & 1-q < x. \end{cases}$$



- Distribution of winning percentage is uniform

$$f(x) = F'(x) = \begin{cases} 0 & 0 < x < q \\ \dfrac{1}{1-2q} & q < x < 1-q \\ 0 & 1-q < x. \end{cases}$$

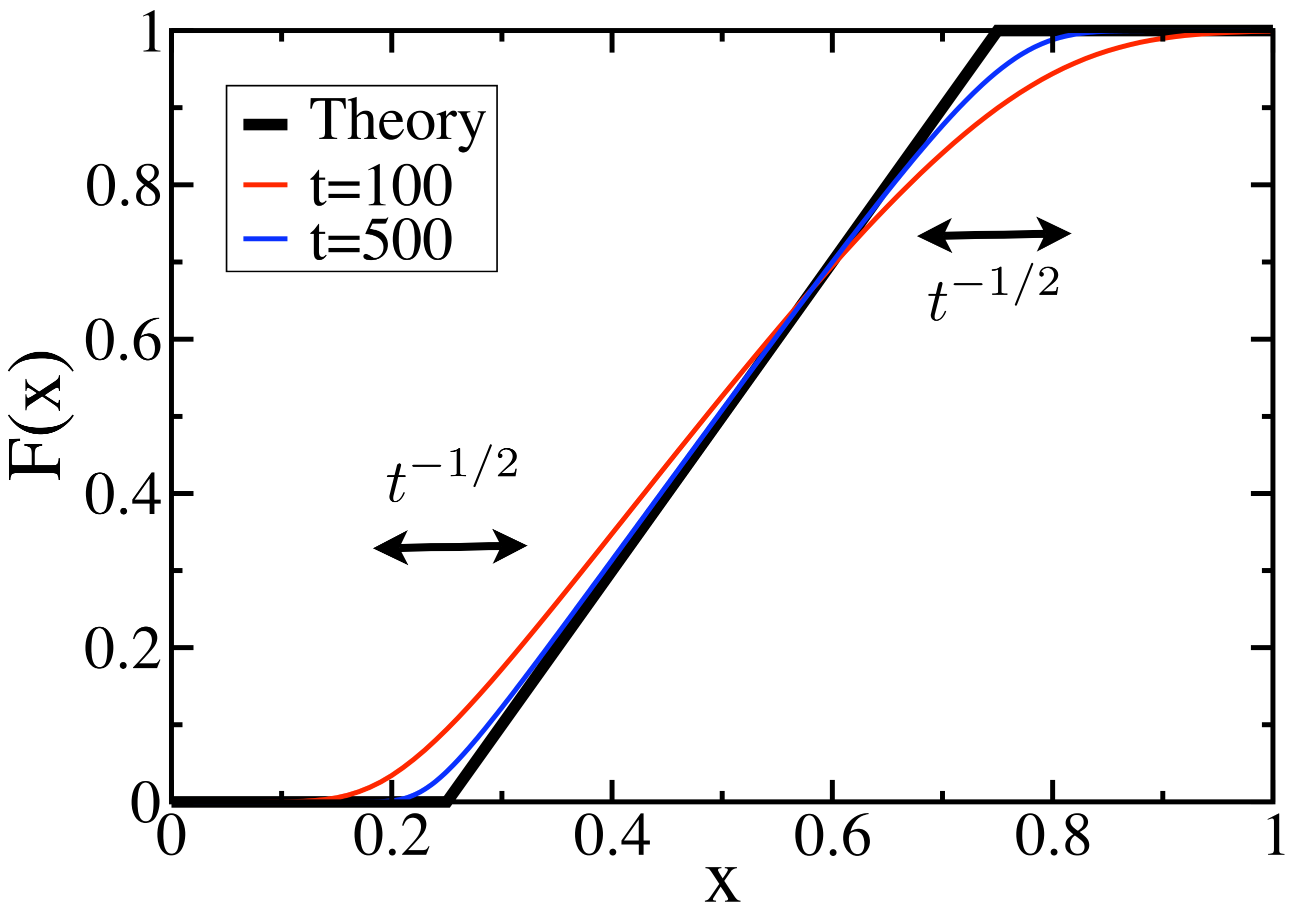


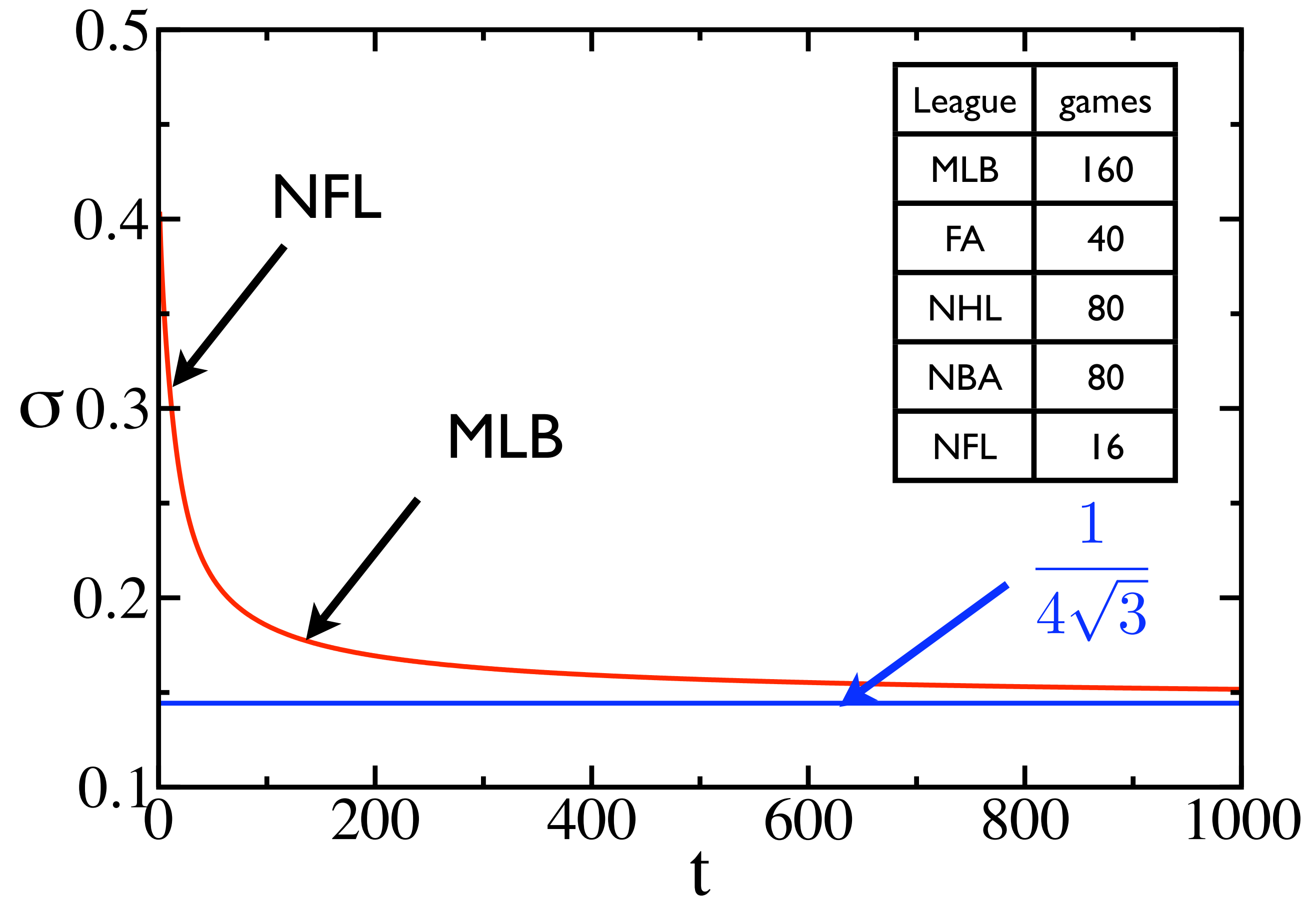
- Variance in winning percentage

$$\sigma = \frac{1/2-q}{\sqrt{3}} \quad \longrightarrow \begin{cases} q = 1/2 & \text{perfect parity} \\ q = 0 & \text{maximum disparity} \end{cases}$$

