

Bifurcations and Patterns in Opinion Dynamics

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Thanks

Paul Krapivsky and Sidney Redner (*Boston*)

<http://cnls.lanl.gov/~ebn>



Plan

1. **Motivation: modeling social systems**
2. **Continuous opinions: simulations, scaling**
3. **Discrete opinions: general features, theory**
4. **Pattern selection: linear stability analysis**
5. **Extensions: initial conditions, 2D, noise**

Modeling social dynamics

- ◆ Goal: predictive models of human opinions
- ◆ Relevance: politics, economics, consumer, sports

Questions

- Are “physics concepts useful?”
- Are human interactions predictable?

This should help

- Large data sets available
- Large number of humans $N \sim 10^9$
- Human opinions are quantitative

Quantifying opinions


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Humans interact, opinions evolve

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Rankings

Division I Polls

AP Top 25

1. [Connecticut](#) (37)
2. [Duke](#) (5)
3. [Georgia Tech](#) (21)
4. [Arizona](#) (4)
5. [Stanford](#) (3)
6. [Wake Forest](#) (1)
7. [Oklahoma](#)
8. [Kentucky](#)
9. [North Carolina](#)
10. [St. Joseph's \(PA\)](#)

USA Today/ESPN

1. [Connecticut](#) (18)
2. [Duke](#) (6)
3. [Georgia Tech](#) (5)
4. [Arizona](#)
5. [Wake Forest](#) (1)
6. [Kentucky](#)
7. [Stanford](#)
8. [Oklahoma](#)
9. [North Carolina](#)
10. [St. Joseph's \(PA\)](#) (1)

tendency to reach consensus?

The Compromise Process

- ◆ **Opinion measured by continuum variable**

$$-\Delta < x < \Delta$$

- ◆ **Compromise: reached via pairwise interactions**

$$(x_1, x_2) \rightarrow \left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2} \right)$$

- ◆ **Conviction: restricted interaction range**

$$|x_1 - x_2| < 1$$

- ◆ **Minimal, one parameter model**
- ◆ **Mimics competition between compromise and conviction**

Weisbuch 2001

Problem

- ◆ **Given initial distribution**

$$P_0(x) = \begin{cases} 1 & |x| < \Delta \\ 0 & |x| > \Delta \end{cases}$$

- ◆ **Find final distribution**

$$P_\infty(x) = ?$$

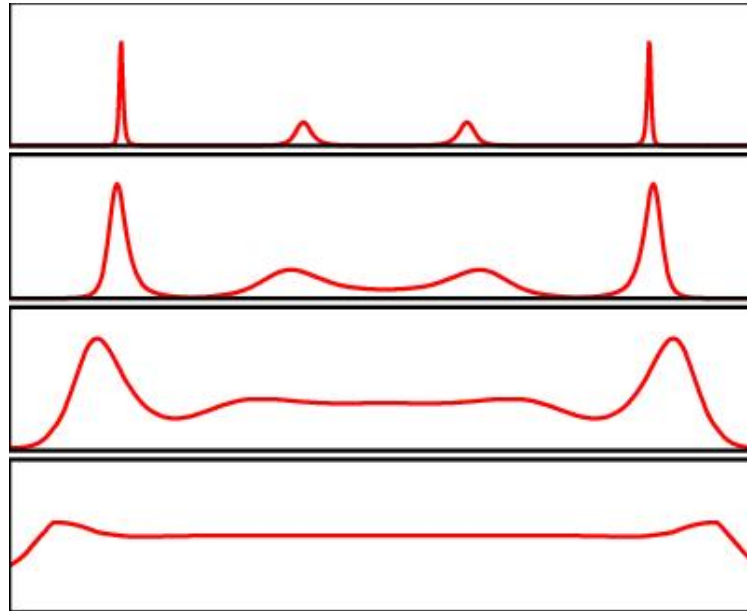
- ◆ **Multitude of final states**

$$P_\infty(x) = \sum_{i=1}^N m_i \delta(x - x_i) \quad |x_i - x_j| > 1$$

- ◆ **Dynamics selects one (deterministically)**

Multiple localized clusters (parties)

Numerical methods, kinetic theory



✓ Numerical integration of probability distribution

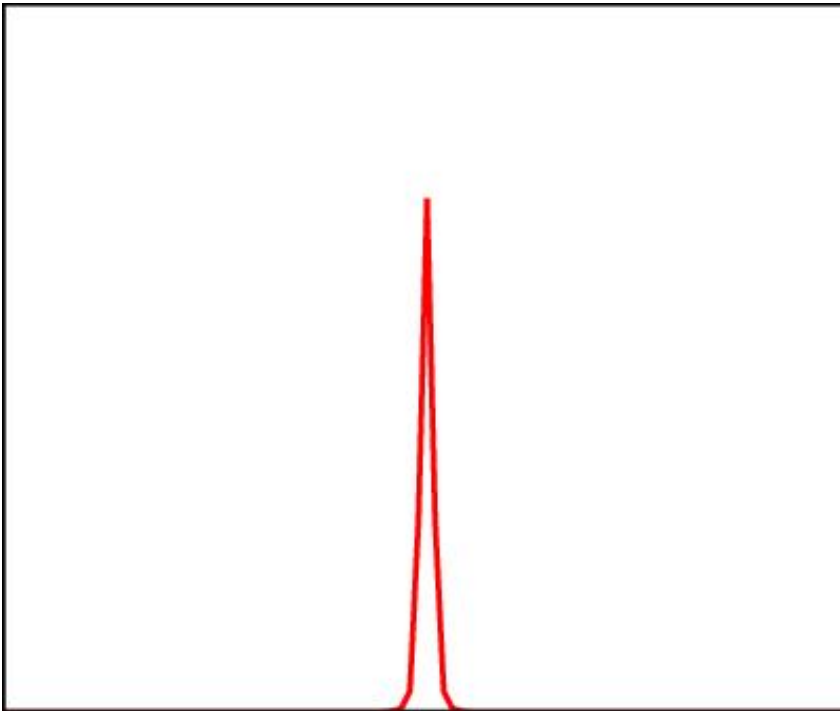
Kinetic theory: nonlinear rate equations

$$\frac{\partial}{\partial t} P(x, t) = \iint_{|x_1 - x_2| < 1} dx_1 dx_2 P(x_1, t) P(x_2, t) [2\delta(x - (x_1 + x_2)/2) - \delta(x - x_1) - \delta(x - x_2)]$$

