# Energy Cascades in Granular Gases


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talk, papers available: http://cnls.lanl.gov/~ebn

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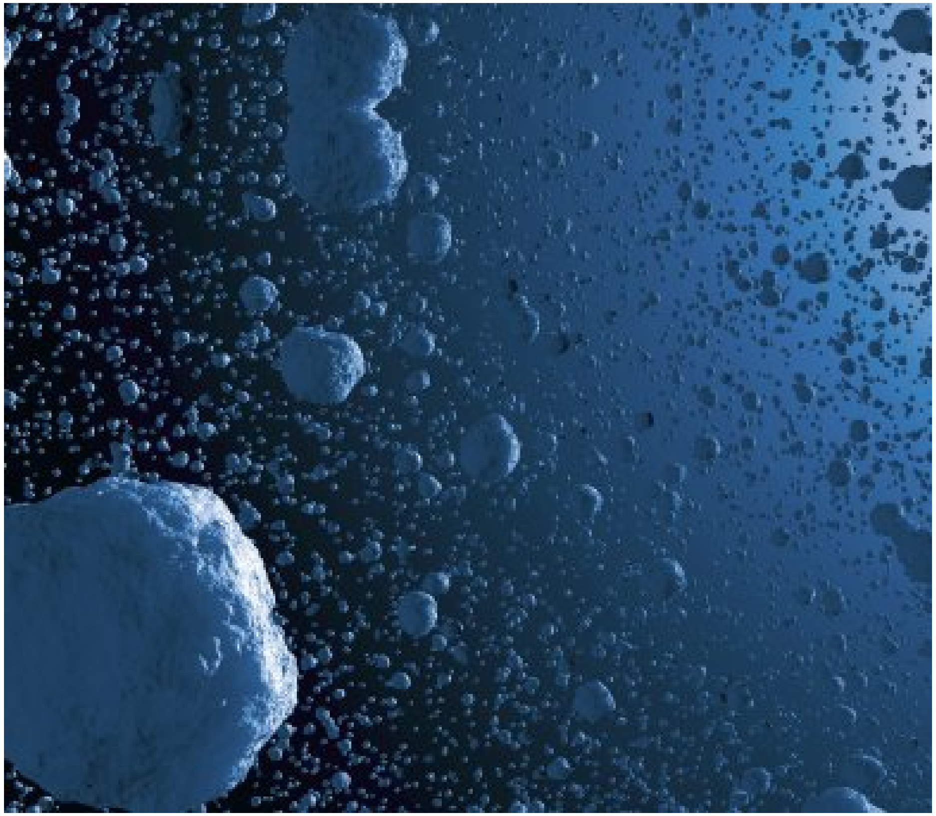
# Plan

1. Granular gases in nature
2. Nonequilibrium distributions
3. Driven steady-states
4. Cascade dynamics and stationary states
5. Associated time dependent states
6. Conclusions & outlook

