On the Solutions of the Inelastic Boltzmann Equation

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Talk, papers available from: http://cnls.lanl.gov/~ebn

Analytical and numerical issues on quantum, kinetic, and statistical evolution

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Plan

I. One Dimension

   A. Similarity solutions

   B. Stationary solutions

   C. Hybrid solutions

II. General Dimension

   A. Stationary solutions

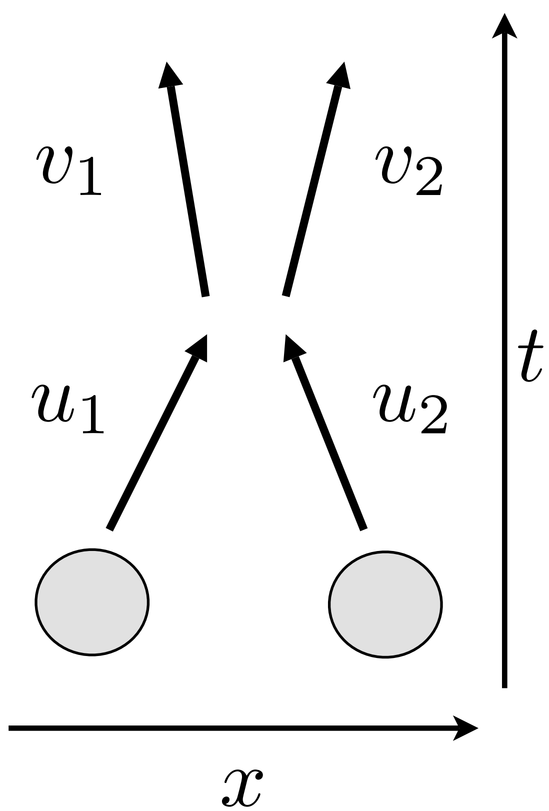
   B. Similarity solutions

Part I: One Dimension

Inelastic collisions

- Relative velocity reduced by $0 \le r < 1$

$$v_1 - v_2 = -r(u_1 - u_2)$$



- Momentum is conserved

$$v_1 + v_2 = u_1 + u_2$$

- Energy is dissipated

$$\Delta E = \frac{1 - r^2}{4}(u_1 - u_2)^2$$

- Limiting cases

$$r = \begin{cases} 0 & \text{completely inelastic } (\Delta E = \text{max}) \\ 1 & \text{elastic } (\Delta E = 0) \end{cases}$$

Inelastic collisions: symmetries

- Galliean invariance

$$v \to v + v_0$$

- Set average velocity is zero

$$\langle v \rangle = 0$$

- Scale invariance

$$v \to \gamma v$$

- Stationary solution

$$P(v) \to \gamma P(\gamma v)$$

The inelastic Boltzmann equation

- Collision rule $\quad r = 1 - 2p \qquad p + q = 1 \qquad 0 < p \le 1/2$

$$(u_1, u_2) \to (pu_1 + qu_2, pu_2 + qu_1)$$

- General collision rate

$$K(v_1, v_2) = |v_1 - v_2|^\lambda \qquad \lambda = \begin{cases} 0 & \text{Maxwell molecules} \\ 1 & \text{Hard spheres} \end{cases}$$

- Boltzmann equation (nonlinear and nonlocal)

$$\frac{\partial P(v)}{\partial t} = \iint du_1 du_2 P(u_1) P(u_2) |u_1 - u_2|^\lambda \left[\delta(v - pu_1 - qu_2) - \delta(v - u_2)\right]$$

collision rate $\qquad$ gain $\qquad$ loss

Theory: non-linear, non-local
energy dissipation, no explicit forcing

The inelastic Boltzmann equation

$$\frac{\partial P(v)}{\partial t} = \iint du_1 du_2 P(u_1) P(u_2) |u_1 - u_2|^{\lambda} \left[\delta(v - pu_1 - qu_2) - \delta(v - u_2)\right]$$

collision rate gain loss

What is the solution of this equation?
What is the nature of the velocity distribution?

The inelastic Maxwell Model (λ=0)

- Constant collision rate

$$\frac{\partial P(v)}{\partial t} = \iint du_1 du_2 P(u_1) P(u_2) |u_1 - u_2|^\lambda \left[\delta(v - pu_1 - qu_2) - \delta(v - u_2)\right]$$

- Moments obey closed equations

EB, Krapivsky 00
Carrillo, Gamba 00

$$T = \langle v^2 \rangle \qquad \frac{dT}{dt} = -\lambda_2 T \qquad \lambda_n = 1 - p^n - q^n$$

- Temperature decays exponentially with time

$$T = T_0\, e^{-\lambda_2 t}$$

- All energy is eventually dissipated

- Trivial steady-state

$$P(v) \to \delta(v)$$

The Fourier transform

- The Fourier transform $F(k) = \int dv\, e^{ikv} P(v,t)$
- Obeys closed, nonlinear, nonlocal equation Krup 67

$$\frac{\partial F(k)}{\partial t} + F(k) = F(pk)F(qk)$$

- Scaling behavior, scale set by temperature

$$F(k,t) \to f\left(ke^{-\lambda t}\right) \qquad \lambda = \frac{\lambda_2}{2}$$

- Nonlinear differential equation

$$-\lambda\, z\, f'(z) + f(z) = f(pz)f(qz) \qquad \begin{aligned} f(0) &= 1 \\ f'(0) &= 0 \end{aligned}$$

- Exact solution

$$f(z) = (1+|z|)e^{-|z|}$$

EB, Krapivsky 00

Similarity solution

- Self-similar form

$$P(v,t) \to e^{\lambda t} p\left(v e^{\lambda t}\right)$$

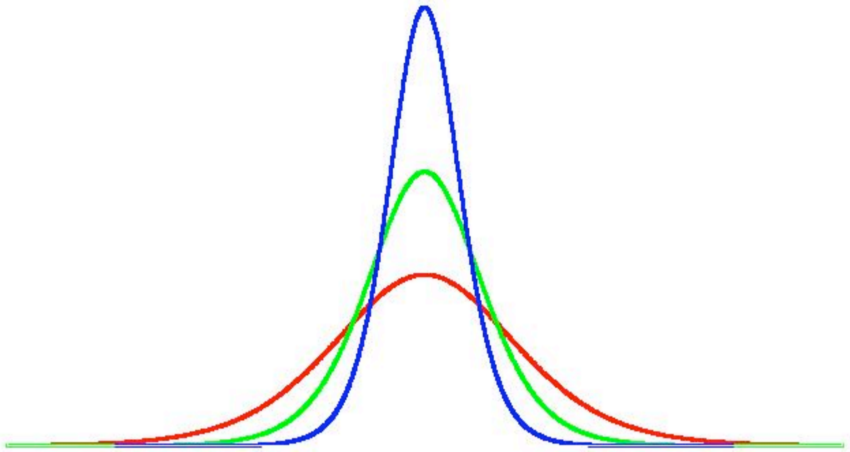
- Obtained by inverse Fourier transform

$$p(w) = \frac{2}{\pi} \frac{1}{\left(1 + w^2\right)^2}$$

- Power-law tail

$$p(w) \sim w^{-4}$$

1. Self-similar solution
2. Power-law tail



Baldassari 02

Homogeneous cooling state: temperature decay (λ>0)