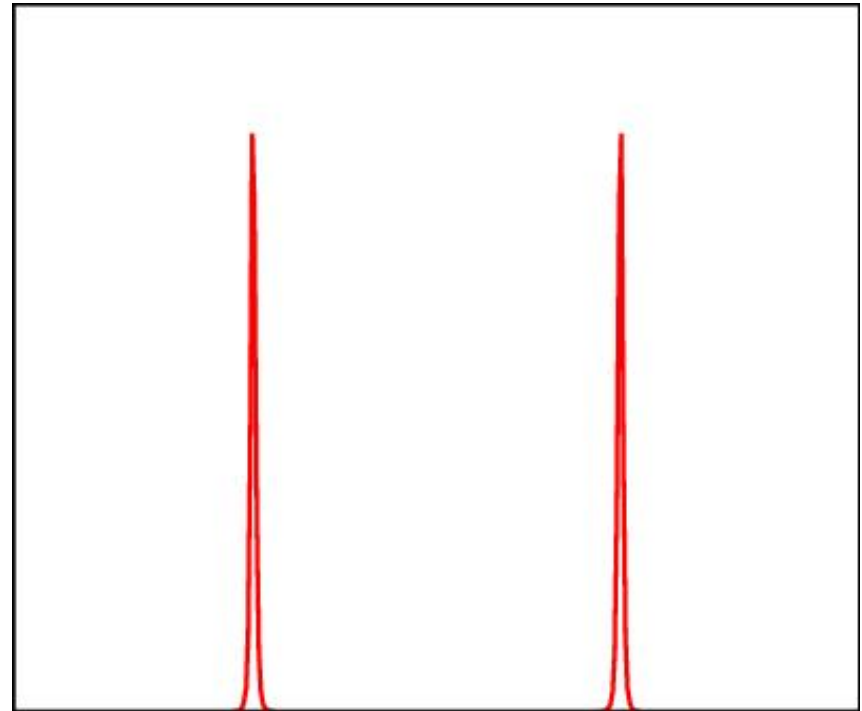
✗ Direct simulation of stochastic process