# Frozen granular gases

Saturn's rings



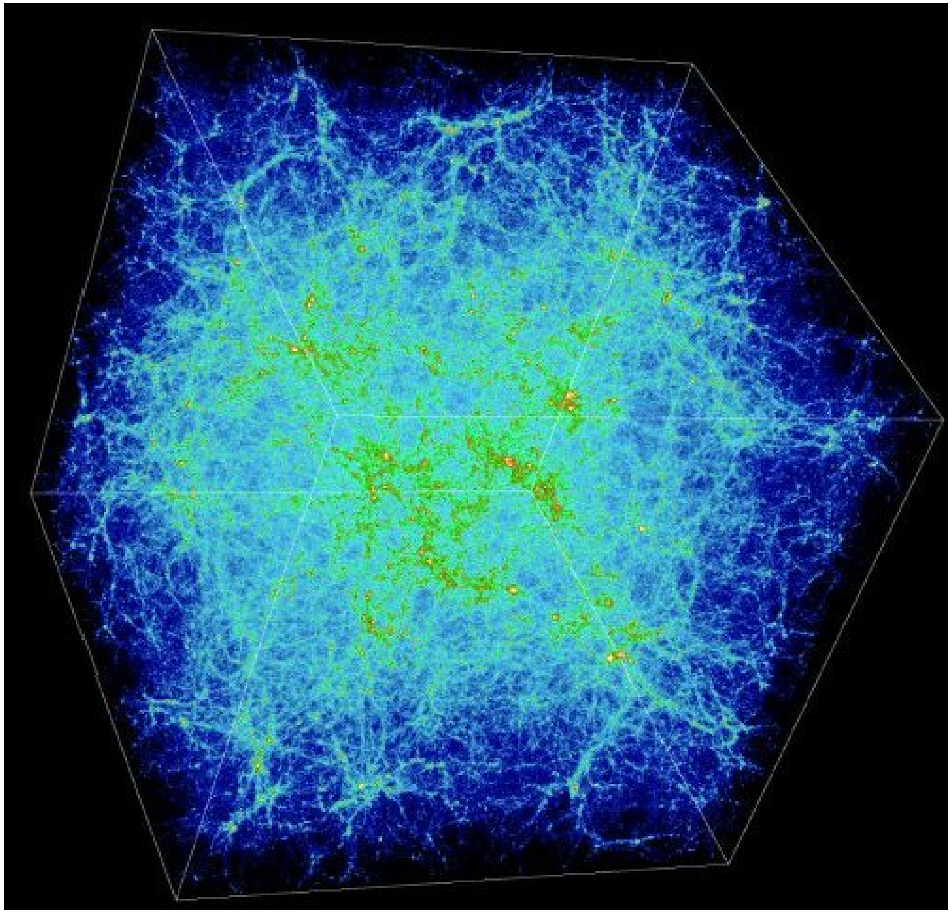
Christoph Hormann

snow avalanche

Swiss institute for snow and avalanch research

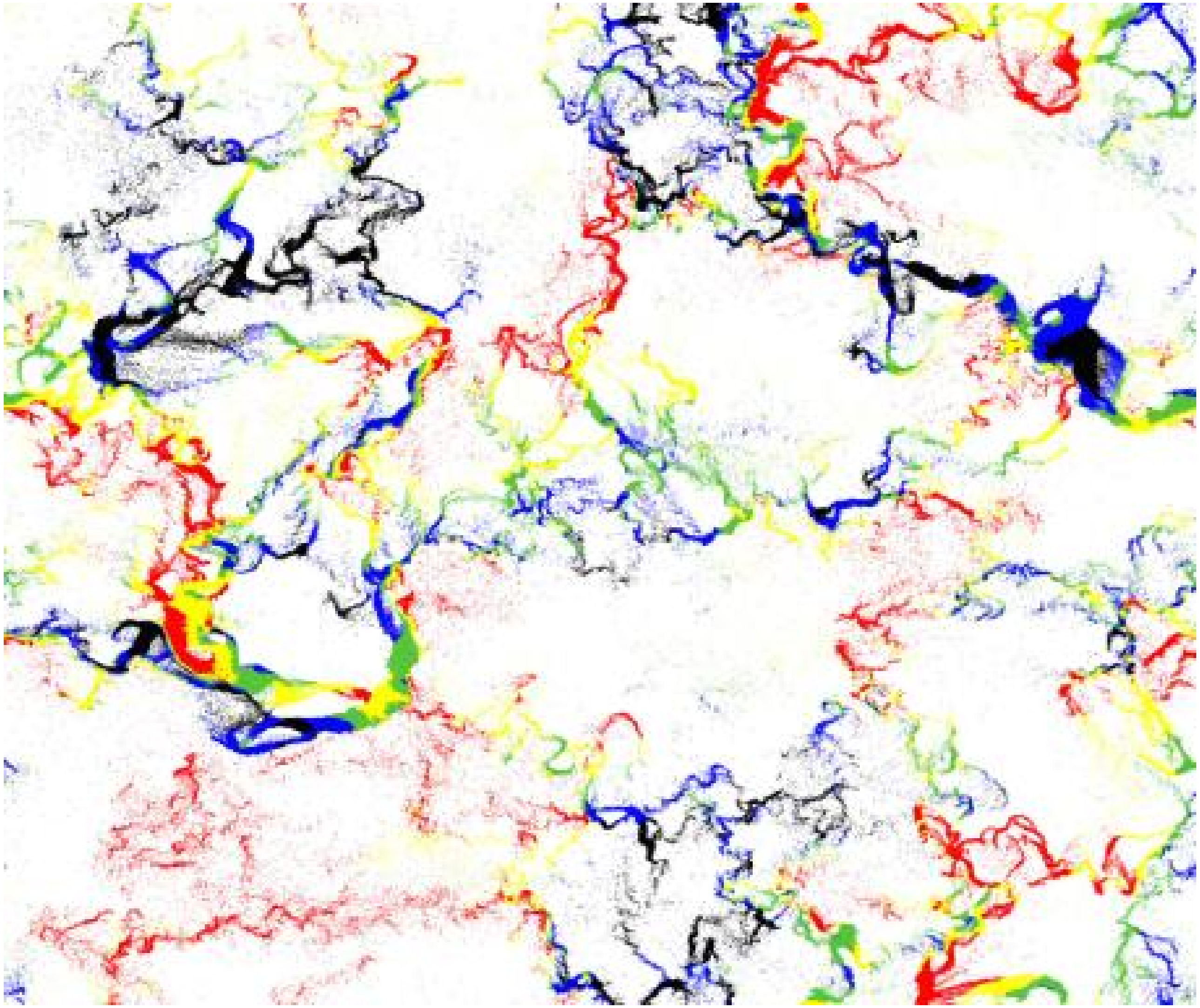
# Large scale formation of matter in the universe

Max Tegmark, UPenn
Sloan Digital Survey



Michael Warren
Los Alamos

# Filaments in granular gases



X Nie, S Chen, EB 02

# Energy dissipation in granular matter

- **Responsible for collective phenomena**
  - » **Clustering** I Goldhirsch, G Zanetti 93
  - » **Hydrodynamic instabilities** E Khain, B Meerson 04
  - » **Pattern formation** P Umbanhower, H Swinney 96
- **Anomalous statistical mechanics**
  - ➢ **No energy equipartition** R Wildman, D Parker 02
  - ➢ **Nonequilibrium energy distributions**

$$P(E) \neq \exp\left(-E/kT\right)$$

# Experiments

- **Friction**

  D Blair, A Kudrolli 01
- **Rotation**

  K Feitosa, N Menon 04
- **Driving strength**

  W Losert, J Gollub 98
- **Dimensionality**

  J Urbach & Olafsen 98
- **Boundary**

  J van Zon, H Swinney 04
- **Fluid drag**

  K Kohlstedt, I Aronson, EB 05
- **Long range interactions**

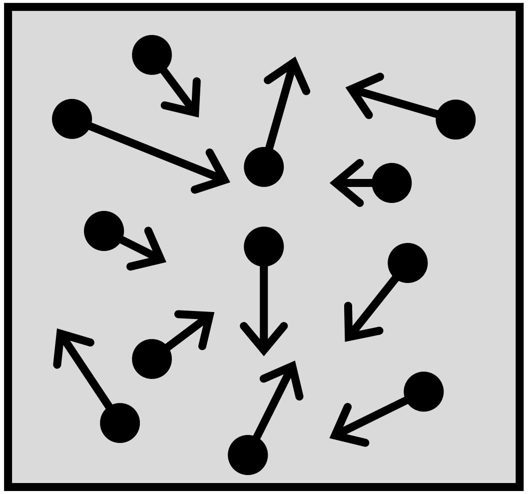
  D Blair, A Kudrolli 01; W Losert 02

  K Kohlstedt, J Olafsen, EB 05
- **Substrate**

  G Baxter, J Olafsen 04

**Deviations from equilibrium distribution**

# Driven Granular gas



- Vigorous driving
- Spatially uniform system
- Particles undergo binary collisions
- Velocities change due to
  1. <u>Collisions: lose energy</u>
  2. <u>Forcing: gain energy</u>
- What is the typical velocity (granular "temperature")?

$$T = \langle v^2 \rangle$$

- What is the velocity distribution?

$$f(v)$$

# Non-Maxwellian velocity distributions

JC Maxwell, Phil Trans Roy. Soc. **157** 49 (1867)