- Energy loss

$$\Delta T \sim (\Delta v)^2$$

- Collision rate

$$\Delta t \sim 1/(\Delta v)^\lambda$$

- Energy balance equation

$$\frac{dT}{dt} \sim -(\Delta v)^{2+\lambda} \qquad \Longrightarrow \qquad \frac{dT}{dt} = -T^{1+\lambda/2}$$

- Temperature decays, system comes to rest

$$T \sim t^{-2/\lambda} \qquad \Longrightarrow \qquad P(v) \to \delta(v)$$

Trivial stationary solution

Haff 82

Homogeneous cooling states: similarity solutions (λ>0)

- Similarity solution

$$P(v,t) = t^{1/\lambda} p(vt^{1/\lambda})$$

- Scaling function: stretched exponential

$$p(w) \sim \exp\left(-|w|^{\lambda}\right)$$

- Overpopulated (with respect to Maxwellian) tails

Esipov, Poeschel 97
Ernst, van Noije 97

Are there nontrivial stationary solutions?

- Stationary Boltzmann equation

$$\frac{\partial P(v)}{\partial t} = \iint du_1 du_2 P(u_1) P(u_2) \underbrace{|u_1 - u_2|^\lambda}_{\text{collision rate}} [\underbrace{\delta(v - pu_1 - qu_2)}_{\text{gain}} - \underbrace{\delta(v - u_2)}_{\text{loss}}]$$

Naive answer: NO!

- According to the energy balance equation

$$\frac{dT}{dt} = -\Gamma$$

- Dissipation rate is positive

$$\Gamma > 0$$

Stationary solutions (λ=0)

- Stationary solutions do exist!

$$F(k) = F(pk)F(qk)$$

- Family of exponential solutions, parametrized by v_0

$$F(k) = \exp(-|k|v_0)$$

- Lorentz/Cauchy distribution:

$$P(v) = \frac{1}{\pi v_0} \frac{1}{1 + (v/v_0)^2}$$

Divergent energy, divergent dissipation rate

EB, Machta 04

Properties of stationary solution

- Perfect balance between collisional loss and gain
- Purely collisional dynamics (no source term)
- Family of solutions: scale invariance $v \to v/v_0$
- Power-law high-energy tail
- Divergent energy, divergent dissipation rate!

Questions about stationary solutions

- How is a steady state consistent with dissipation?
- Are these stationary solutions physical?
- How to simulate numerically?
- How to realize experimentally?
- A family of solutions: which one is selected by dynamics?

The answers to **all** of these questions require understanding dynamics of extreme velocities!

Extreme statistics

- When $v_1 \to \infty$ the binary collision process

$$(v_1, \cancel{v_2}) \to (pv_1 + \cancel{qv_2}, \cancel{pv_2} + qv_1)$$

turns into the linear cascade process

$$v \to (pv, qv)$$

- Cascade: conserves momentum, dissipates energy, doubles number of particles!

- Linear Boltzmann equation for extreme velocities

$$\frac{\partial P(v)}{\partial t} = \frac{1}{p} P\left(\frac{v}{p}\right) + \frac{1}{q} P\left(\frac{v}{q}\right) - P(v)$$

- Steady-state: power-law tail

$$P(v) \sim v^{-2}$$

The linear Boltzmann equation

- For extreme velocities, double integral factorizes

$$
\begin{aligned}
\frac{\partial P(v)}{\partial t} &= \iint du_1 du_2 P(u_1)P(u_2)|u_1-u_2|^\lambda \left[\delta(v-pu_1-pu_2)-\delta(v-u_1)\right] \\
&= \int du_< P(u_<) \int du_> P(u_>)|u_>|^\lambda \left[\delta(v-pu_>)+\delta(v-qu_>)-\delta(v-u_>)\right]
\end{aligned}
$$

- Extreme velocities: linear but nonlocal equation

$$
\frac{\partial P(v)}{\partial t} = |v|^\lambda \left[\frac{1}{p^{1+\lambda}} P\left(\frac{v}{p}\right) + \frac{1}{q^{1+\lambda}} P\left(\frac{v}{q}\right) - P(v)\right]
$$