Approach to scaling

Numerical integration of the rate equations, q=1/4



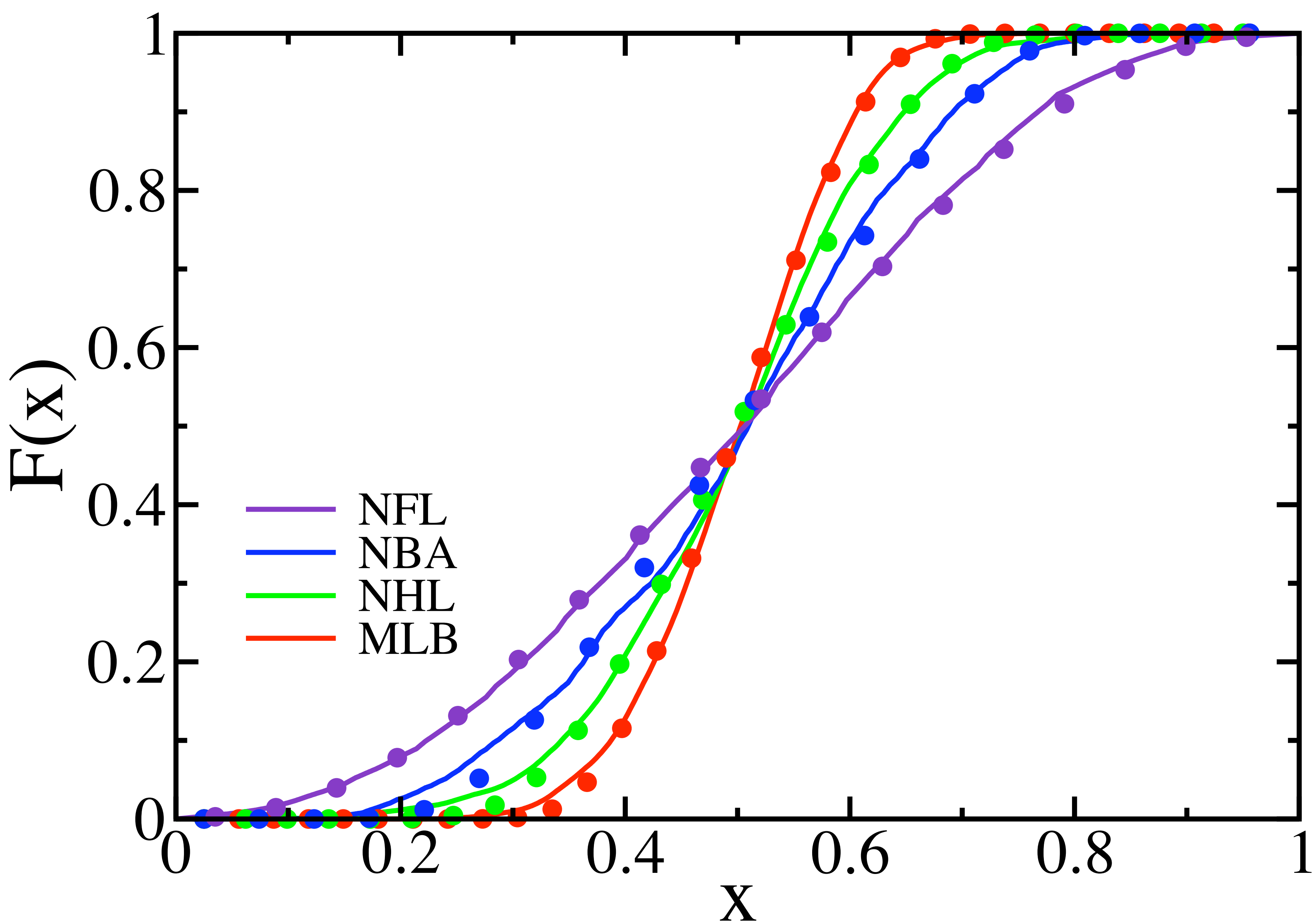


- Winning percentage distribution approaches scaling solution
- Correction to scaling is very large for realistic number of games
- Large variance may be due to small number of games

$$\sigma(t) = \frac{1/2 - q}{\sqrt{3}} + f(t)$$ Large!

Variance inadequate to characterize competitiveness!

The distribution of win percentage



- Treat q as a fitting parameter, time=number of games
- Allows to estimate q_{model} for different leagues