1. **Velocity distribution is isotropic**

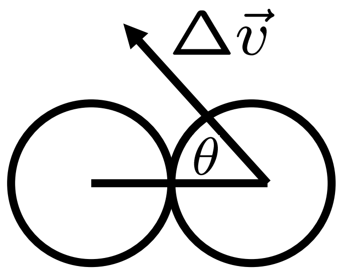
$$f(v_x, v_y, v_z) = f(|v|)$$

2. **No correlations between velocity components**

$$f(v_x, v_y, v_z) \neq f(v_x) f(v_y) f(v_z)$$

**Only possibility is Maxwellian**

$$f(v_x, v_y, v_z) \neq C\ \exp\left(-\frac{v_x^2 + v_y^2 + v_z^2}{2T}\right)$$




**Granular gases: collisions create correlations**

# Deviation from Maxwell-Boltzmann

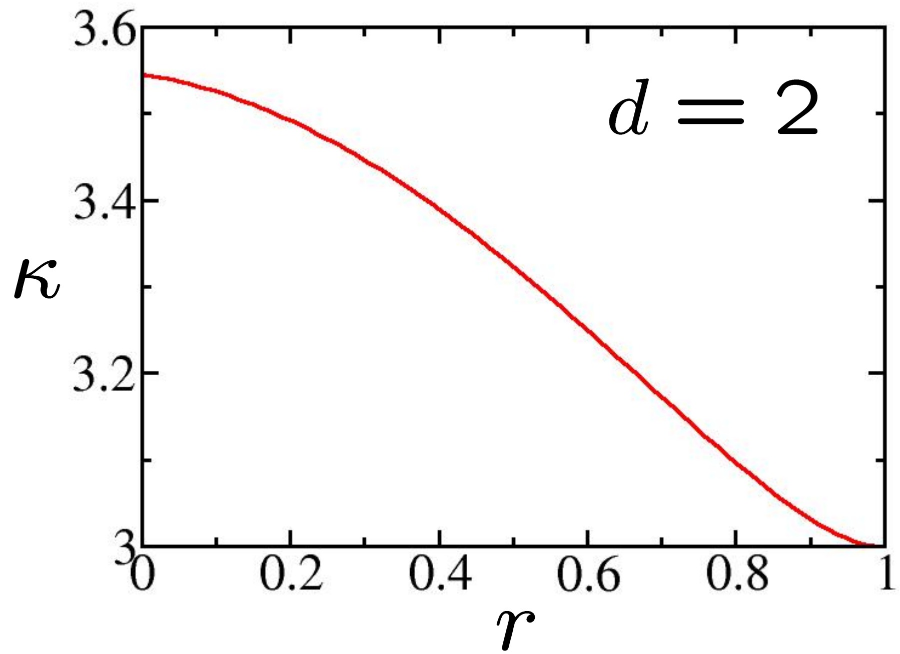
- **Kurtosis** $\kappa$

$$\kappa = \langle v^4 \rangle / \langle v^2 \rangle^2$$

$$\kappa = 3 + \frac{18(1-r)^2(1+r)}{33 - 25r + 3r^2 - 3r^3}$$

- **Restitution coefficient** $r$

$$\Delta E \propto \left(1 - r^2\right)(\Delta v)^2$$




**1. Velocity distribution independent of driving strength**

**2. Stronger dissipation yields stronger deviation**

Exact solution of Maxwell's kinetic theory: thermal forcing balances dissipation

EB, Krapivsky 02

# Nonequilibrium velocity distributions

A **Mechanically vibrated beads**

F Rouyer & N Menon 00

B **Electrostatically driven powders**

I Aronson & J Olafsen 05

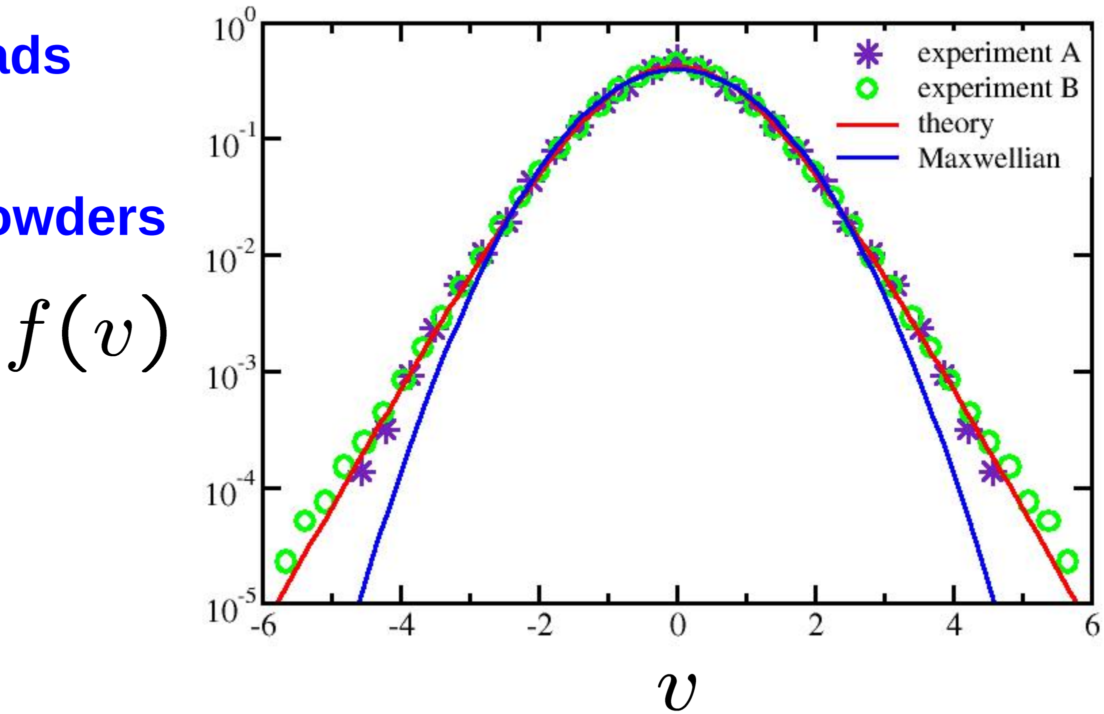
- **Gaussian core**
- **Overpopulated tail**

$$f(v) \sim \exp\left(-|v|^{\delta}\right)$$

$$1 \leq \delta \leq 3/2$$

- **Kurtosis**

$$\kappa = \begin{cases} 3.55 & \text{theory} \\ 3.6 & \text{experiment} \end{cases}$$