- Stationary solution: power-law distribution

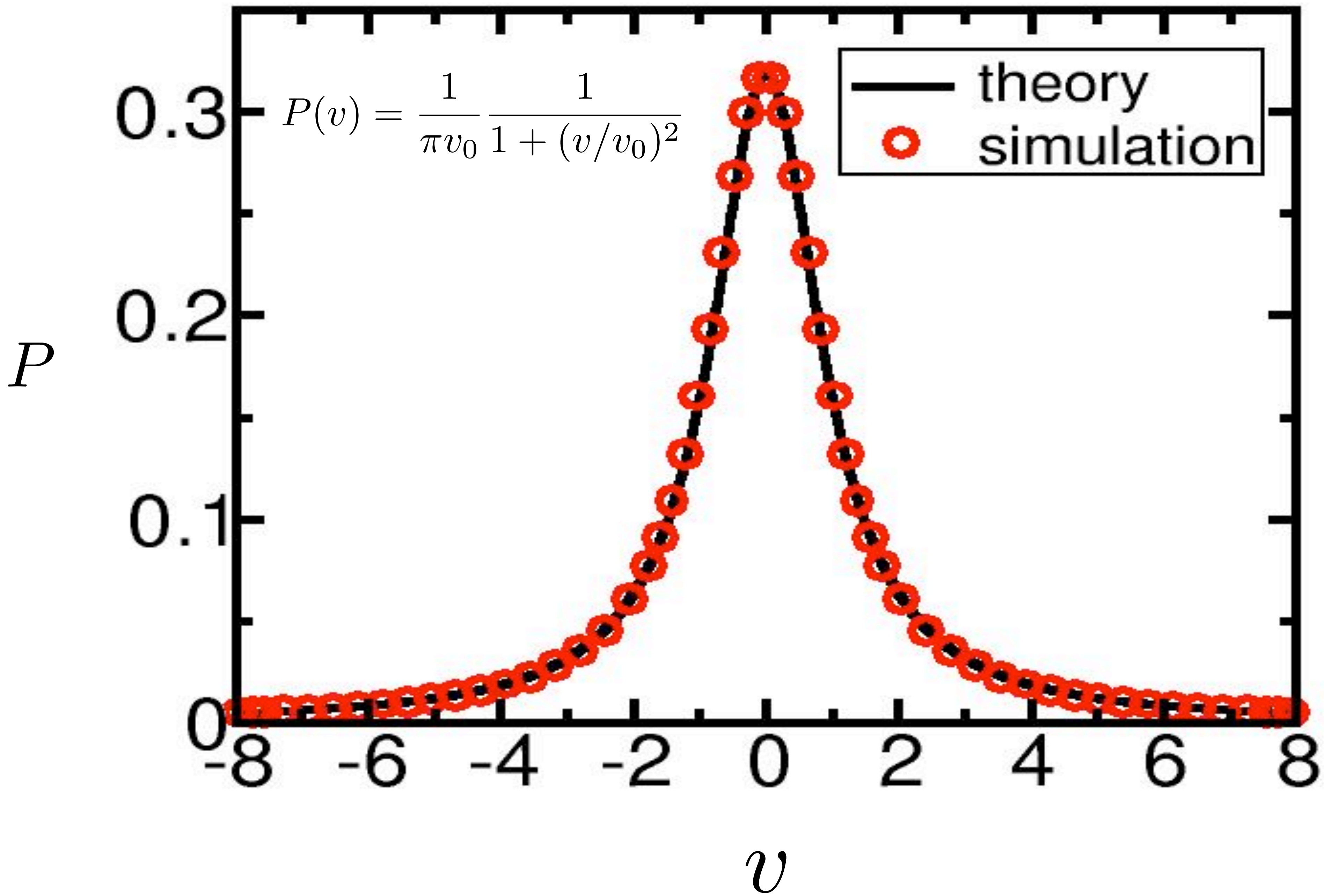
$$
P(v) \sim v^{-2-\lambda}
$$

Stationary solution: always power-law
Hard spheres and Maxwell Molecules

Numerical solution

- Force constant energy
- Inject energy:
  - At extremely large scales
  - With extremely small rate
- "Lottery" implementation:
  - Keep track of total energy dissipated, E_T
  - With small rate, boost **one** particle by E_T

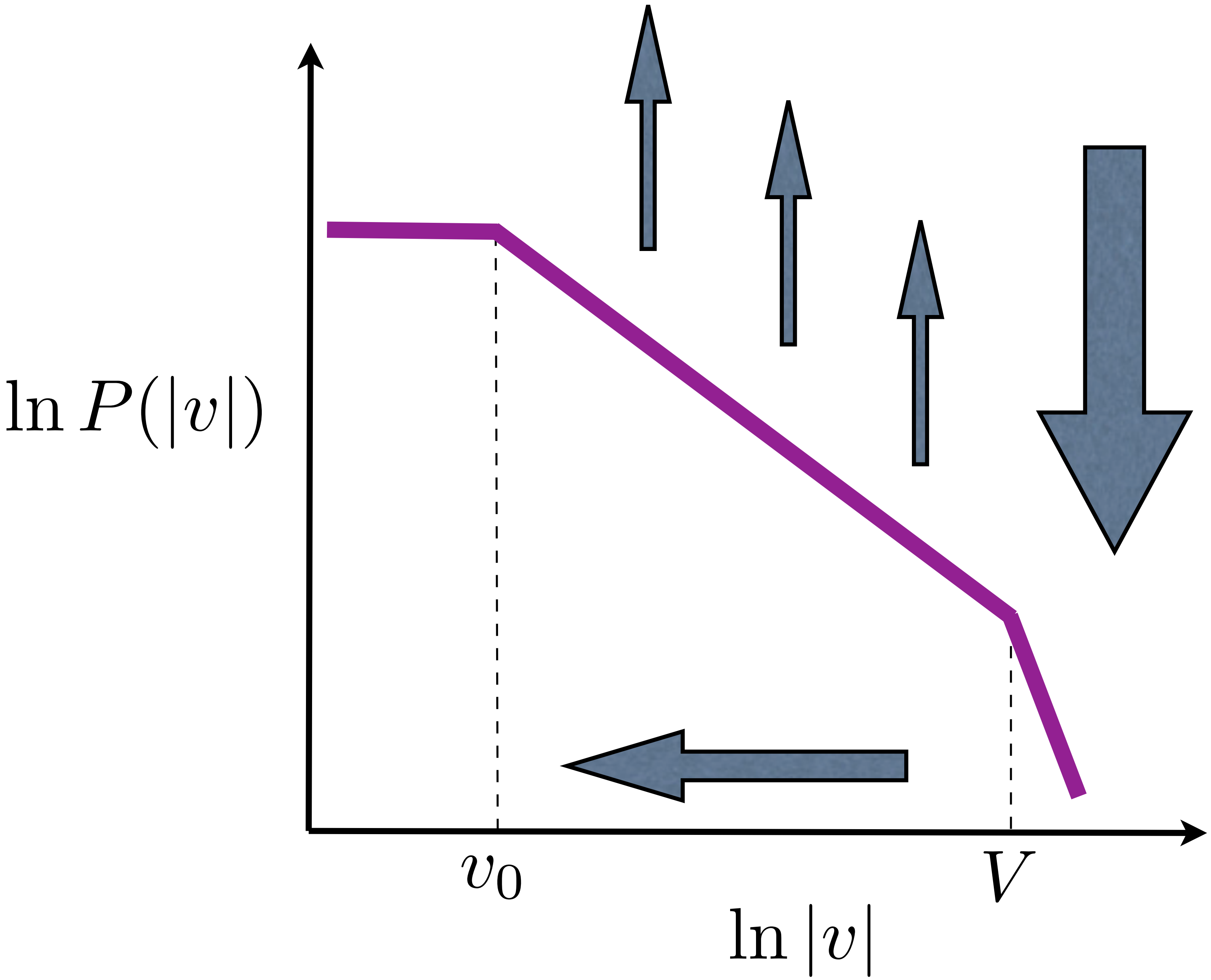
Lottery Monte Carlo simulation



Excellent agreement between theory and simulation

Injection selects one solution with one particular v_0 !!!