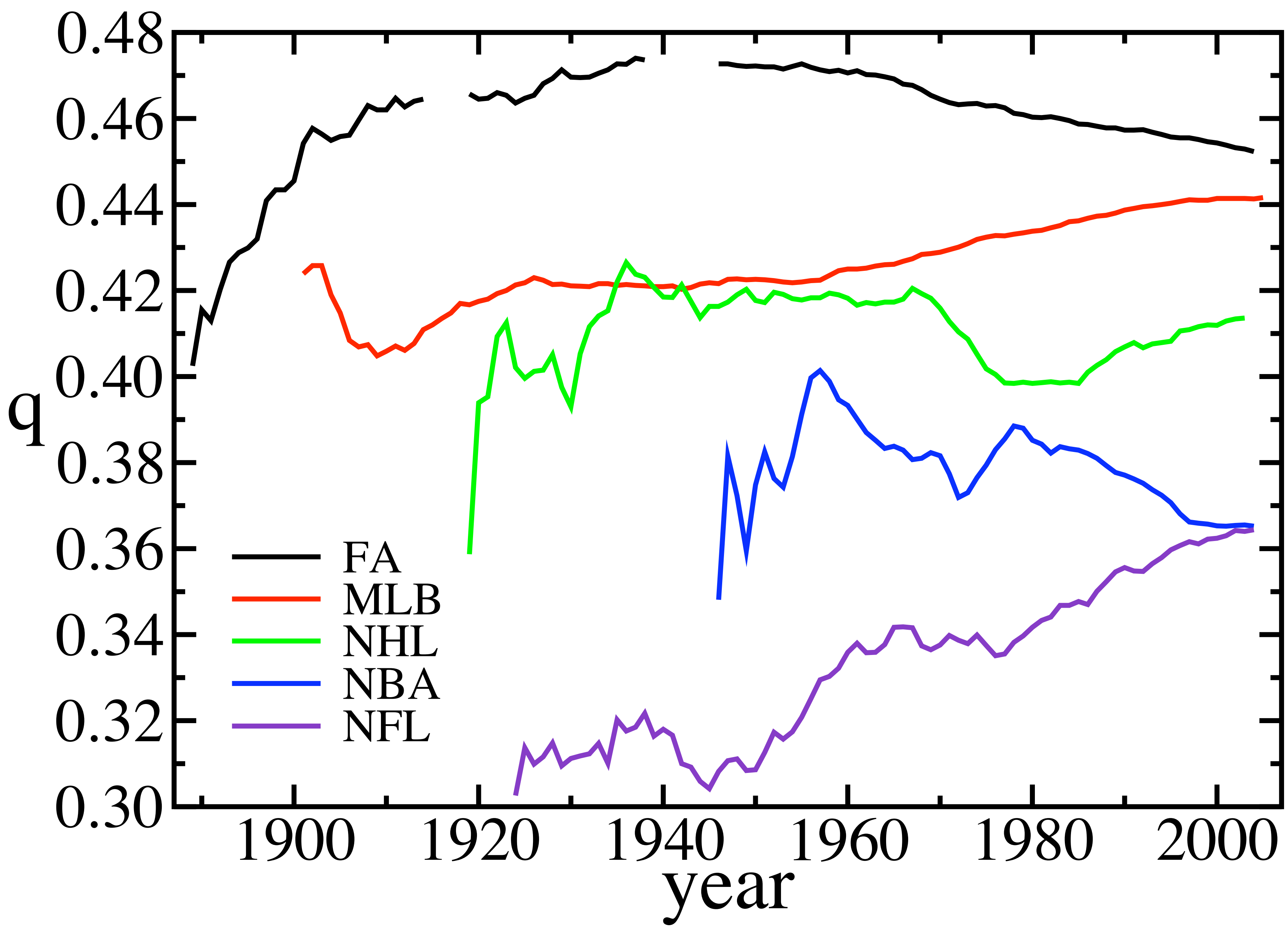
The upset frequency

- Upset frequency as a measure of predictability

$$q = \frac{\text{Number of upsets}}{\text{Number of games}}$$

- Addresses the variability in the number of games
- Measure directly from game-by-game results
  - Ties: count as 1/2 of an upset (small effect)
  - Ignore games by teams with equal records
  - Ignore games by teams with no record

The upset frequency



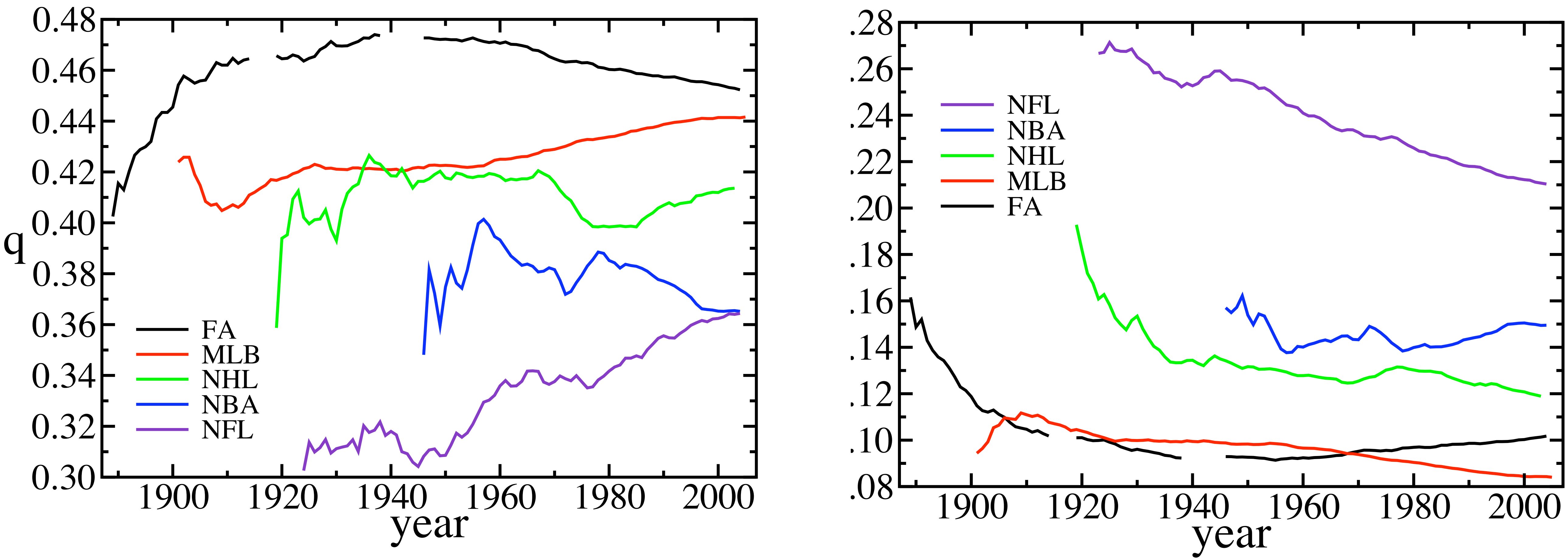
League	q	q_{model}
FA	**0.452**	0.459
MLB	**0.441**	0.413
NHL	**0.414**	0.383
NBA	**0.365**	0.316
NFL	**0.364**	0.309

q differentiates the different sport leagues!

Soccer, baseball most competitive

Basketball, football least competitive

Evolution with time



- Parity, predictability mirror each other $\sigma = \frac{1/2 - q}{\sqrt{3}}$
- Football, baseball increasing competitiveness
- Soccer decreasing competitiveness (past 60 years)