**Excellent agreement between theory and experiment**

**balance between collisional dissipation, energy injection from walls**

# Inelastic Collisions

- **Relative velocity reduced by** $0 < r < 1$

$$v_1 - v_2 = -r(u_1 - u_2)$$

- **Momentum is conserved**

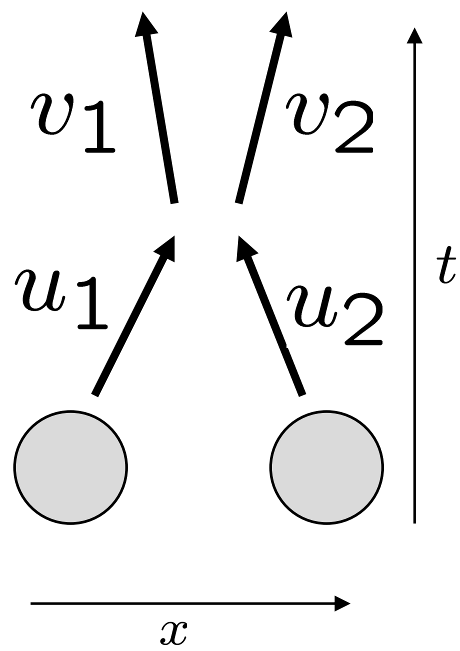
$$v_1 + v_2 = u_1 + u_2$$

- **Energy is dissipated**

$$\Delta E \propto (1 - r)(\Delta v)^2$$

- **Limiting cases**

$$r = \begin{cases} 0 & \text{completely inelastic } (\Delta E = \text{max}) \\ 1 & \text{elastic } (\Delta E = 0) \end{cases}$$



# Time dependent states

Haff, JFM 1982

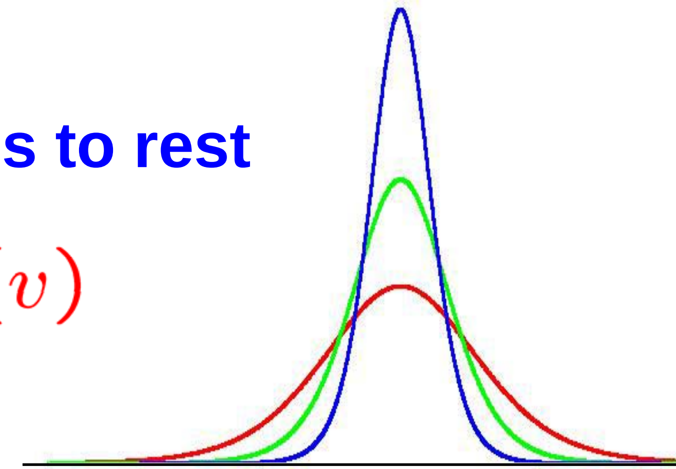
- **Energy loss** $\Delta T \sim (\Delta v)^2$
- **Collision rate** $\Delta t \sim 1/(\Delta v)^\lambda$
- **Energy balance equation**

$$\frac{\Delta T}{\Delta t} \sim -(\Delta v)^{2+\lambda} \quad \Rightarrow \quad \frac{dT}{dt} \sim -T^{1+\lambda/2}$$

- **Temperature decays, system comes to rest**

$$T \sim t^{-2/\lambda} \quad \Rightarrow \quad f(v) \to \delta(v)$$

**Trivial steady-state**



# Kinetic Theory

- **Collision rule (linear)** $r = 1 - 2p, \quad p + q = 1$

$$(u_1, u_2) \to (pu_1 + qu_2, pu_2 + qu_1)$$

- **Boltzmann equation (nonlinear and nonlocal)**

$$\frac{\partial P(v)}{\partial t} = \iint du_1 du_2 f(u_1) f(u_2) \underbrace{|u_1 - u_2|^\lambda}_{\text{collision rate}} [\underbrace{\delta(v - pu_1 - qu_2)}_{\text{gain}} - \underbrace{\delta(v - u_2)}_{\text{loss}}]$$

- **Collision rate related to interaction potential**

$$U(r) \sim r^{-\gamma} \qquad \lambda = 1 - 2\frac{d-1}{\gamma} = \begin{cases} 0 & \text{Maxwell molecules} \\ 1 & \text{Hard spheres} \end{cases}$$

**Theory: non-linear, non-local, dissipative**

# Are there nontrivial steady states?

- **Stationary Boltzmann equation**

$$0 = \iint du_1 du_2 f(u_1) f(u_2) |u_1 - u_2|^\lambda [\delta(v - pu_1 - qu_2) - \delta(v - u_2)]$$

collision rate gain loss

## Naive answer: NO!

- **According to the energy balance equation**

$$\frac{dT}{dt} = -\Gamma$$

- **Dissipation rate is positive**

$$\Gamma > 0$$

# An exact solution

- One-dimensional Maxwell molecules
- Fourier transform obeys a closed equation $F(k) = \int dv\, e^{ikv} f(v)$

$$F(k) = F(pk)F(qk)$$

- Exponential solution

$$F(k) = \exp(-v_0|k|)$$

- Lorentzian velocity distribution

$$f(v) = \frac{1}{\pi v_0} \frac{1}{1 + (v/v_0)^2}$$

**Nontrivial stationary states do exist!**

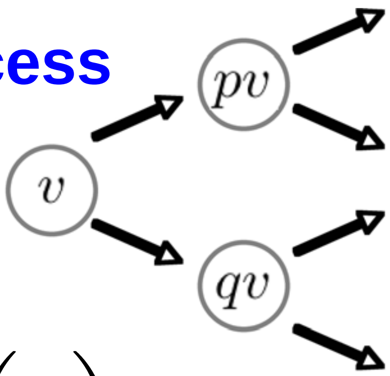
# Cascade Dynamics (1D)

- **Collision rule: arbitrary velocities**

$$(u_1, u_2) \to (pu_1 + qu_2, pu_2 + qu_1)$$

- **Large velocities: linear but nonlocal process**

$$v \to (pv, qv)$$




- **High-energies: linear equation**

$$f(v) = \frac{1}{p^{1+\lambda}} f\left(\frac{v}{p}\right) + \frac{1}{q^{1+\lambda}} f\left(\frac{v}{q}\right)$$