Injection, Cascade, Dissipation



Experimental realization?
Energetic particle "shot" into static medium

Energy balance
$\Gamma \sim \gamma V^2$

- Energy is injected only at large velocity scales!
- Energy cascades from large velocities to small velocities
- Energy dissipated at small velocity scales

Energy balance

- Energy injection rate γ
- Energy injection scale V
- Typical velocity scale v_0
- Balance between energy injection and dissipation

$$\gamma \sim V^{\lambda}(V/v_0)^{d-\sigma}$$

- For "lottery" injection: injection scale diverges with injection rate

$$V \sim \begin{cases} \gamma^{-1/(2-\lambda)} & \sigma < d+2 \\ \gamma^{-1/(\sigma-d-\lambda)} & \sigma > d+2 \end{cases}$$

Energy injection selects stationary solution

Hybrid solutions

EB, Machta 05

- Suppose the system is stationary; then, we turn off energy injection. The system will start cooling
- Hybrid solution
  - Stationary at small velocities $v \ll V(t)$
  - Self-similar at large velocities $v \gg V(t)$

$$P(v,t) \sim v^{-2-\lambda} \phi\left(v t^{1/\lambda}\right)$$

- Cutoff velocity decays following Haff law $V(t) \sim t^{-1/\lambda}$
- Scaling solution $p = q = 1/2$

$$\phi(x) = \sum_{n=1}^{\infty} a_n \exp\left[-(2^n x)^\lambda\right] \qquad a_n = \prod_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{1}{1 - 2^{\lambda(n-k)}}$$

Hybrid between steady-state and time dependent state

Extreme statistics

- Scaling function

$$\phi(x) = \sum_{n=1}^{\infty} a_n \exp\left[-(2^n x)^\lambda\right] \qquad a_n = \prod_{\substack{k=1 \\ k\neq n}}^{\infty} \frac{1}{1-2^{\lambda(n-k)}}$$

- Large velocities: stretched exponential (like free cooling)

$$\phi(x) \sim \exp\left(-x^\lambda\right) \qquad \text{as} \qquad x \to \infty$$

- Small velocities: log-normal distribution

$$1-\phi(x) \sim \exp\left[-(\ln x)^2\right] \qquad \text{as} \qquad x \to 0$$

Hybrid between steady-state and time dependent state

Maxwell Model (λ=0) only unsolved case!

Obtaining the scaling function (λ=0, p=1/2)

- Substitute scaling form into linear equation

$$\phi'(x) = 2\left[\phi(2x) - \phi(x)\right]$$

- Use Laplace transform

$$(2+s)\phi(s) = 1 + \phi(s/2) \qquad \phi(s) = \int dx\, e^{-sx}\, \phi(x)$$

- Make a further transformation

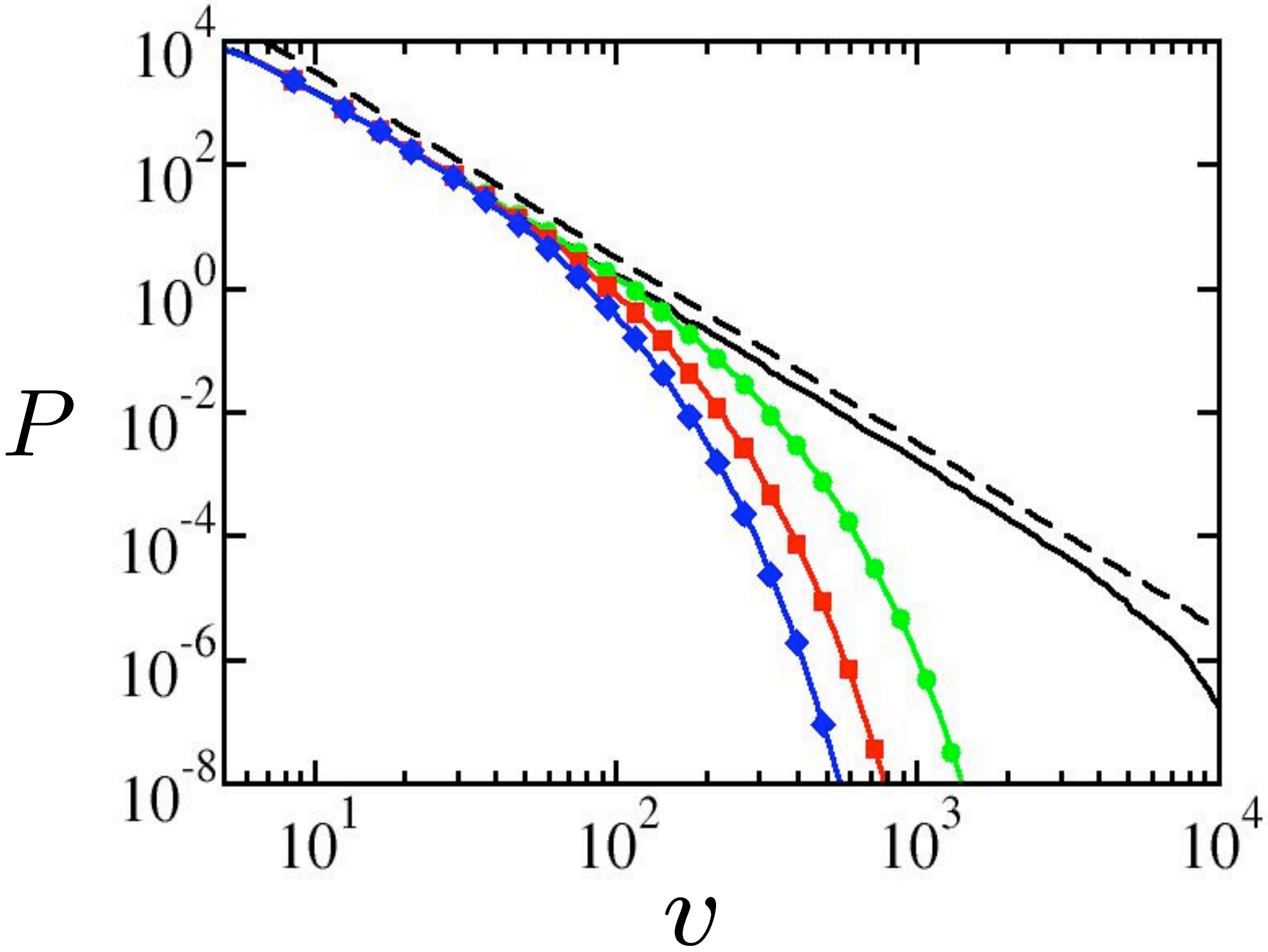
$$u(s) = \frac{1}{1+s/2}\, u(s/2) \qquad u(s) = \frac{1-\phi(s)}{s}$$