S.J. Gould, Full House, The spread of excellence from Pluto to Darwin, 1996

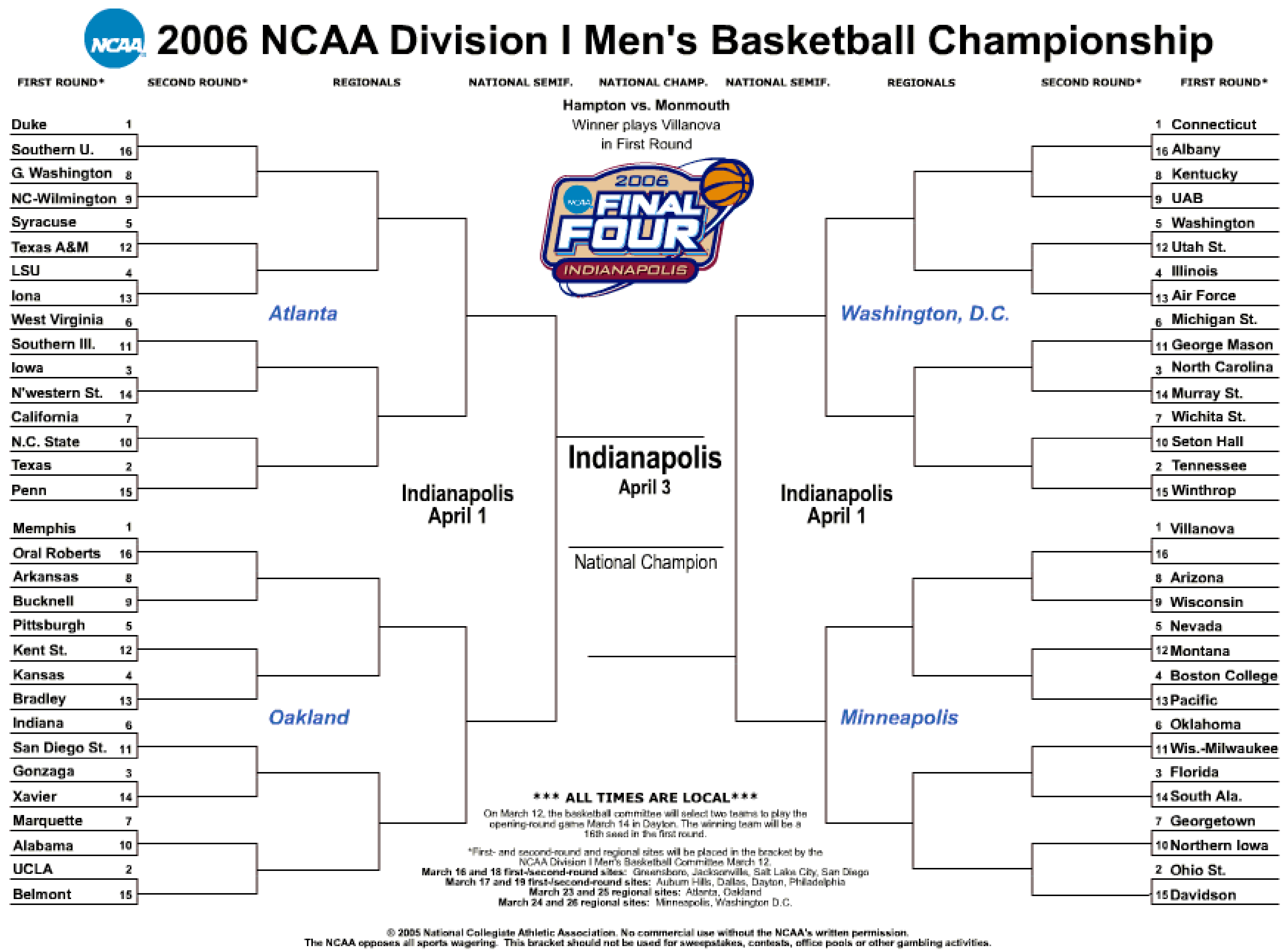
Recap

- Randomness crucial for modeling competitions
- Basic competition model incorporates upsets
- 1 parameter model
- Captures major statistical characteristics of sports leagues
- Enables quantitative theoretical analysis

1. Tournaments

(trees)

Single-elimination Tournaments



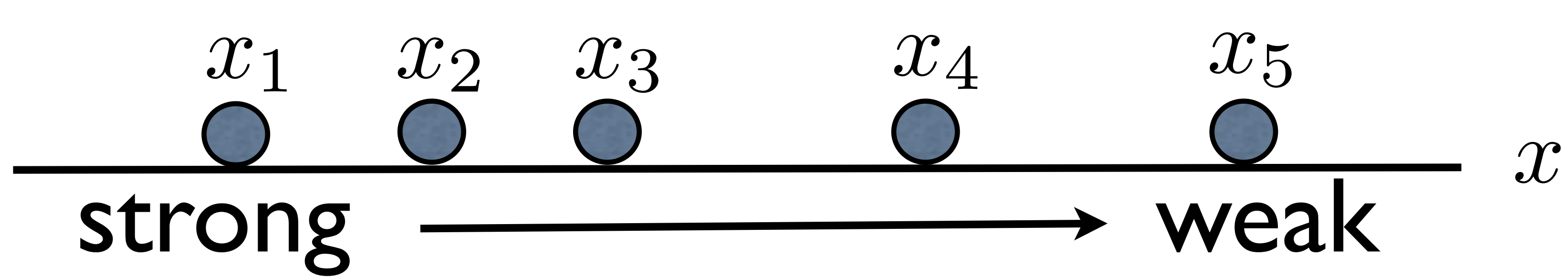
Binary Tree Structure

The competition model

- Two teams play, loser is eliminated

$$N \to N/2 \to N/4 \to \cdots \to 1$$

- Teams have inherent strength (or fitness) x



- Outcome of game depends on team strength

$$(x_1, x_2) \to \begin{cases} x_1 & \text{probability } 1-q \\ x_2 & \text{probability } q \end{cases} \qquad x_1 < x_2$$

Recursive approach

- Number of teams

$$N = 2^k = 1, 2, 4, 8, \ldots$$

- $G_N(x)$ = Cumulative probability distribution function for teams with fitness less than x to win an N-team tournament