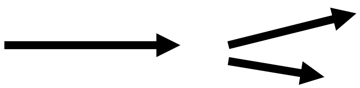
loss gain gain

- **Power-law tail**

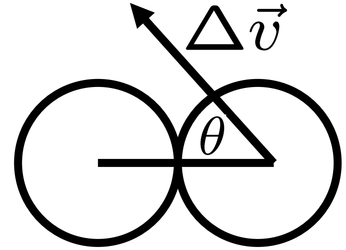
$$f(v) \sim v^{-2-\lambda}$$

# Cascade Dynamics (any D)

- **Collision process: large velocities**



$$v \to (\alpha v, \beta v)$$




- **Stretching parameters related to impact angle**

$$\alpha = (1-p)\cos\theta \qquad \beta = \sqrt{1-(1-p^2)\cos^2\theta}$$

- **Energy decreases, velocity magnitude increases**

$$\alpha^2 + \beta^2 \geq 1 \qquad \alpha + \beta \leq 1$$

- **Steady state equation**

$$f(v) = \left\langle \frac{1}{\alpha^{d+\lambda}} f\left(\frac{v}{\alpha}\right) + \frac{1}{\beta^{d+\lambda}} f\left(\frac{v}{\beta}\right) \right\rangle$$

# Power-laws are generic

- **Velocity distributions always has power-law tail**

$$f(v) \sim v^{-\sigma}$$

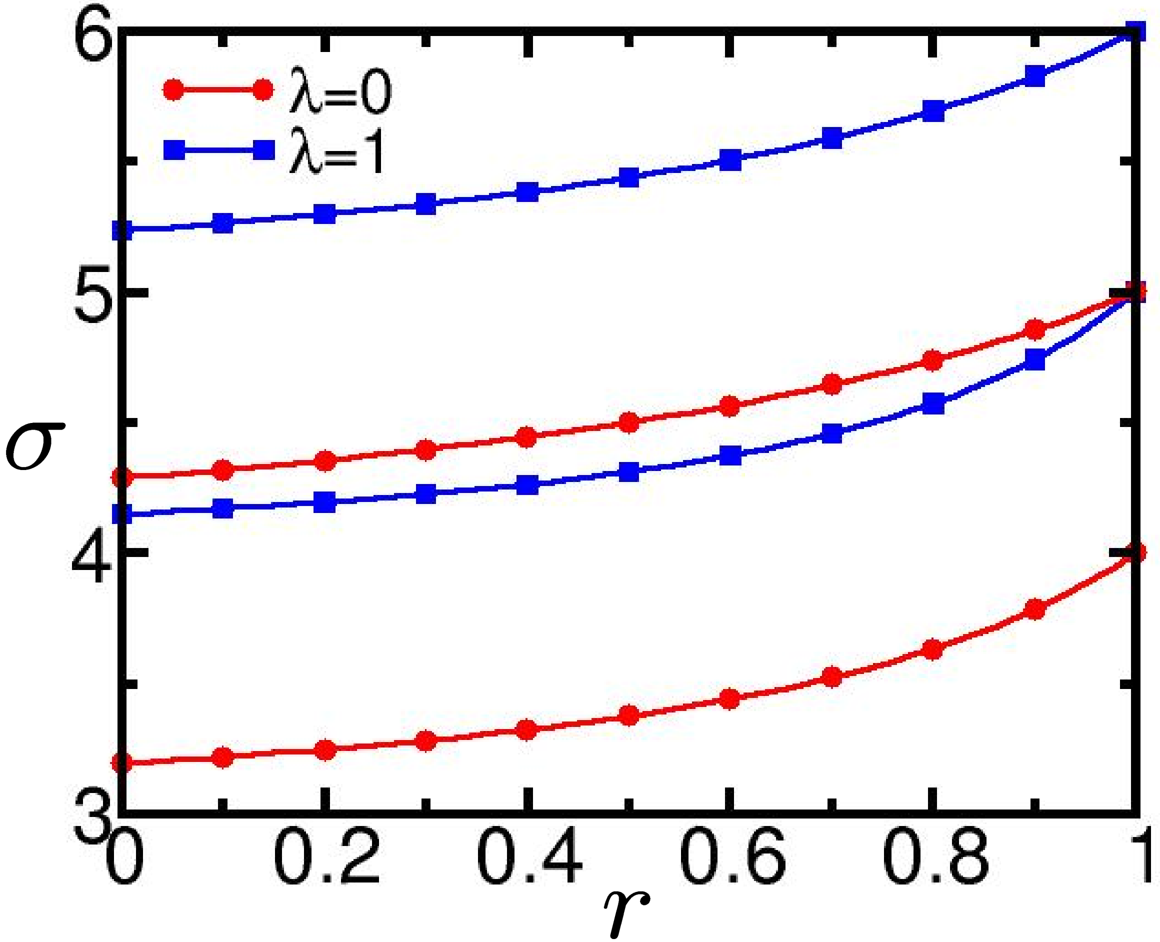
- **Exponent varies with parameters**

$$\frac{1 - {}_2F_1\left(\frac{d+\lambda-\sigma}{2}, \frac{\lambda+1}{2}, \frac{d+\lambda}{2}, 1-p^2\right)}{(1-p)^{\sigma-d-\lambda}} = \frac{\Gamma(\frac{\sigma-d+1}{2})\Gamma(\frac{d+\lambda}{2})}{\Gamma(\frac{\sigma}{2})\Gamma(\frac{\lambda+1}{2})}$$

- **Tight bounds** $1 \leq \sigma - d - \lambda \leq 2$
- **Elastic limit is singular** $\sigma \to d + 2 + \lambda$