Rise and fall of central party

$$0 < \Delta < 1.871$$



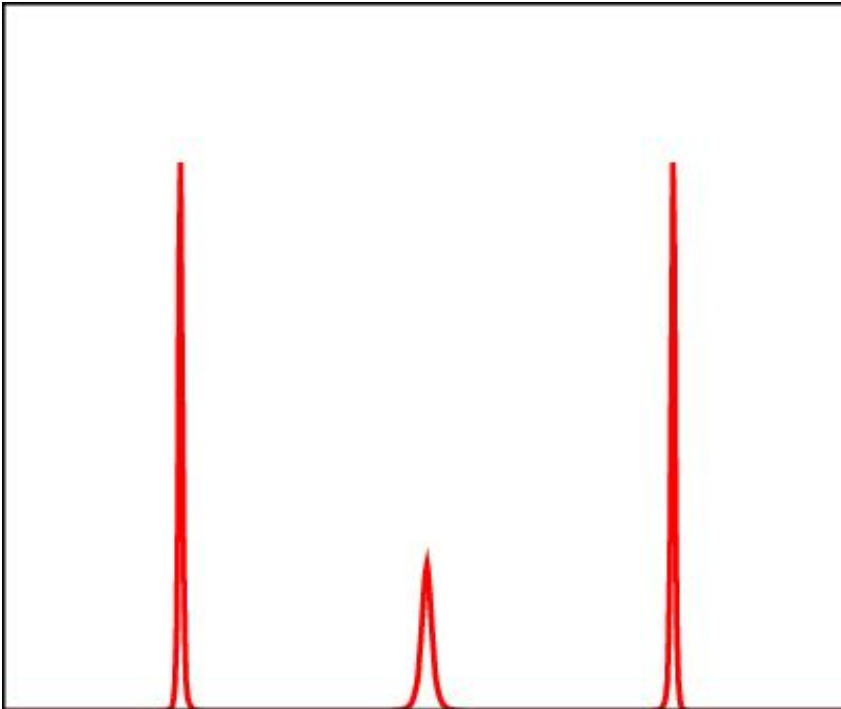
$$1.871 < \Delta < 2.724$$



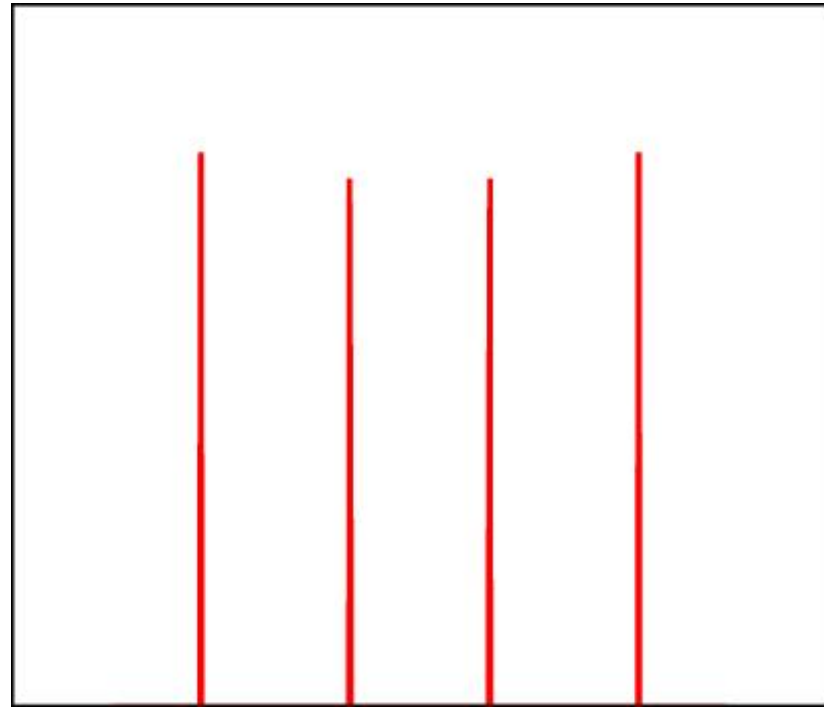
Central party may or may not exist!