- Closed equations for the cumulative distribution

$$G_{2N}(x) = 2p\, G_N(x) + (1-2p)\, [G_N(x)]^2$$

Nonlinear Recursion Equation

Scaling properties

1. Scale of Winner

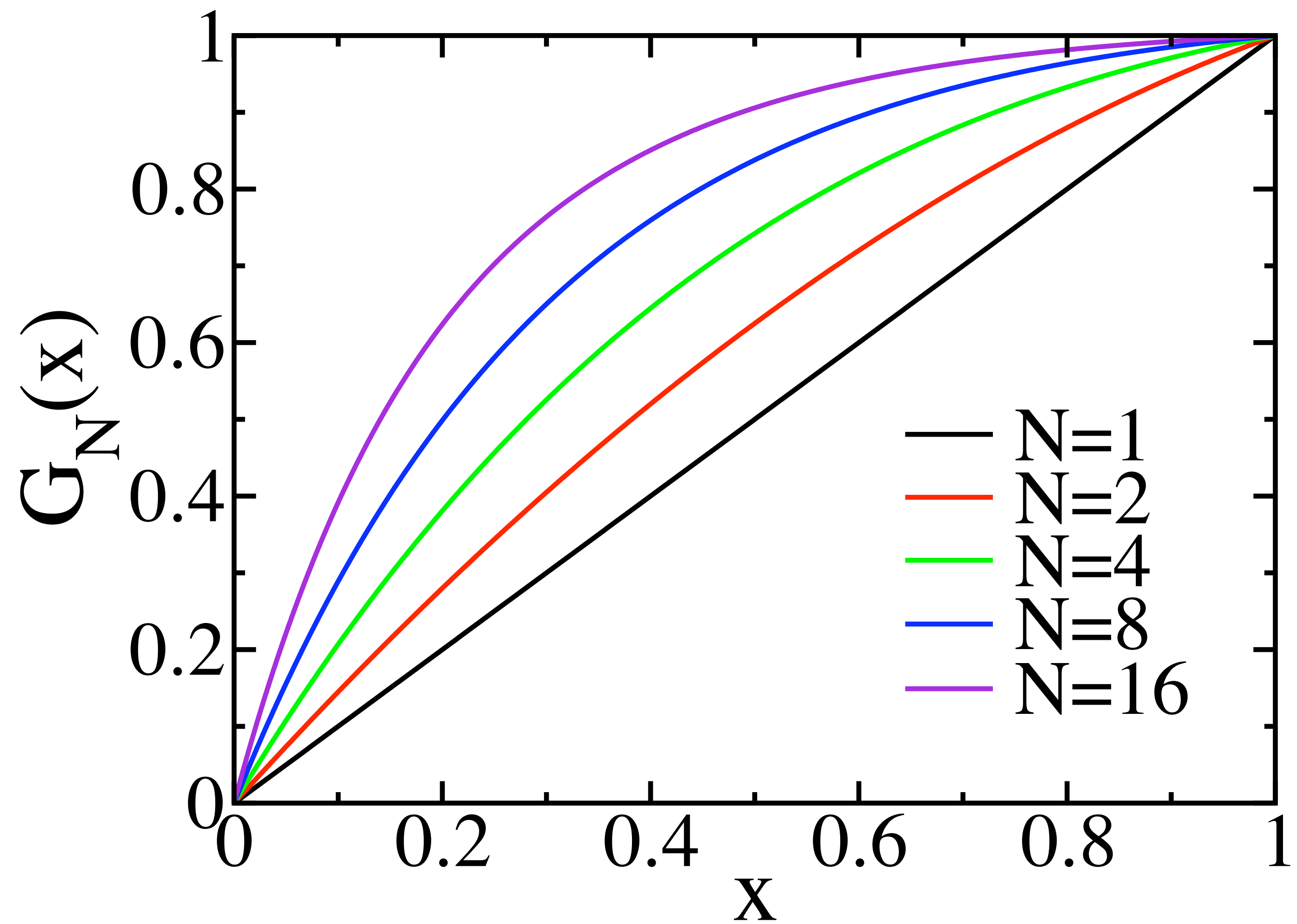
$$x_* \sim N^{-\ln 2p/\ln 2}$$

2. Scaling Function

$$G_N(x) \to \Psi\left(x/x_*\right)$$

3. Algebraic Tail

$$1 - \Psi(z) \sim z^{\ln 2p/\ln 2q}$$

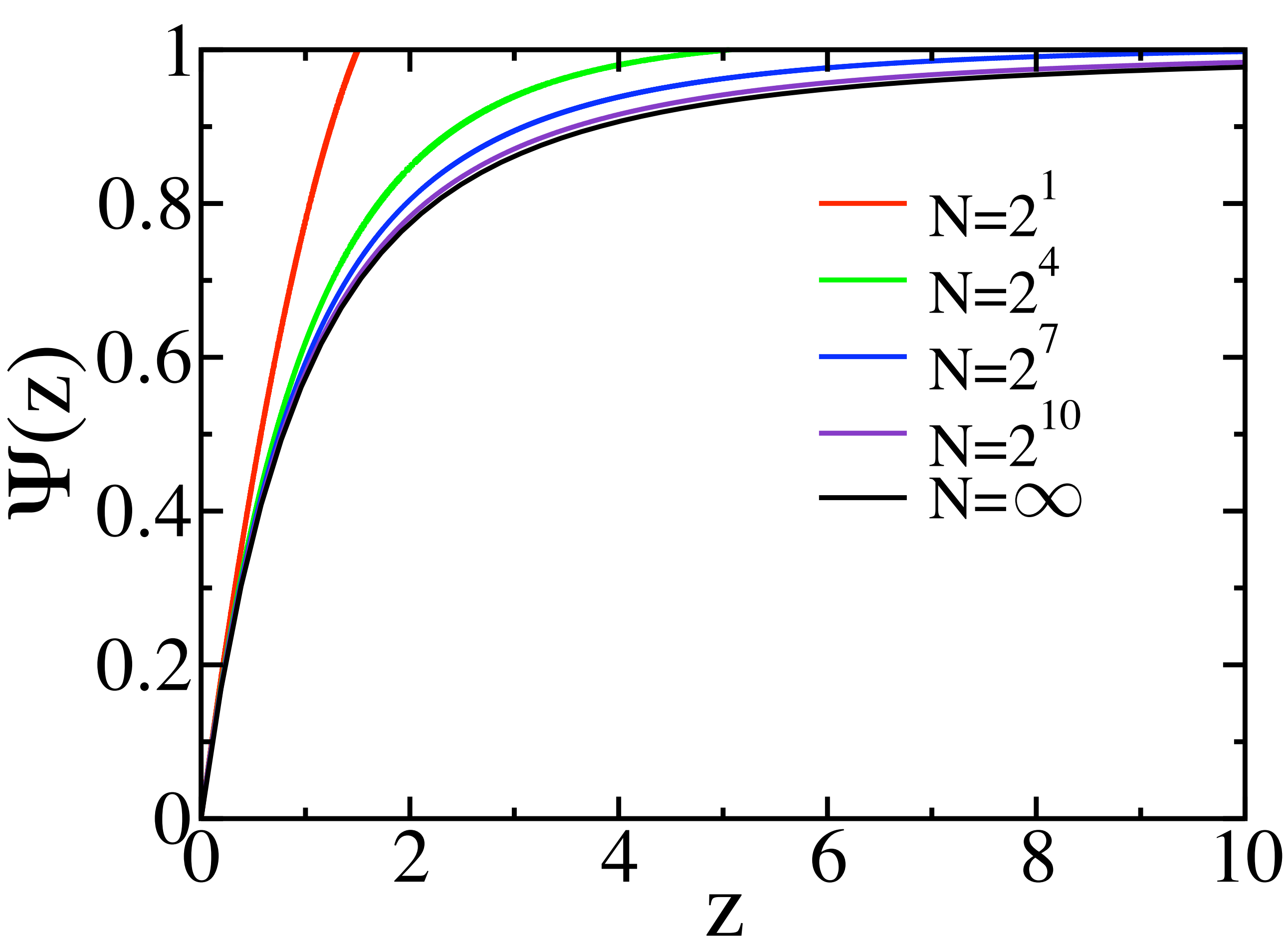


1. Large tournaments produce strong winners
3. High probability for an upset

The scaling function

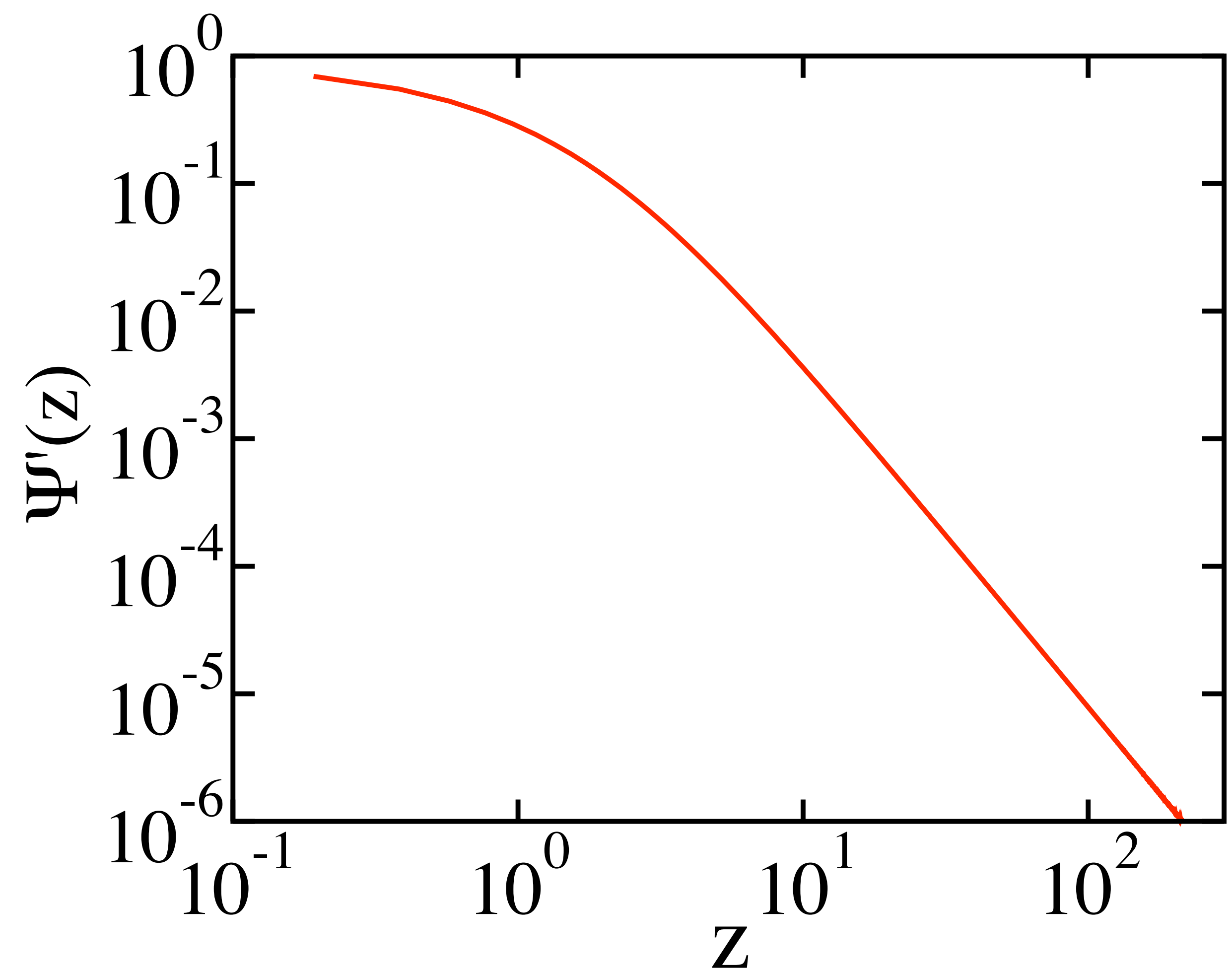
Universal shape

$$\Psi(2pz) = 2p\Psi(z) + (1-2p)\Psi^2(z)$$

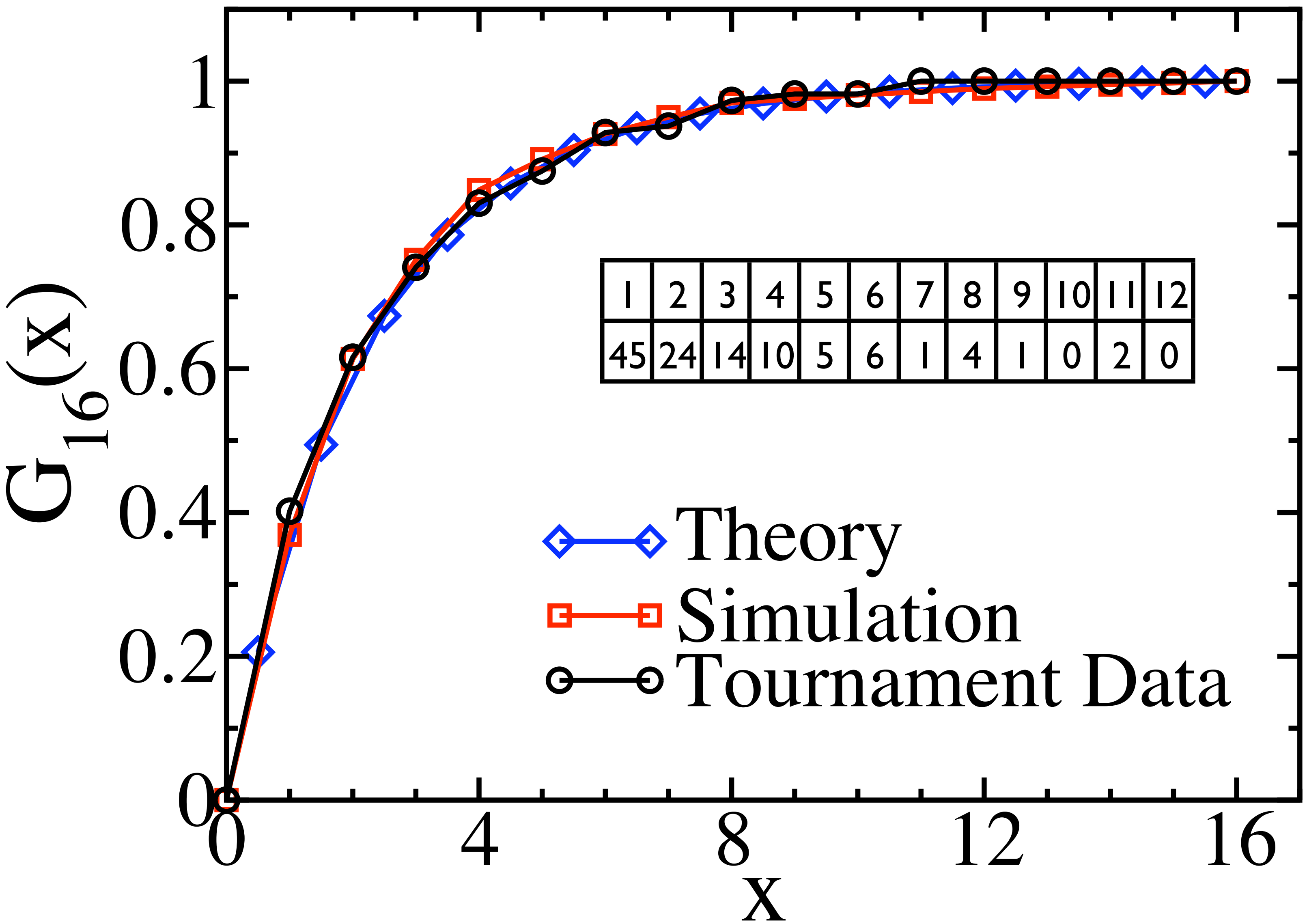


Broad tail

$$\Psi'(z) \sim z^{\ln 2p/\ln 2q - 1}$$



College Basketball



- Teams ranked 1-16