- Iterative solution through an infinite product

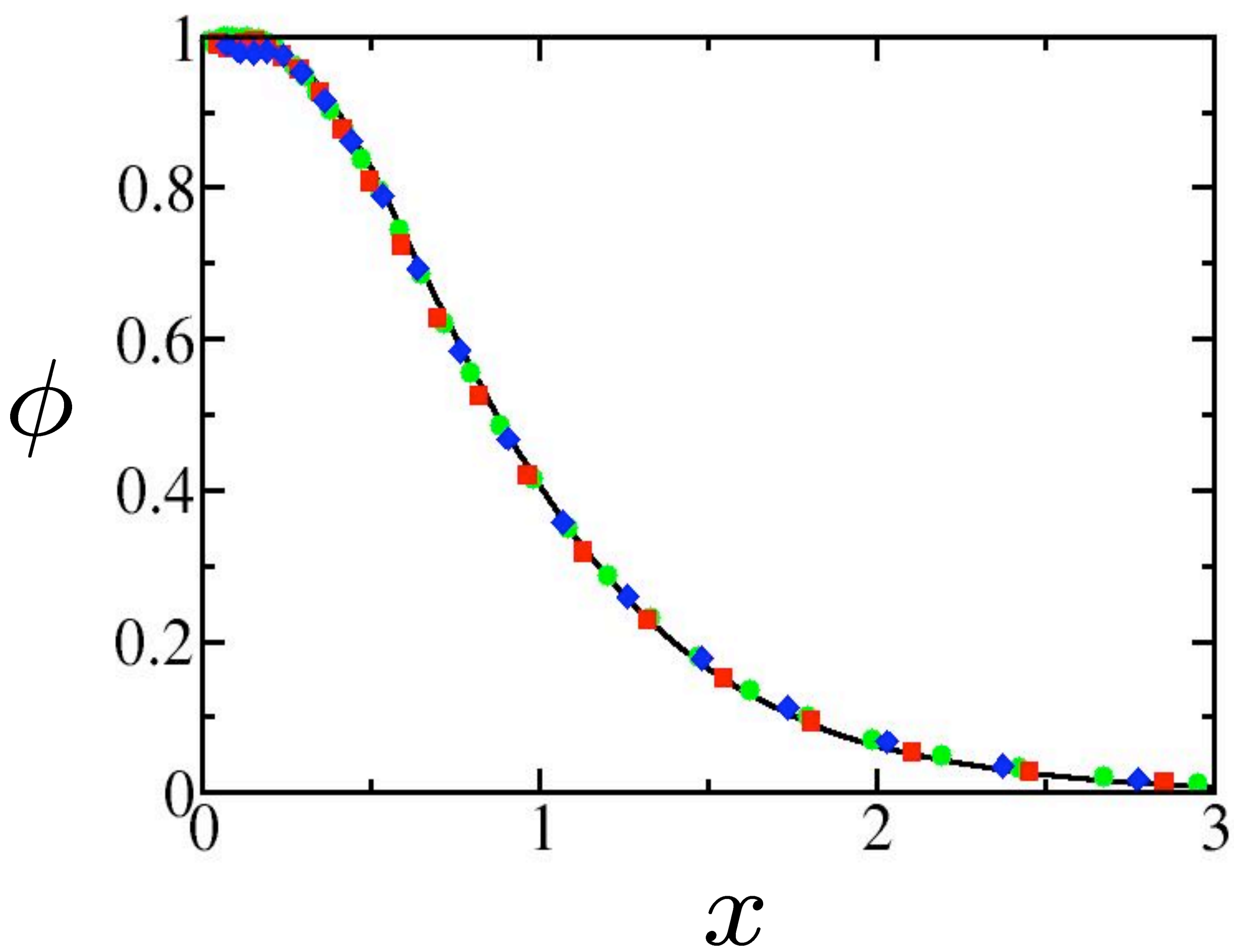
$$\phi(s) = \frac{1}{s}\left(1 - \prod_{n=1}^{\infty} \frac{1}{1+\frac{s}{2^n}}\right)$$

Numerical confirmation

Velocity distribution



Scaling function



A third family of solutions does exist

Part II: General Dimensions

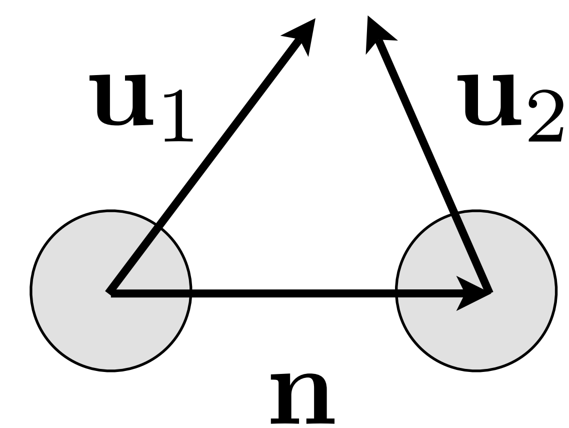
Inelastic collisions

- Normal relative velocity reduced by $0 \leq r < 1$

$$(\mathbf{v}_1 - \mathbf{v_2}) \cdot \mathbf{n} = -r(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{n} \qquad r = 1 - 2p$$

- Momentum conservation

$$\mathbf{v}_1 + \mathbf{v_2} = \mathbf{u}_1 + \mathbf{u}_2$$



- Energy loss

$$\Delta E = \frac{1 - r^2}{4} \left[(\mathbf{u}_1 - \mathbf{u_2}) \cdot \mathbf{n}\right]$$

- Collision rate

$$K(v_1, v_2) = |(\mathbf{v}_1 - \mathbf{v}_2) \cdot \mathbf{n}|^\lambda \qquad \lambda = \begin{cases} 0 & \text{Maxwell molecules} \\ 1 & \text{Hard spheres} \end{cases}$$

Collision rules

- Collision process

$$(\mathbf{u}_1, \mathbf{u}_2) \to (\mathbf{v}_1, \mathbf{v}_2)$$

- Explicit collision rule for all velocities

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{u}_1 - (1-p)(\mathbf{u}_1 - \mathbf{u}_2) \cdot \hat{\mathbf{n}}\, \hat{\mathbf{n}} \\ \mathbf{v}_2 &= \mathbf{u}_2 - (1-p)(\mathbf{u}_2 - \mathbf{u}_1) \cdot \hat{\mathbf{n}}\, \hat{\mathbf{n}} \end{aligned}$$