Reemergence of central party

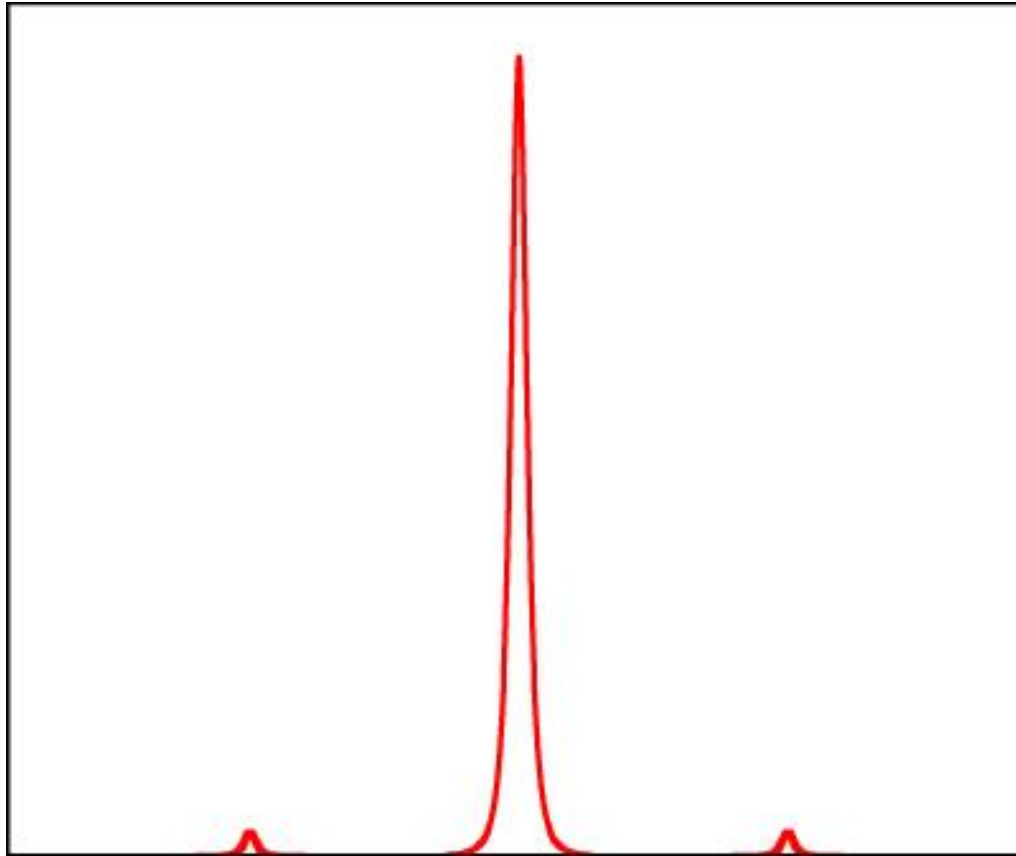
$$2.724 < \Delta < 4.079$$



$$4.079 < \Delta < 4.956$$

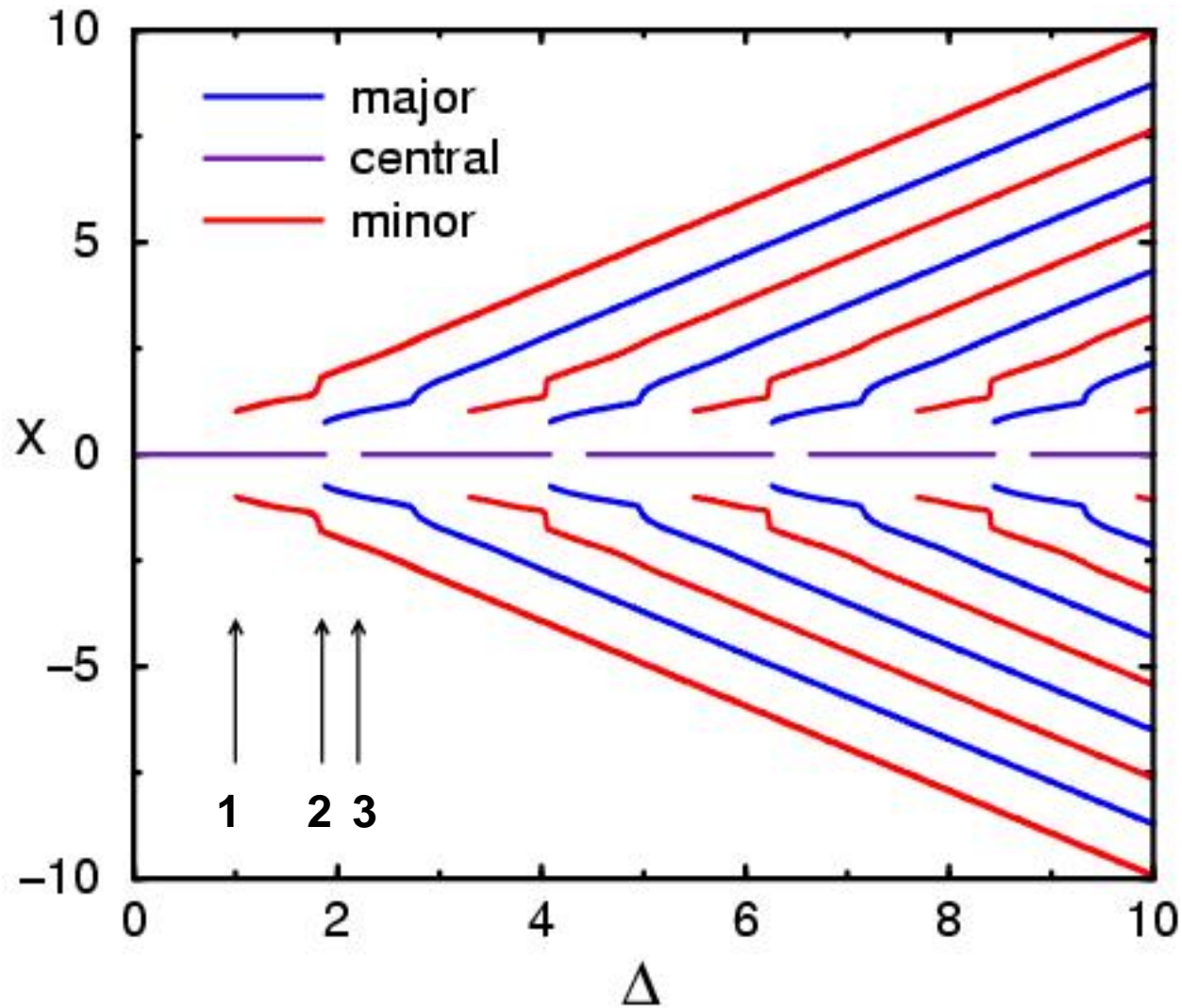


Hidden clusters



Tiny parties (mass $<10^{-3}$), extremists

Bifurcations and Patterns



Self-similar structure, universality

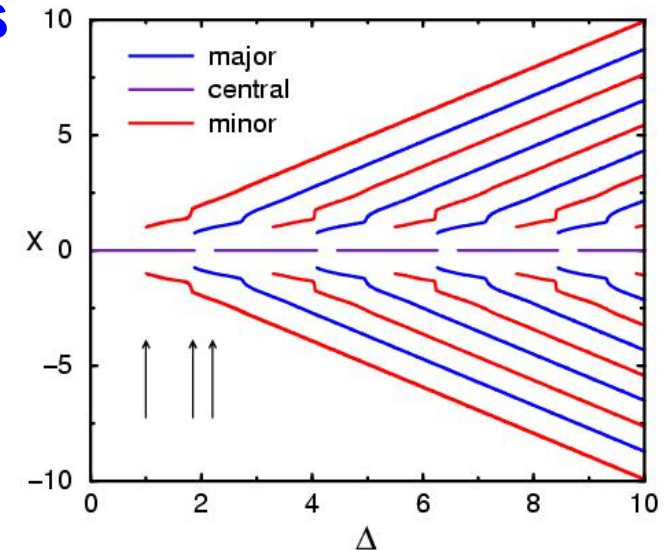