**Dissipation rate always divergent**
**Energy finite or infinite**

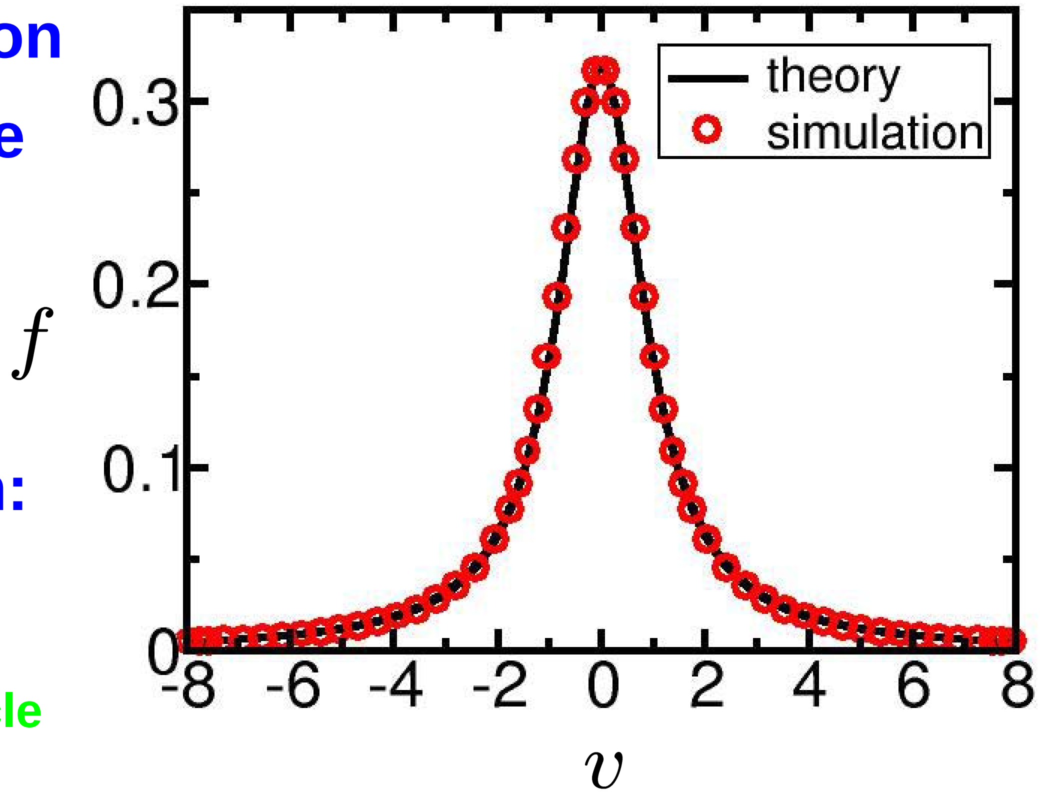
# The characteristic exponent σ




**σ varies with spatial dimension, collision rules**

# Monte Carlo Simulations

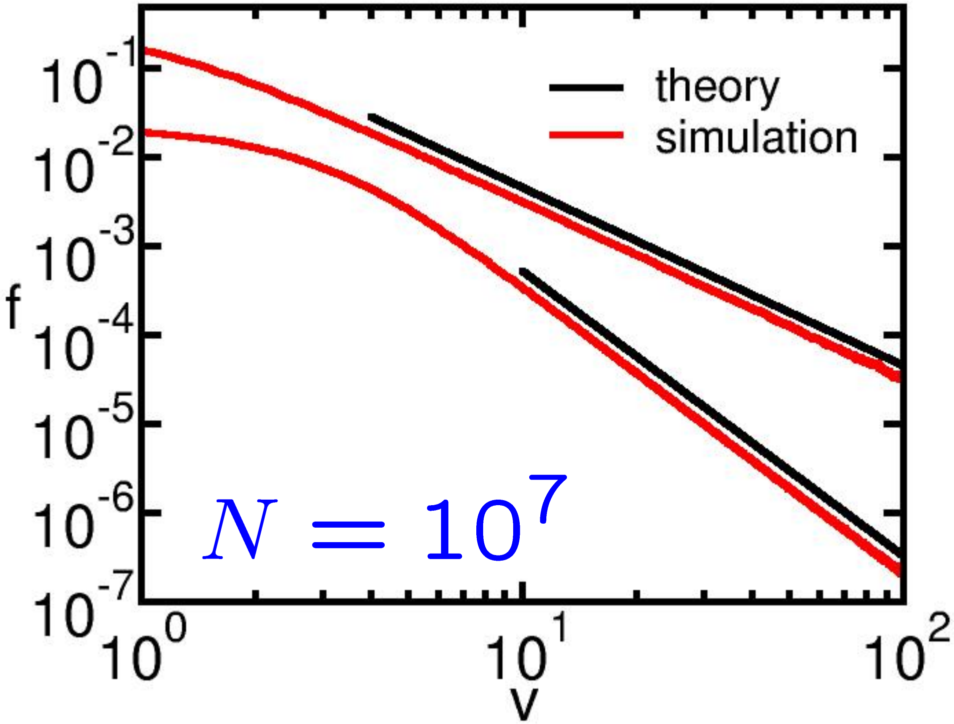
- Compact initial distribution
- Inject energy at very large velocity scales only
- Maintain constant total energy
- "Lottery" implementation:
  - Keep track of total energy dissipated, $E_T$
  - With small rate, boost a particle by $E_T$