- Cascade process

$$\mathbf{u} \to (\mathbf{v}_1, \mathbf{v}_2)$$

- Explicit cascade rules for extremely large velocities

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{u} - (1-p)\, \mathbf{u} \cdot \hat{\mathbf{n}}\, \hat{\mathbf{n}} \\ \mathbf{v}_2 &= (1-p)\mathbf{u} \cdot \hat{\mathbf{n}}\, \hat{\mathbf{n}} \end{aligned}$$

The Boltzmann equation

- Full nonlinear equation

$$\frac{\partial P(\mathbf{v})}{\partial t} = \iiint d\hat{\mathbf{n}}\, d\mathbf{u}_1\, d\mathbf{u}_2 |(\mathbf{u}_1 - \mathbf{u}_2)\cdot\hat{\mathbf{n}}|^{\lambda} P(\mathbf{u}_1)P(\mathbf{u}_2)[\delta(\mathbf{v}-\mathbf{v}_1) - \delta(\mathbf{v}-\mathbf{u}_1)]$$

angular integration with uniform measure

- Linear equation for large velocities

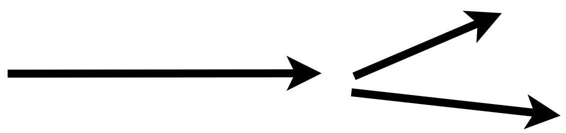
$$\frac{\partial P(\mathbf{v})}{\partial t} = \iint d\hat{\mathbf{n}}\, d\mathbf{u}\, |\mathbf{u}\cdot\hat{\mathbf{n}}|^{\lambda} P(\mathbf{u})[\delta(\mathbf{v}-\mathbf{v}_1) + \delta(\mathbf{v}-v_2) - \delta(\mathbf{v}-\mathbf{u})]$$

- Formulate linear equation for velocity magnitude

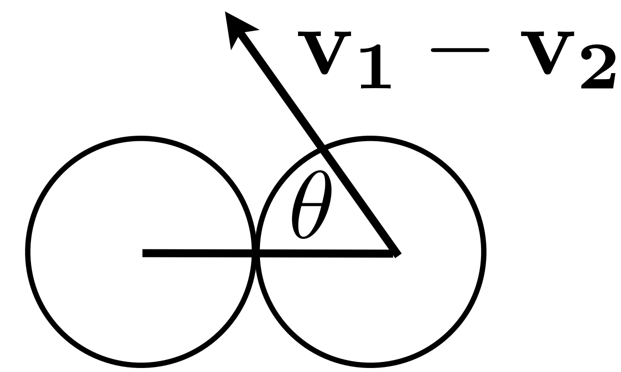
$$P(v) \qquad v \equiv |\mathbf{v}|$$

Extreme statistics

- Collision process: large velocities



$$v \to (\alpha v, \beta v)$$



- Stretching parameters related to impact angle

$$\alpha = (1-p)\cos\theta \qquad \beta = \left[1-(1-p^2)\cos^2\theta\right]^{1/2}$$

- Energy decreases, velocity magnitude increases

$$\alpha^2+\beta^2 \leq 1 \qquad \alpha+\beta \geq 1$$

- Linear Boltzmann equation $\qquad \langle\,\rangle \equiv \int d\mathbf{n}$

$$\frac{\partial P(v)}{\partial t} = \left\langle v^\lambda \cos^{\lambda/2}\theta \left(\frac{1}{\alpha^{d+\lambda}} P\left(\frac{v}{\alpha}\right) + \frac{1}{\beta^{d+\lambda}} P\left(\frac{v}{\beta}\right) - P(v) \right) \right\rangle.$$

Similarity solutions

- Velocity distribution always has power-law tail

$$P(v) \sim v^{-\sigma}$$

$$\langle \big(\alpha^{\sigma-d-\lambda} + \beta^{\sigma-d-\lambda} - 1\big) \cos^{\lambda/2}\theta \rangle = 0$$

- Characteristic exponent varies with **all** parameters

$$\frac{1 - {}_2F_1\left(\frac{d+\lambda-\sigma}{2}, \frac{\lambda+1}{2}, \frac{d+\lambda}{2}, 1-p^2\right)}{(1-p)^{\sigma-d-\lambda}} = \frac{\Gamma(\frac{\sigma-d+1}{2})\Gamma(\frac{d+\lambda}{2})}{\Gamma(\frac{\sigma}{2})\Gamma(\frac{\lambda+1}{2})}$$