◆ Periodic sequence of bifurcations

1. Nucleation of minor cluster branch
2. Nucleation of major cluster branch
3. Nucleation of central cluster

◆ Alternating major-minor pattern

◆ Clusters are equally spaced

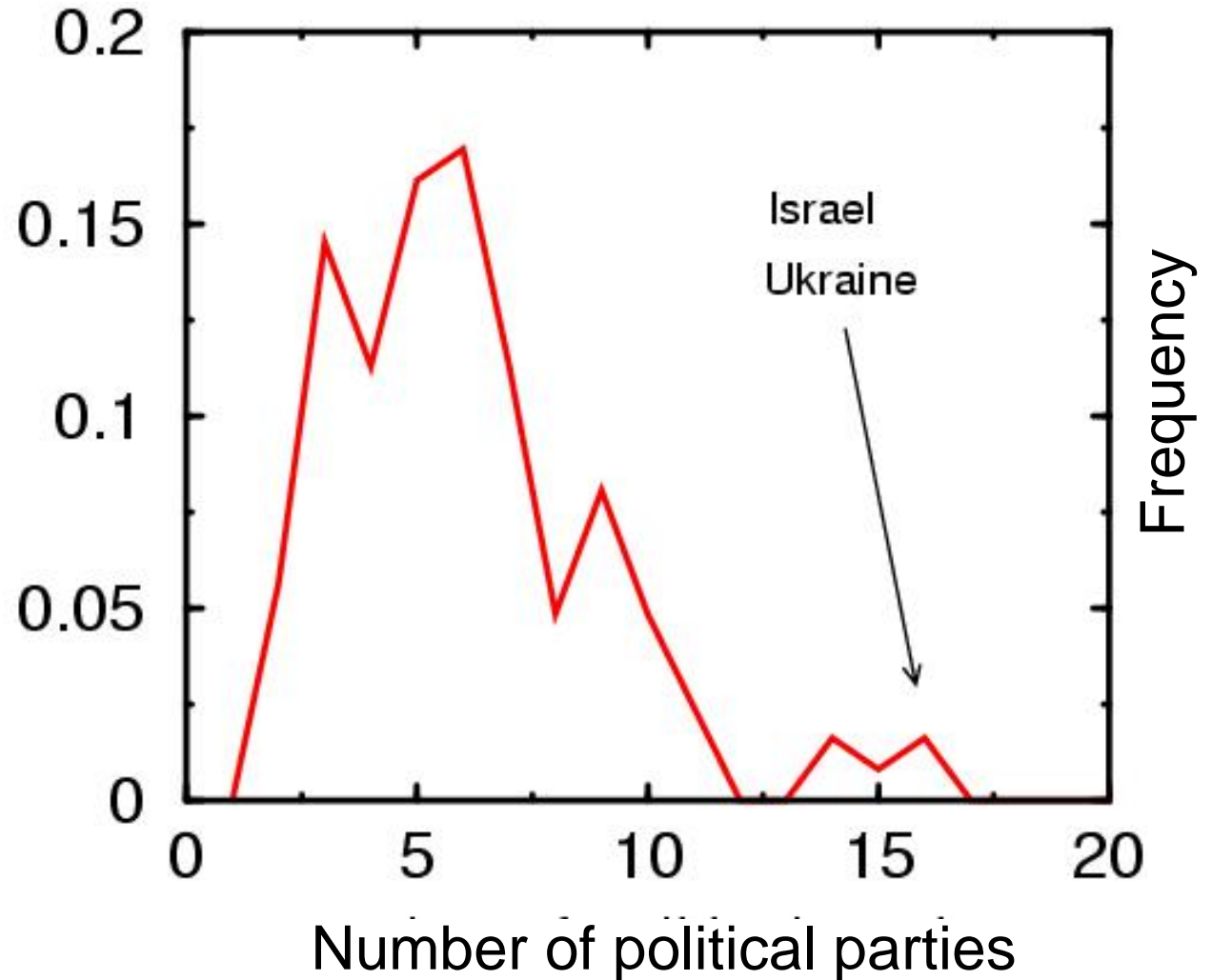
◆ Period gives major cluster mass, separation