  Well defined favorite
  Well defined underdog
- 4 winners each year
- Theory: q=0.18
- Simulation: q=0.22
- Data: q=0.27
- Data: 1978-2006
- 1600 games

I. Conclusions

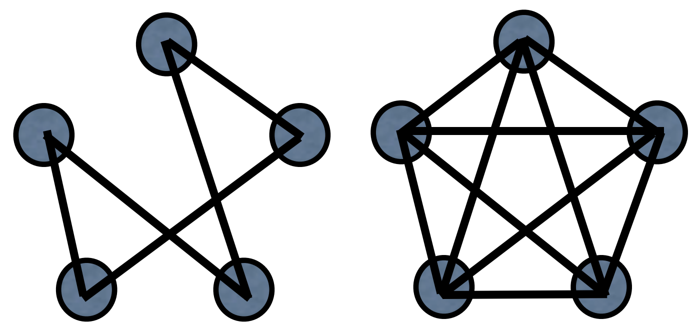
- Tournaments are efficient but not fair
- Strong teams fare better in large tournaments
- Tournaments can produce major upsets
- Distribution of winner relates parity with predictability

II. Leagues

(complete graphs)

League champions

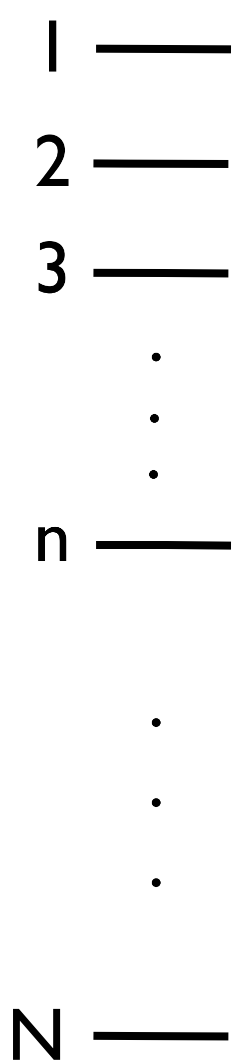
- N teams with fixed ranking
- In each game, favorite and underdog are well defined
- Favorite wins with probability $p>1/2$
  Underdog wins with probability $q<1/2$ $\quad p+q=1$
- Each team plays t games against random opponents
  - Regular random graph