Excellent agreement between theory and simulation

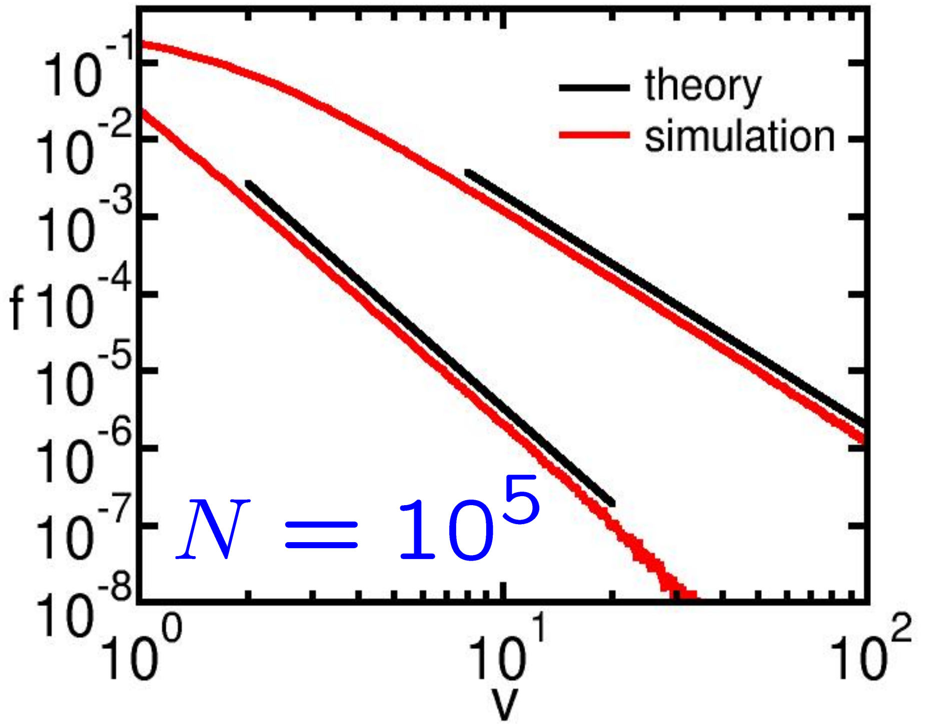
# Further confirmation

## Maxwell molecules (1D, 2D)



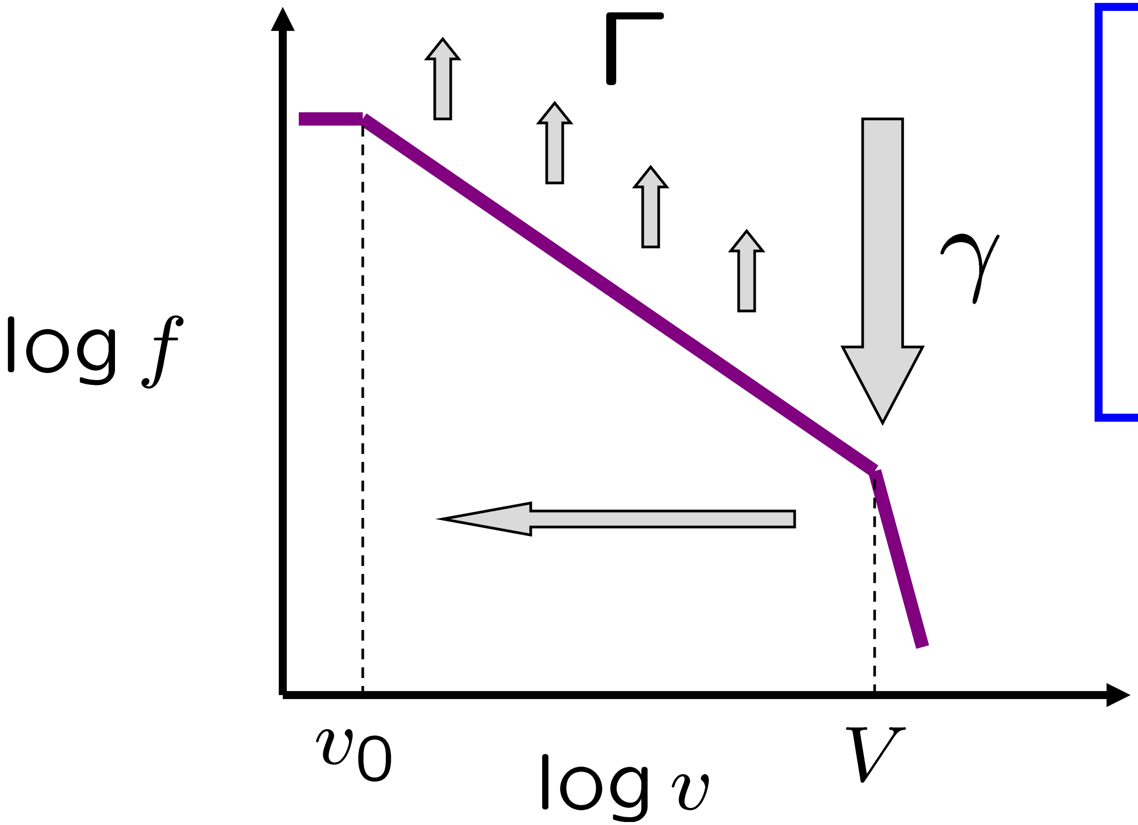

| d | theory | simulation |
|---|---|---|
| 1 | 2 | 1.995 |
| 2 | 3.19520 | 3.19 |

## Hard spheres (1D, 2D)




| d | theory | simulation |
|---|---|---|
| 1 | 3 | 2.994 |
| 2 | 4.14922 | 4.15 |

# Injection, cascade, dissipation




**Experimental realization?**
Energetic particle "shot" into static medium

**Energy balance**
$$\Gamma \sim \gamma V^2$$

- **Energy is injected at large velocity scales**
- **Energy cascades from large velocities to small velocities**
- **Energy dissipated at small velocity scales**

# Energy balance

- **Energy injection rate** $\gamma$
- **Energy injection scale** $V$
- **Typical velocity scale** $v_0$
- **Balance between energy injection and dissipation**

$$\gamma \sim V^{\lambda}(V/v_0)^{d-\sigma}$$

- **For "lottery" injection: injection scale diverges with injection rate**

$$V \sim \begin{cases} \gamma^{-1/(2-\lambda)} & \sigma < d+2 \\ \gamma^{-1/(\sigma-d-\lambda)} & \sigma > d+2 \end{cases}$$

# Traditional forcing: Injection, dissipation

T van Noije, M Ernst 97

- **Energy injection: thermal forcing (at all scales)**

$$dv/dt = \eta$$

- **Energy dissipation: inelastic collision**

$$v \longrightarrow (pv, qv)$$

- **Steady state equation**

$$0 = D\frac{d^2 f(v)}{d^2 v} + v^\lambda \left[\xcancel{\frac{1}{p^{1+\lambda}} f\left(\frac{v}{p}\right) + \frac{1}{q^{1+\lambda}} f\left(\frac{v}{q}\right)} - f(v)\right]$$

- **Stretched exponentials**

$$f(v) \sim \exp\left(-v^{1+\lambda/2}\right)$$

with Ben Machta (Brown)