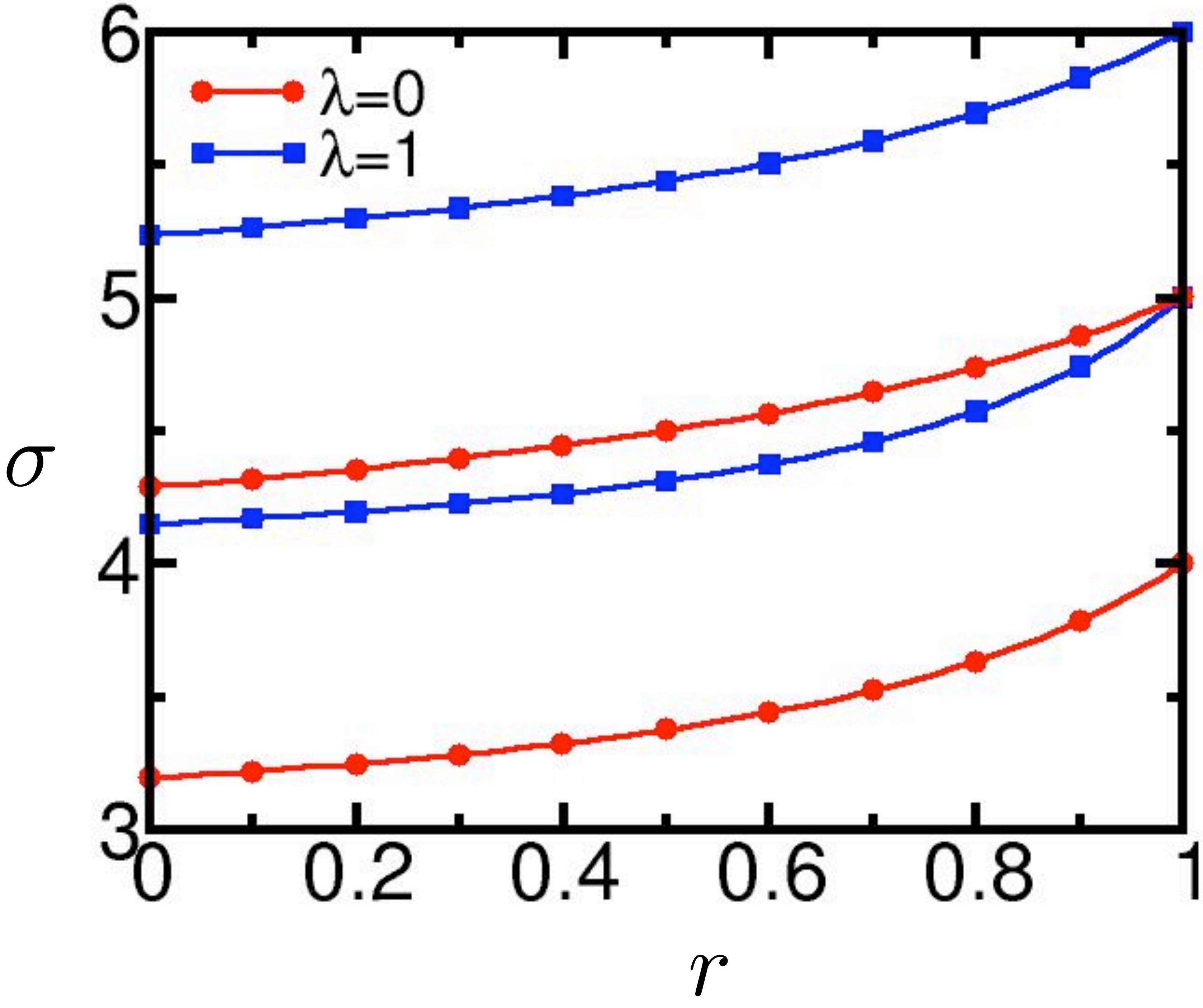
- Range of exponent

$$1 \le \sigma - d - \lambda \le 2$$

Dissipation rate is always divergent!
Energy may be finite or infinite

EB, Machta 05

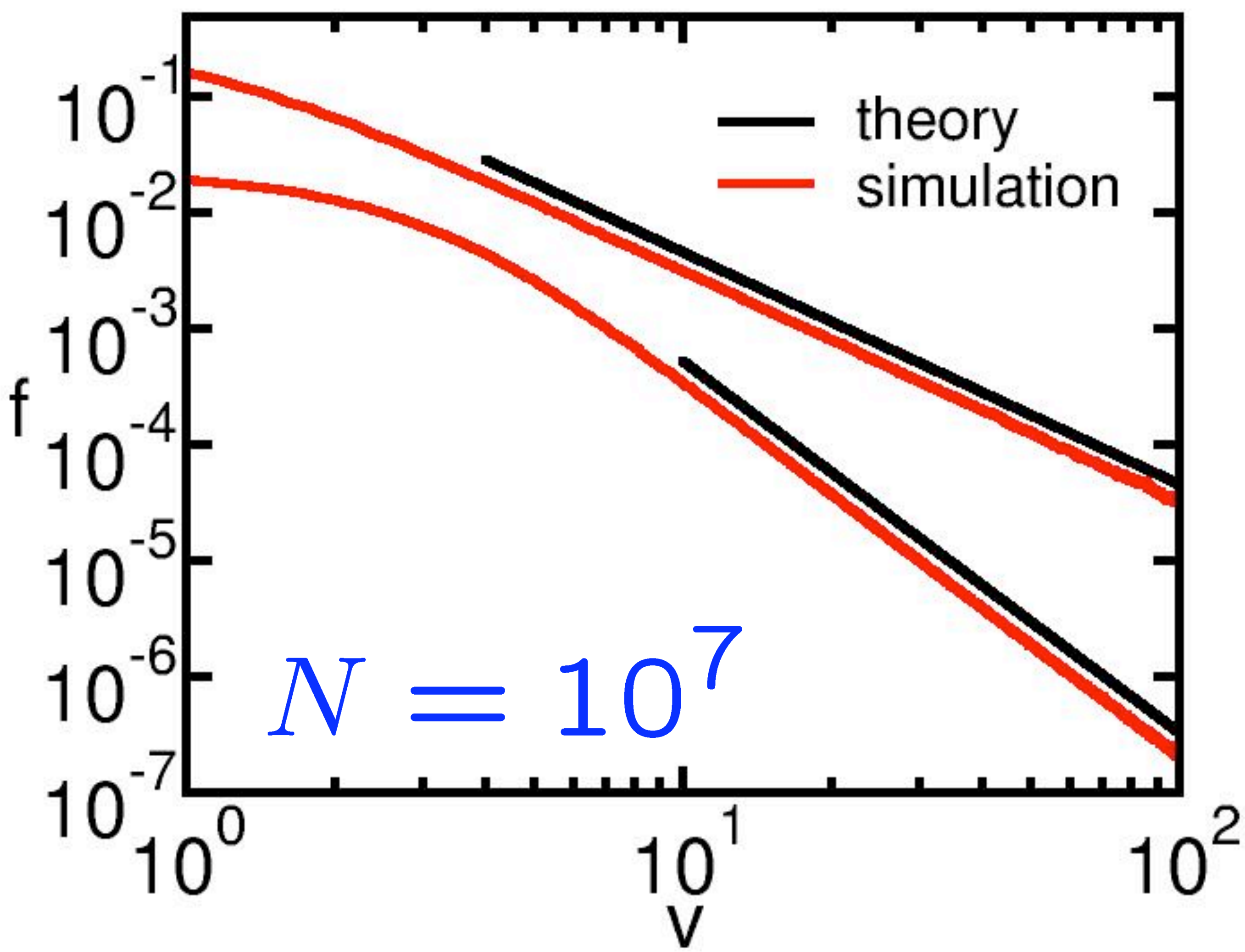
The characteristic exponent σ (d=2,3)