$$x(\Delta) = x(\Delta + L) \quad L = 2.155$$

How many political parties?

- ◆ Data: CIA world factbook 2002
- ◆ 120 countries with multi-party parliaments
- ◆ Average=5.8
standard deviation=2.9



Cluster mass

- ◆ Masses are periodic

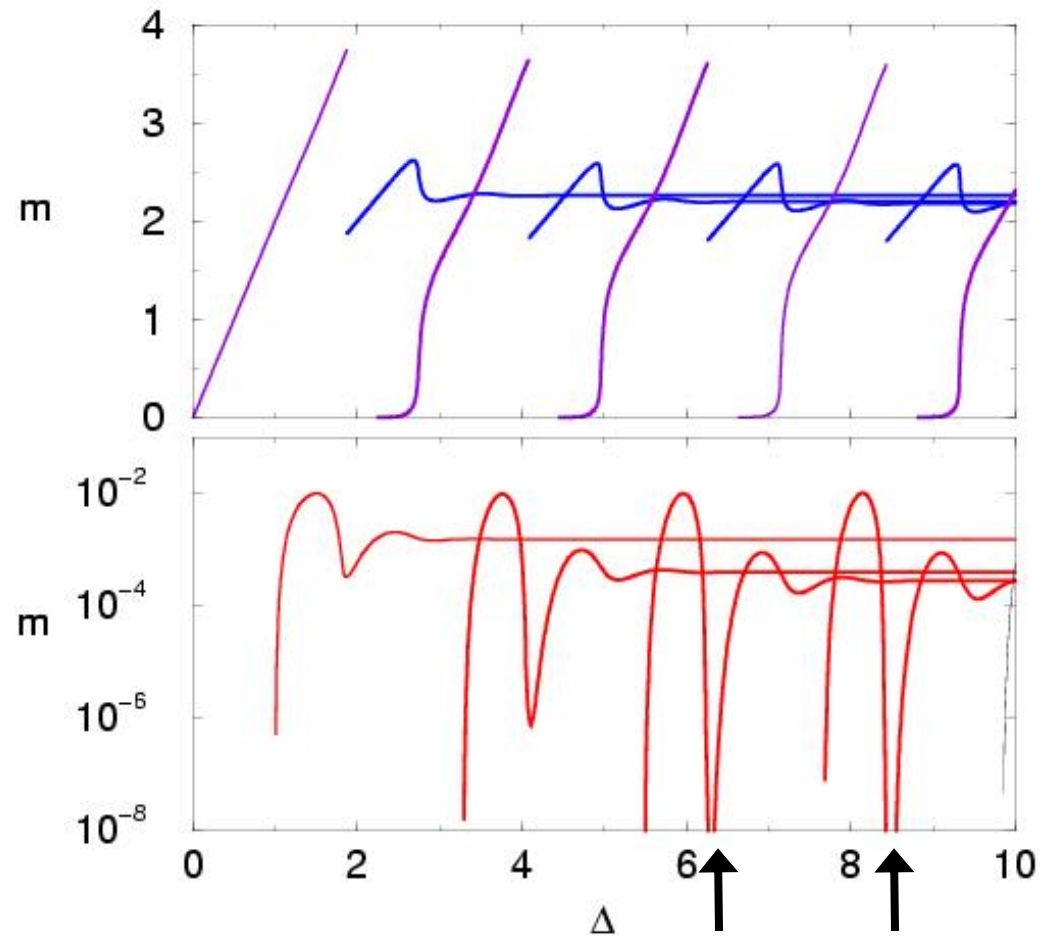
$$m(\Delta) = m(\Delta + L)$$

- ◆ Major mass

$$M \rightarrow L = 2.155$$

- ◆ Minor mass

$$m \rightarrow 3 \times 10^{-4}$$



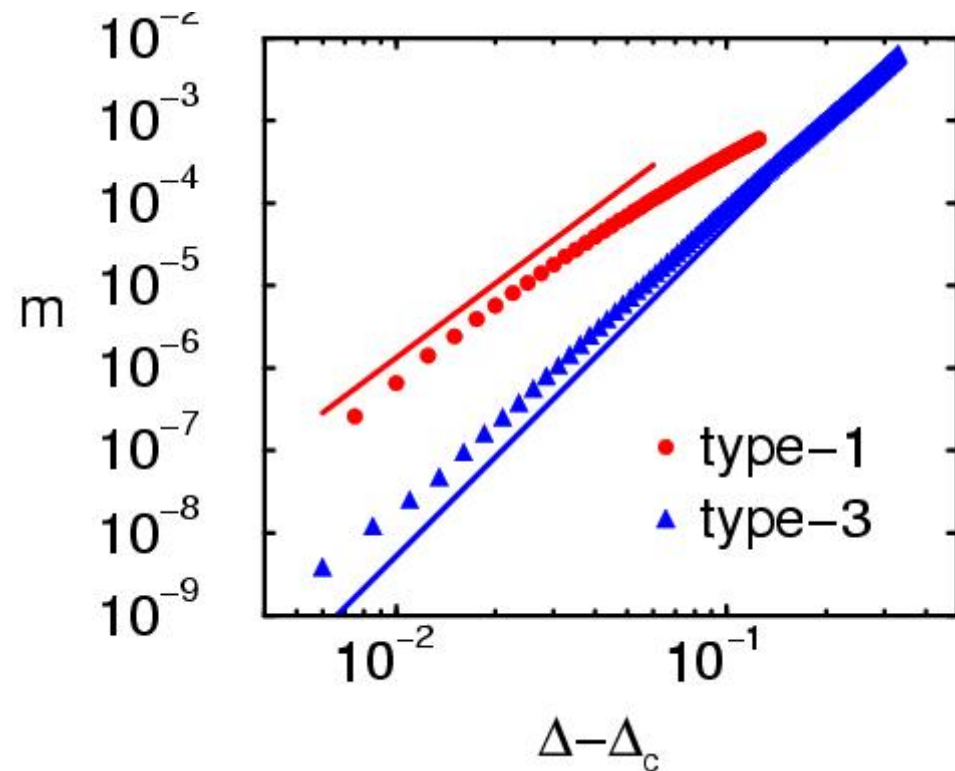
Scaling near bifurcation points

- ◆ Minor mass vanishes

$$m \sim (\Delta - \Delta_c)^\alpha$$

- ◆ Universal exponents

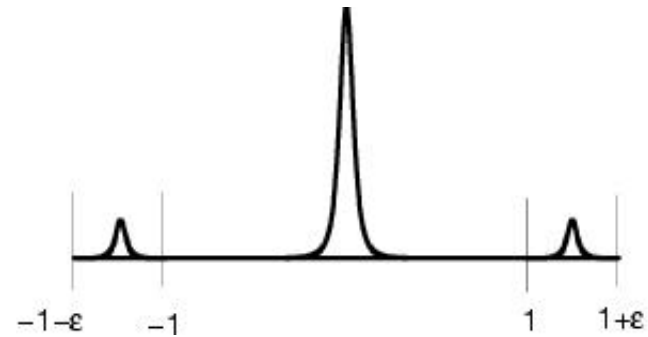
$$\alpha = \begin{cases} 3 & \text{type 1} \\ 4 & \text{type 3} \end{cases}$$



L-2 is the small parameter
explains small saturation mass

Heuristic derivation of exponents

- Perturbation theory $\Delta = 1 + \varepsilon$
- Central cluster $x(\infty) = 0$
- Extremist minor cluster $x(\infty) = 1 + \varepsilon/2$