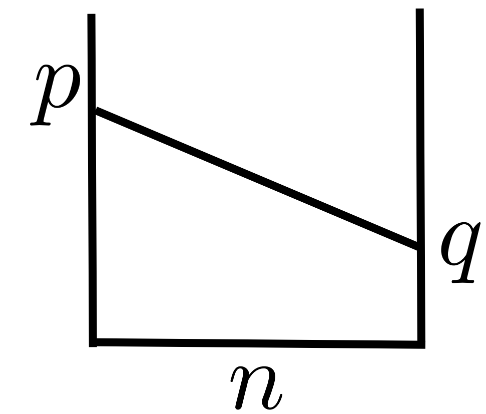


- Team with most wins is the champion

How many games are needed for best team to win?

Random walk approach





- Probability team ranked n wins a game

$$P_n = p\frac{n-1}{N-1} + q\frac{N-n}{N-1}$$

- Number of wins performs a biased random walk

$$w_n = P_n\, t \pm \sqrt{D_n\, t}$$

- Team n can finish first at early times as long as

$$(2p-1)\frac{n}{N}\, t \sim \sqrt{t}$$

- Rank of champion as function of N and t

$$n_* \sim \frac{N}{\sqrt{t}}$$

Length of season

- For best team to finish first

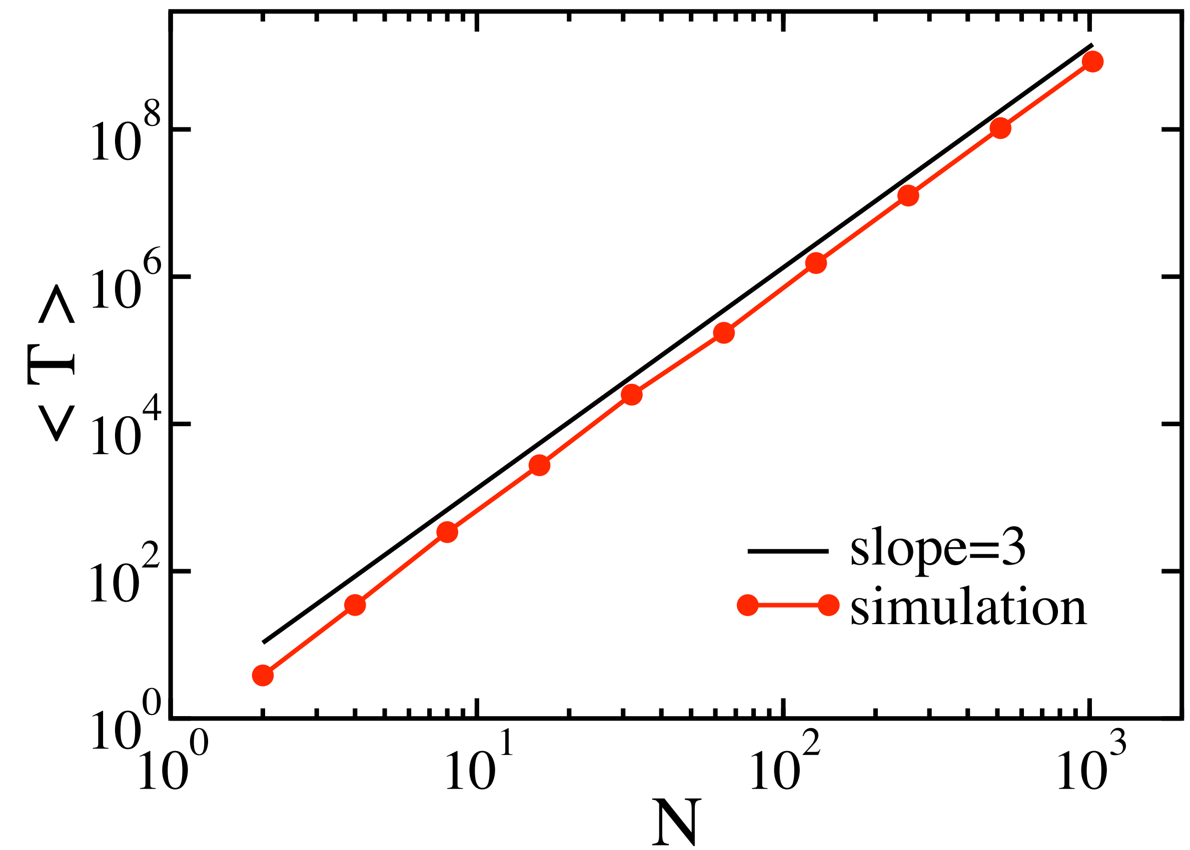
$$1 \sim \frac{N}{\sqrt{t}}$$

- Each team must play

$$t \sim N^2$$

- Total number of games

$$T \sim N^3$$



1. Normal leagues are too short
2. Normal leagues: rank of winner $\sim \sqrt{N}$
3. League champions are a transient!

Distribution of outcomes

- Scaling distribution for the rank of champion

$$Q_n(t) \sim \frac{1}{n_*}\psi\left(\frac{n}{n_*}\right) \qquad\qquad n_* \sim \frac{N}{\sqrt{t}}$$