σ varies with spatial dimension, collision rules

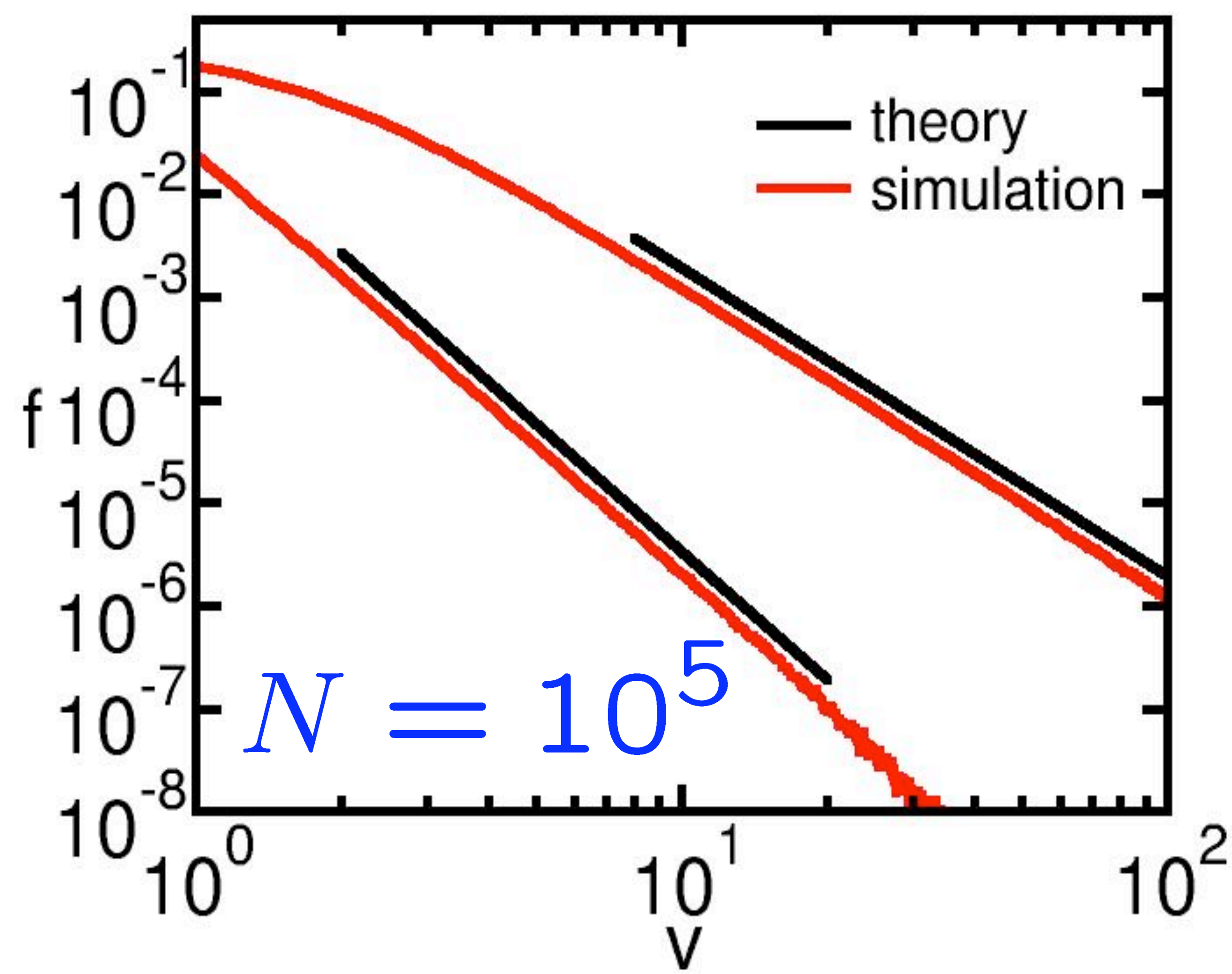
Monte Carlo simulations

Maxwell molecules (1D, 2D) infinite energy



d	theory	simulation
1	2	1.995
2	3.19520	3.19

Hard spheres (1D, 2D) finite energy



d	theory	simulation
1	3	2.994
2	4.14922	4.15

Similarity solution (Maxwell Molecules)

- Temperature follows from full nonlinear equation

$$T = T_0 \exp(-\lambda_2 t) \qquad \lambda_2 = \frac{2p(1-p)}{d}$$

- Substitute similarity form

$$P(v,t) \to e^{(d-1)\lambda t} p\left(v e^{\lambda t}\right) \qquad \lambda = \lambda_2/2$$

- Into linear Boltzmann equation

$$\frac{\partial P(v)}{\partial t} = \left\langle \frac{1}{\alpha^d} P\left(\frac{v}{\alpha}\right) + \frac{1}{\beta^d} P\left(\frac{v}{\beta}\right) - P(v) \right\rangle$$

- Linear equation for scaling function

$$\lambda(d-1)p(w) + \lambda\, w\, p'(w) = \left\langle \frac{1}{\alpha^d} p\left(\frac{w}{\alpha}\right) + \frac{1}{\beta^d} p\left(\frac{w}{\beta}\right) - p(w) \right\rangle$$

- Power-law tail

$$p(w) \sim w^{-\sigma}$$

Characteristic exponent

EB, Krapivsky 01
Brito, Ernst 01

- Velocity distribution always has power-law tail

$$p(w) \sim w^{-\sigma}$$

- Exponent is solution of transcendental equation

$$1 - p(1-p)\frac{\sigma - d}{d} = {}_2F_1\left[\frac{d-\sigma}{2}, \frac{1}{2}; \frac{d}{2}; 1-p^2\right] + (1-p)^{\sigma-d}\,\frac{\Gamma\left(\frac{\sigma-d+1}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{\sigma}{2}\right)\,\Gamma\left(\frac{1}{2}\right)}$$

EB, Krapivsky 01

- Transparent in terms of stretching parameters

$$\lambda\left[(d-1)-\sigma\right] = \left\langle \alpha^{\sigma-d} + \beta^{\sigma-d} - 1 \right\rangle$$

- Energy is finite

Linear analysis for large velocities transparent
(compare small wave number Fourier analysis)

Similarity solutions (λ>0)

- Similarity solution

$$P(v,t) \simeq t^{(d-1)/\lambda} p\left(vt^{1/\lambda}\right)$$

- Scaling function: stretched exponential

$$p(w) \sim \exp\left(-|w|^{\lambda}\right)$$

- Overpopulated (with respect to Maxwellian) tails

Esipov, Poeschel 97
Ernst 97

Summary

- Time dependent solution $P(v,t) \simeq t^{1/\lambda} p\left(vt^{1/\lambda}\right)$
  - Temperature characterizes the distribution, free cooling
  - Shape of velocity distribution invariant after suitable rescaling
  - Straightforward numerical implementation, questionable relevance to experiments
- Stationary solution $P_s(v) \sim v^{-\sigma}$
  - Dissipation rate divergent, energy finite or divergent
  - Can be realized using energy injection but only up to large scale
  - Numerically: lottery monte carlo
  - Experiment: rare but powerful injection of energetic particles
- Hybrid solution $P(v,t) \simeq P_s(v)\phi\left(vt^{1/\lambda}\right)$
  - Stationary at small scales
  - Self-similar at large scales

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