- ◆ Rate of transfer from minor cluster to major cluster

$$dm / dt = -mM \quad \rightarrow \quad m(t) \sim \varepsilon e^{-t}$$

- ◆ Process stops when

$$x \sim e^{-t_f/2} \sim \varepsilon$$

- ◆ Final minor cluster mass

$$m(\infty) \sim m(t_f) \sim \varepsilon^3$$

Consensus

- ◆ **Integrable for** $\Delta < 1/2$

$$\langle x^2(t) \rangle = \langle x^2(0) \rangle e^{-\Delta t}$$

- ◆ **Final state: localized**

$$P_\infty(x) = 2\Delta\delta(x)$$

- ◆ **Rate equations in Fourier space**

$$P_t(k) + P(k) = P^2(k/2)$$

- ◆ **Self-similar collapse dynamics**

$$\Phi(z) \propto (1 + z^2)^{-2} \quad z = \frac{x}{\langle x^2(t) \rangle}$$

Discrete opinions

- ◆ **Compromise process**

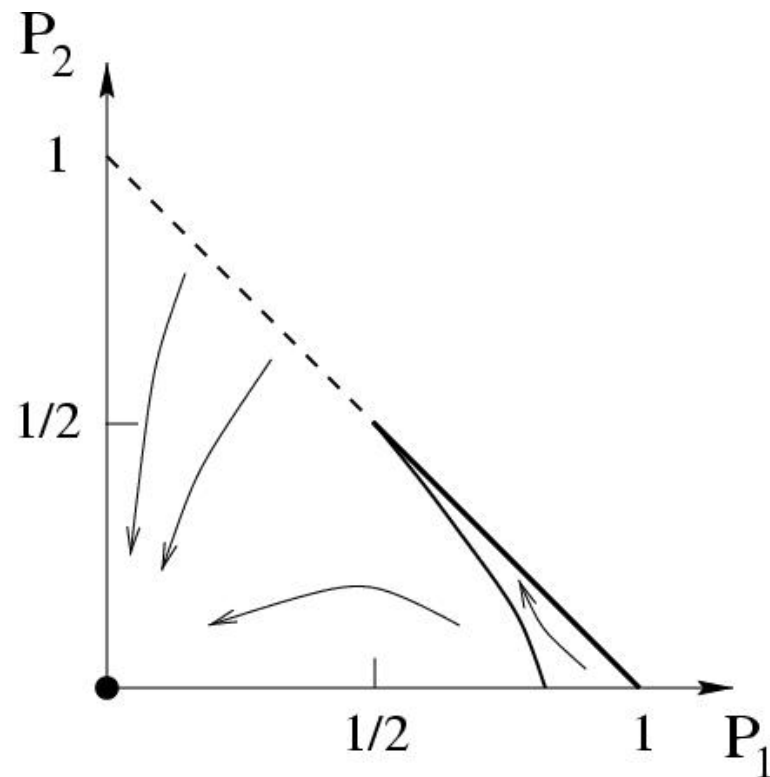
$$(i-1, i+1) \rightarrow (i, i)$$

- ◆ **Master equation**

$$dP_i / dt = 2P_{i-1}P_{i+1} - P_i(P_{i-2} + P_{i+2})$$

- ◆ **Example: 6 states**

- ◆ **Symmetry + normalization:
two-dimensional problem**

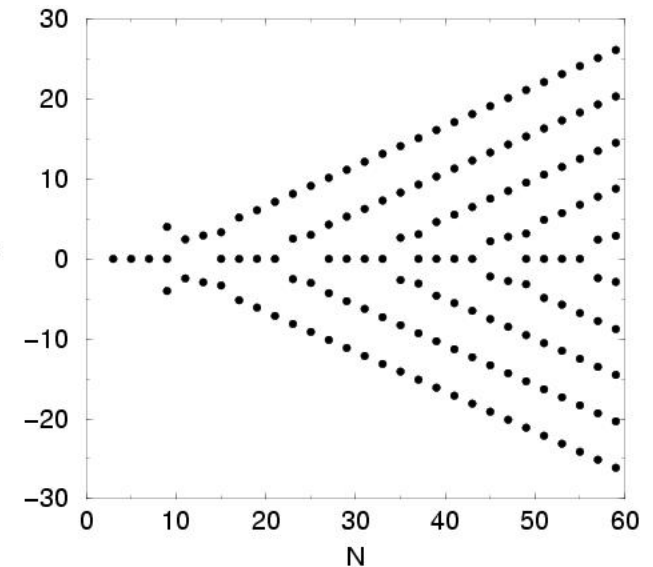


Initial conditions determine final state

Isolated fixed points, lines of fixed points

General features

- ◆ Dissipative system, volume contracts
- ◆ Energy (Lyapunov) function exists: $\langle x^2 \rangle$
- ◆ No cycles or strange attractors
- ◆ Uniform state is unstable (Cahn-Hilliard)



$$P_i = 1 + \phi_i \quad \phi_t + \left(\phi + a\phi_{xx} + b\phi^2 \right)_{xx} = 0$$

Discrete case yields useful insights

Pattern selection

◆ Linear stability analysis

$$P - 1 \propto e^{i(kx + \omega t)} \Rightarrow \omega(k) = \frac{8}{k} \sin \frac{k}{2} - \frac{2}{k} \sin k - 2$$

◆ Fastest growing mode

$$d\omega / dk = 0 \Rightarrow L = 2\pi / k = 2.2515$$

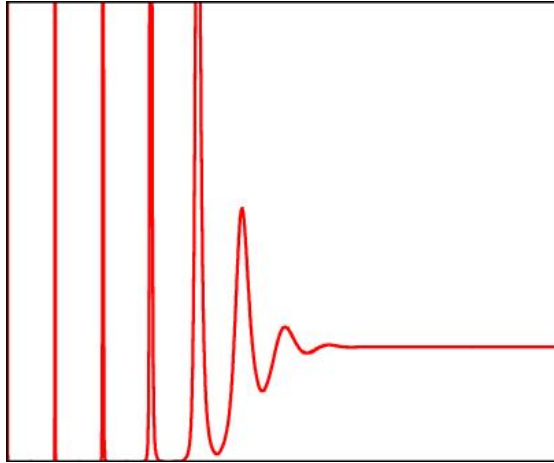
◆ Traveling wave (FKPP extremal selection)

$$d\omega / dk = \text{Im}(\omega) / \text{Im}(k) \Rightarrow L = 2\pi / k = 2.0375$$

Patterns induced by wave propagating from boundary.
However, emerging period is different $L=2.155!$