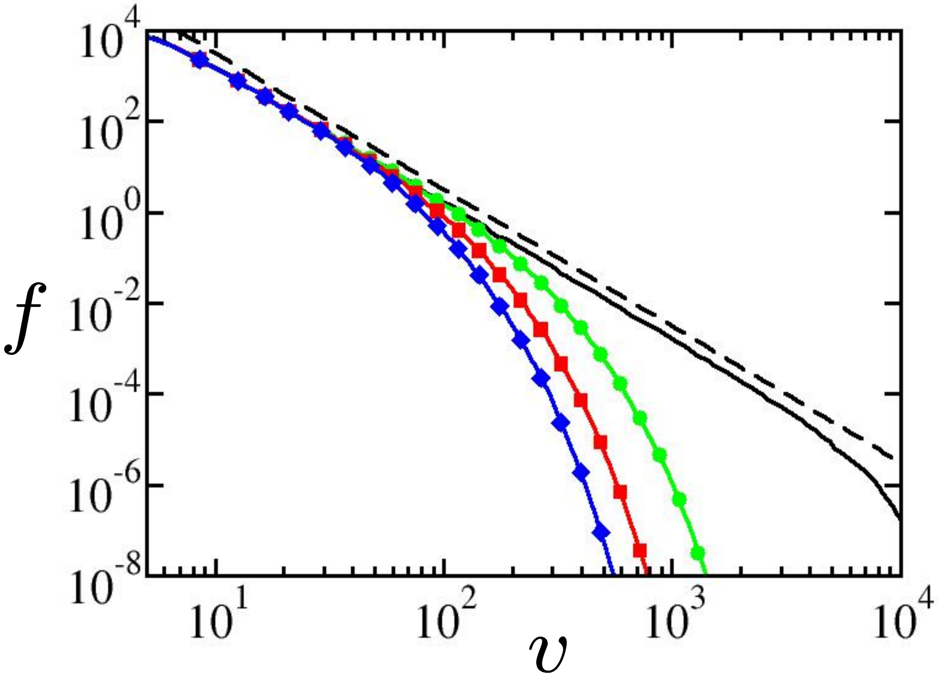
# Self-similar collapse

- **Self-similar distribution**

$$f(v,t) \simeq v^{-\sigma}\Phi\left(\frac{v}{V(t)}\right)$$

- **Cutoff velocity decays**

$$V(t) \sim t^{-1/\lambda}$$




- **Scaling function**

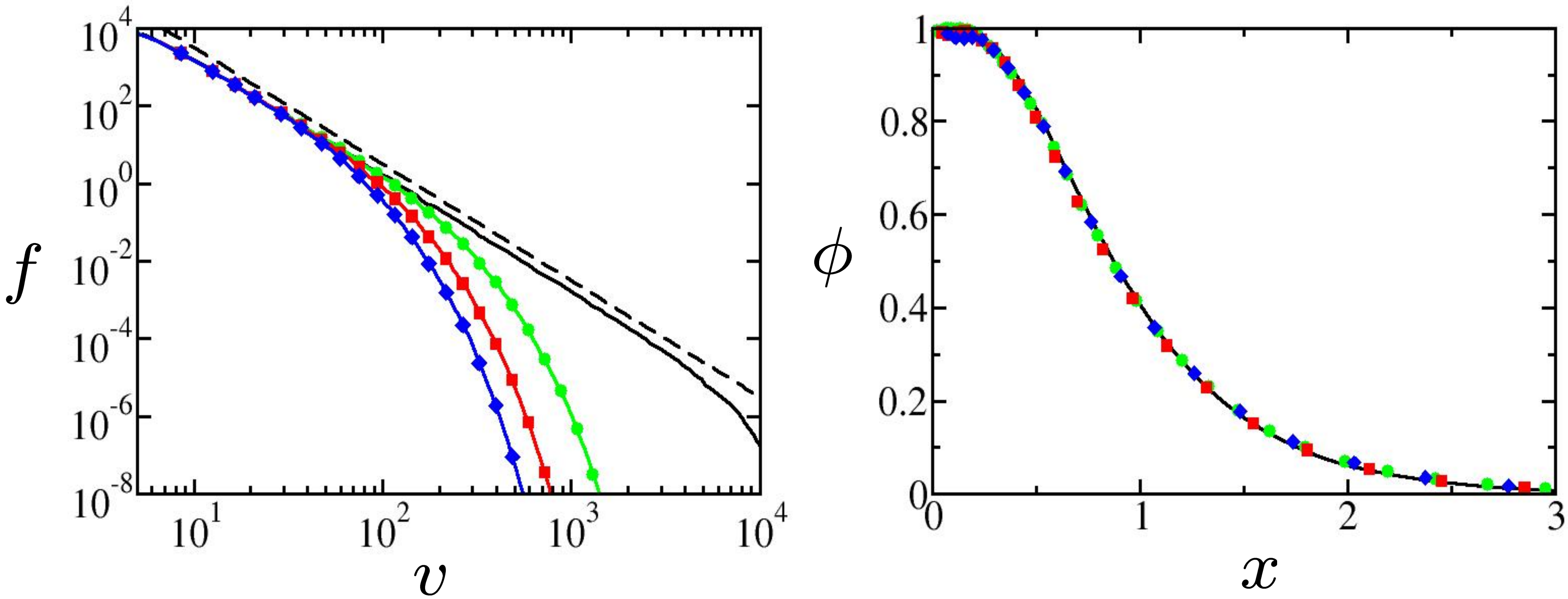
$$\Phi(x) = \sum_{n=1}^{\infty} A_n \exp\left[-(2^n x)^\lambda\right] \qquad A_n = \prod_{\substack{k=1 \\ k\neq n}}^{\infty} \frac{1}{1-2^{\lambda(n-k)}}$$

**Hybrid between steady-state and time dependent state**

# Numerical confirmation

Velocity distribution

Scaling function




**A third family of solutions exists**

# Conclusions

- New class of nonequilibrium stationary states
- Energy cascades from large to small velocities
- Power-law high-energy tail
- Energy input at large scales balances dissipation
- Associated similarity solutions exist as well
- Temperature insufficient to characterize velocities
- Experimental realization: requires a different driving mechanism

# Outlook

- **Spatially extended systems**
- **Spatial structures**
- **Polydisperses granular media**
- **Experimental realization**

E. Ben-Naim and J. Machta, PRL 91 (2005) cond-mat/0411473