- Probability worst team wins decays exponentially

$$Q_N(t) \sim \exp(-\mathrm{const} \times t)$$

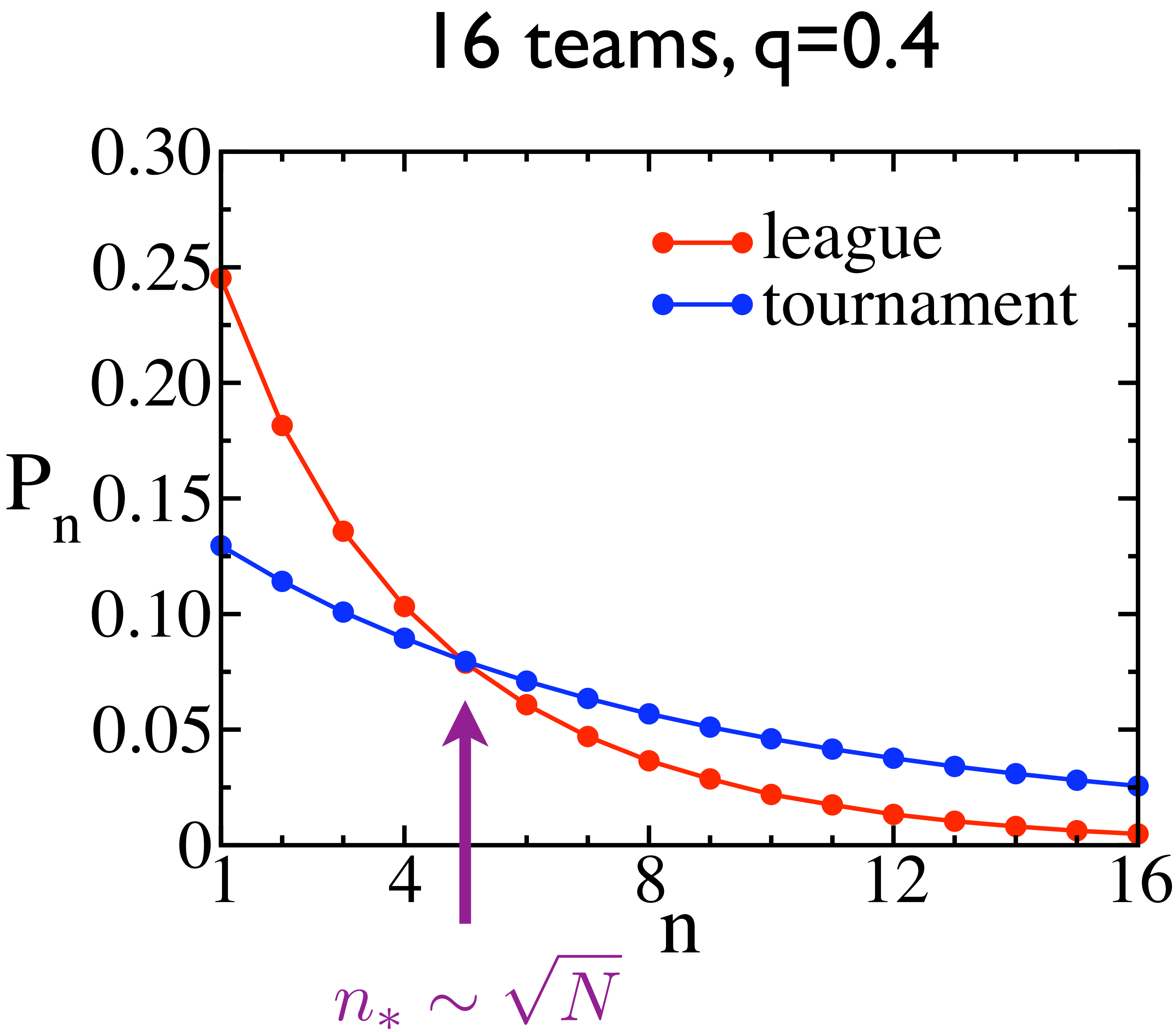
- Gaussian tail because $\psi\left(t^{1/2}\right) \sim \exp(-t)$

$$\psi(z) \sim \exp\left(-\mathrm{const} \times z^2\right)$$

- Normal league: Prob. (weakest team wins) $\sim \exp(-N)$

Leagues are fair: upset champions extremely unlikely

Leagues versus Tournaments



n	league	tournament
1	24.5	12.9
2	18.2	11.4
3	13.6	10.1
4	10.3	8.9
5	7.9	7.9
6	6.1	7.1
7	4.7	6.3
8	3.7	5.7
9	2.9	5.1
10	2.2	4.6
11	1.7	4.2
12	1.3	3.8
13	1.0	3.4
14	0.81	3.1
15	0.63	2.8
16	0.49	2.6

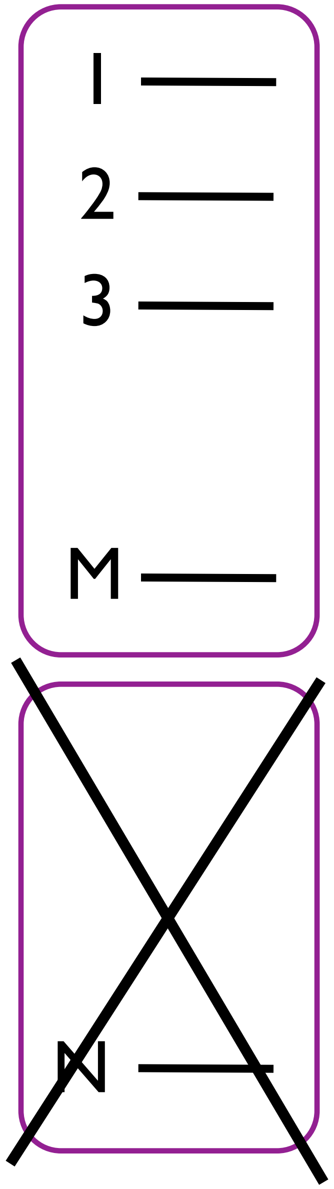
II. Conclusions

- <u>Leagues are fair but inefficient</u>
- Leagues do not produce major upsets

III. Gradual Elimination

(regular random graphs
and complete graphs)

One preliminary round



- Preliminary round
  - Teams play a small number of games $T \sim N\, t$
  - Top M teams advance to championship round $M \sim N^{\alpha}$
  - Bottom N-M teams eliminated
  - Best team must finish no worse than M place $t \sim \dfrac{N^2}{M^2}$
- Championship round: plenty of games $T \sim M^3$
- Total number of games

$$T \sim N^{3-2\alpha} + N^{3\alpha}$$

- Minimal when

$$M \sim N^{3/5} \qquad T \sim N^{9/5}$$

Two preliminary rounds

- Two stage elimination

$$N \to N^{\alpha_2} \to N^{\alpha_2 \alpha_1} \to 1$$

- Second round

$$T_2 \sim N^{3-2\alpha_2} + N^{\alpha_2(3-2\alpha_1)} + N^{3\alpha_1\alpha_2}$$

- Minimize number of games

$$3 - 2\alpha_2 = \alpha_2(3 - 2\alpha_1) \qquad \longrightarrow \qquad \alpha_2 = \frac{15}{19}$$

- Further improvement in efficiency

$$T \sim N^{27/19}$$

Multiple preliminary rounds

- Each additional round further reduces T

$$T_k \sim N^{\gamma_k} \qquad \gamma_k = \frac{1}{1-(2/3)^{k+1}}$$

- Gradual elimination

$$\gamma_k = 3, \frac{9}{5}, \frac{27}{19}, \frac{81}{65}, \cdots$$

$$N \to N^{\frac{57}{65}} \to N^{\frac{57}{65}\frac{15}{19}} \to N^{\frac{57}{65}\frac{15}{19}\frac{3}{5}} \to 1$$

- Teams play a small number of games initially

Optimal linear scaling achieved using many rounds

$$T_\infty \sim N \qquad M_\infty \sim N^{1/3}$$

optimal size of playoffs!

Preliminary elimination is very efficient!

III. Conclusions

- Gradual elimination is fair and efficient
- Preliminary rounds reduce the number of games
- In preliminary round, teams play a small number of games and almost all teams advance to next round
- Gradual elimination is fair and efficient

Publications

- How to Choose a Champion
  E. Ben-Naim, N.W. Hengartner
  Phys. Rev. E, submitted (2007)
- Scaling in Tournaments
  E. Ben-Naim, S. Redner, F. Vazquez
  Europhysics Letters **77**, 30005 (2007)
- What is the Most Competitive Sport?
  E. Ben-Naim, F. Vazquez, S. Redner
  J. Korean Phys. Soc. (2007)
- Dynamics of Multi-Player Games
  E. Ben-Naim, B. Kahng, and J.S. Kim
  J. Stat. Mech. P07001 (2006)
- On the Structure of Competitive Societies
  E. Ben-Naim, F. Vazquez, S. Redner
  Eur. Phys. Jour. B **26** 531 (2006)
- Dynamics of Social Diversity
  E. Ben-Naim and S. Redner
  J. Stat. Mech. L11002 (2005)