Pattern selection intrinsically nonlinear

Traveling waves



$$P-1 \propto \exp[-\lambda(x-vt) + i(kx + \omega t)]$$

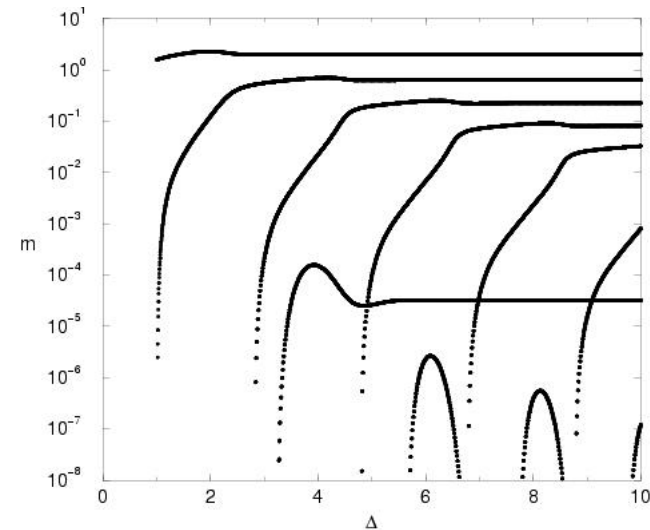
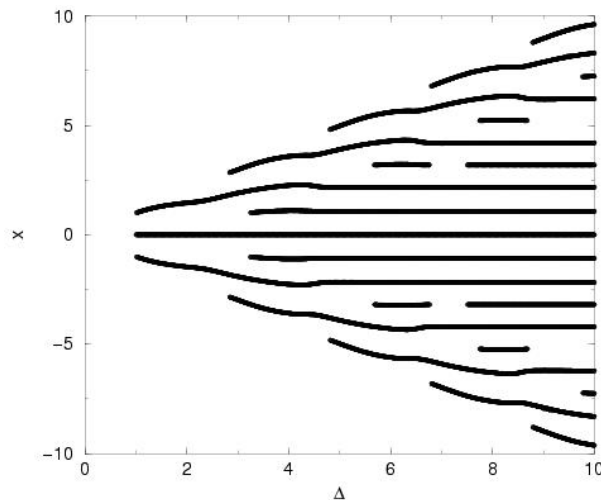
Discrete case

$$L_{\max} = 6$$

$$L = 5.67$$

$$L_{\text{trav wave}} = 5.31$$

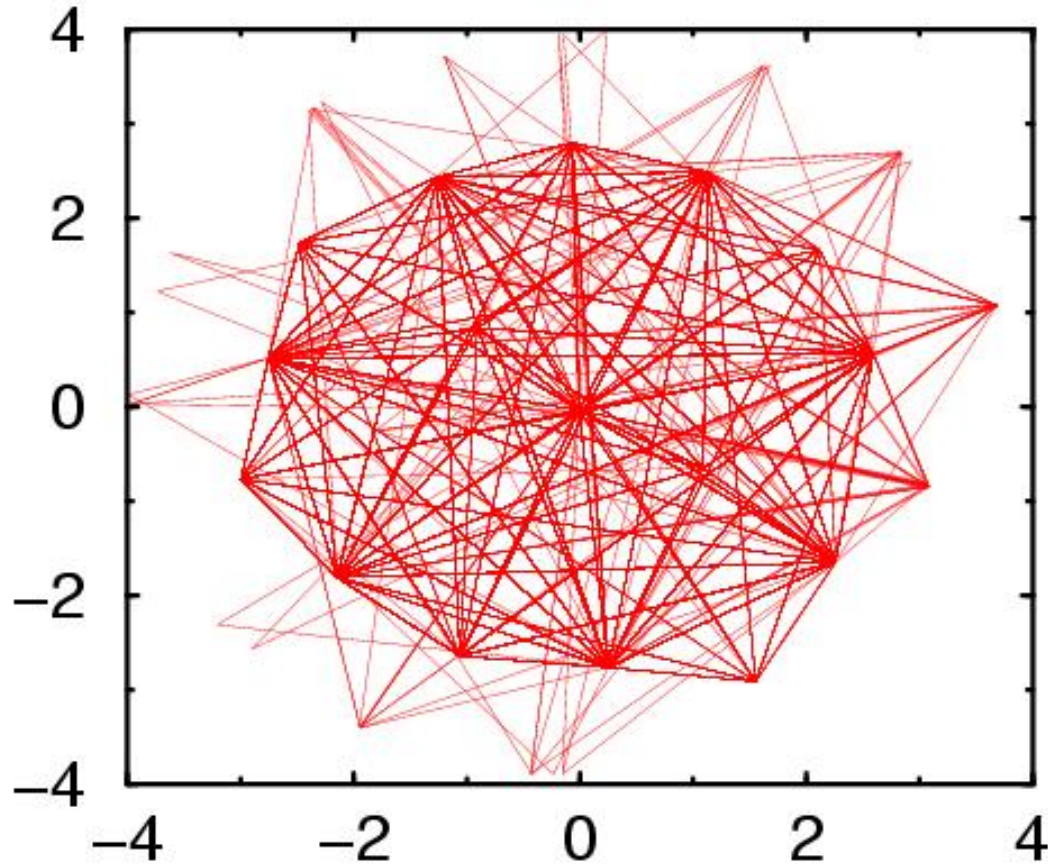
Exponential initial conditions



- ◆ Bifurcations induced at the boundary
- ◆ Periodic structure, nontrivial period
- ◆ Two types of bifurcations
 1. Nucleation of major branch
 2. Nucleation of minor branch

Central cluster is stable

Two kinds of opinions



symmetry breaking, packing

Noisy dynamics

- ◆ Opinions change due to interaction with environment (news, events, editorials)

$$i \rightarrow i \pm 1 \quad \text{with rate } D$$

- ◆ Add diffusion term

$$\lambda \rightarrow \lambda - Dk^2 \quad \Rightarrow \quad \lambda \cong (D_c - D)k^2 + \dots$$

- ◆ Sub-critical noise: slow coarsening (single party)

$$D_m = D / m, \quad m \sim (D_m t)^{1/2} \quad \Rightarrow \quad m \sim (Dt)^{1/3}$$

- ◆ Super-critical noise: flat state stable (no parties)

Conclusions

- ◆ Clusters form via bifurcations
- ◆ Periodic structure
- ◆ Alternating minor-major pattern
- ◆ Central party not always exists
- ◆ Power-law behavior near transitions

E. Ben-Naim, P.L. Krapivsky , and S. Redner, Physica D **183**, 190 (2003)

Outlook

- ◆ Pattern selection criteria
- ◆ Gaps
- ◆ Role of initial conditions, classification
- ◆ Role of spatial dimension, correlations
- ◆ Disorder, noise, inhomogeneities
- ◆ Tiling/Packing in 2D
- ◆ Discord dynamics (seceder model, Halpin-Heally 03)

Many open questions