The Inelastic Maxwell Model

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III Higher dimensions

E. Ben-Naim and P. L. Krapivsky, Lecture Notes in Physics, *cond-mat/0301238*.

The Elastic Maxwell Model

J.C. Maxwell, Phil. Tran. Roy. Soc **157**, 49 (1867)

- Infinite particle system
- Binary collisions
- Random collision partners
- Random impact directions $\mathbf{n}$
- Elastic collisions $(\mathbf{g} = \mathbf{v_1} - \mathbf{v_2})$

$$\mathbf{v_1} \to \mathbf{v_1} - \mathbf{g} \cdot \mathbf{n}\, \mathbf{n}$$

- Mean-field collision process
- Purely Maxwellian velocity distributions

$$P(\mathbf{v}) = \frac{1}{(2\pi T)^{d/2}} \exp\left(-\frac{v^2}{2T}\right)$$

What about inelastic, dissipative collisions?

The Inelastic Maxwell Model (1D)

- **Inelastic collisions** $r = 1 - 2\epsilon$

$$v_1 = \epsilon u_1 + (1-\epsilon) u_2$$

- **Boltzmann equation** (collision rate=1)

$$\frac{\partial P(v,t)}{\partial t} = \int\int du_1 du_2 P(u_1,t) P(u_2,t) \left[\delta(v - v_1) - \delta(v - u_1)\right]$$

- **Fourier transform** $F(k,t) = \int dv e^{ikv} P(v,t)$
- **Evolution**

$$\begin{aligned} &\frac{\partial}{\partial t} F(k,t) + F(k,t) = \\ &= \int\int\int dv du_1 du_2 e^{ikv} P(u_1,t) P(u_2,t) \\ &\qquad \times \delta(v - \epsilon u_1) \delta(v - (1-\epsilon) u_2) \\ &= \int du_1 e^{i\epsilon k u_1} P(u_1,t) \int du_2 e^{i(1-\epsilon) k u_2} P(u_2,t) \end{aligned}$$

- **Closed equations**

$$\frac{\partial}{\partial t} F(k,t) + F(k,t) = F[\epsilon k, t] F[(1-\epsilon) k, t]$$

Similarity solutions

- **Scaling of isotropic velocity distribution**

$$P(\mathbf{v},t) \to \frac{1}{T^{d/2}}\Phi\left(\frac{|\mathbf{v}|}{T^{1/2}}\right) \quad \text{or} \quad F(k,t) \to f\left(kT^{1/2}\right)$$

- **Nonlinear and nonlocal** $(T = T_0 \exp^{-2\epsilon(1-\epsilon)t})$

$$-\epsilon(1-\epsilon)f'(x) + f(x) = f(\epsilon x) f\left((1-\epsilon)x\right)$$

- **Exact solution**

$$f(x) = (1+x)\, e^{-x} \cong 1 - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$$

- **Lorentzian^2 velocity distribution**

$$\Phi(v) = \frac{2}{\pi}\frac{1}{(1+v^2)^2}$$

- **Algebraic tail** Baldassari 2001

$$\Phi(v) \sim v^{-4} \qquad w \gg 1$$

Universal scaling function, exponent

Algebraic Tails

- **Velocity distribution ($v \to \infty$)**

$$P(v,t) \sim v^{-\sigma}$$

- **Fourier transform ($k \to 0$)**

$$\begin{aligned} F(k,t) &= \int dv e^{ikv} v^{-\sigma} \\ &\sim k^{\sigma-1} \int d(kv) e^{ikv} (kv)^{-\sigma} \\ &\sim \text{const}\, k^{\sigma-1} \end{aligned}$$

- **Non-analytic small-k behavior**

$$F(k,t) = F_{\text{reg}}(k) + F_{\text{sing}}(k) \qquad F_{\text{sing}}(k) \sim k^{\sigma-1}$$

The Forced Case

- Add white noise

$$\frac{dv_j}{dt}\Big|_{\text{heat}} = \eta_j(t) \qquad \langle \eta_i(t)\eta_j(t')\rangle = 2D\delta_{ij}\delta(t-t')$$

- Diffusion in velocity space

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + Dk^2$$

- Steady state solution $\frac{\partial}{\partial t} \equiv 0$

$$(1+Dk^2)P(k) = P(\epsilon k)P((1-\epsilon)k)$$

- Recursive solution

$$\begin{aligned} P(k) &= (1+Dk^2)^{-1}P(\epsilon k)P((1-\epsilon)k) \\ &= (1+Dk^2)^{-1}(1+\epsilon^2 Dk^2)^{-1}(1-(1-\epsilon)^2 Dk^2)^{-1}\cdots \end{aligned}$$

- Product solution

$$\hat{P}_\infty(k) = \prod_{i=0}^{\infty}\prod_{j=0}^{i}\left[1+\epsilon^{2j}(1-\epsilon)^{2(i-j)}Dk^2\right]^{-\binom{i}{j}}.$$

Overpopulated high-energy tails

- Pole closest to origin $k = i/\sqrt{D}$ dominates

$$P(k) \propto \frac{1}{1+Dk^2} \propto \frac{1}{(k+i/\sqrt{D})}\frac{1}{(k-i/\sqrt{D})}$$

- Exponential tail

$$P(v) \simeq A(\epsilon)\exp(-|v|/\sqrt{D}) \qquad |v| \to \infty$$

- Direct from equation (ignore gain term)

$$D\frac{\partial^2}{\partial^2 v}P(v) \cong -P(v) \qquad |v| \to \infty$$

- Residue at pole yields prefactor

$$A(\epsilon) \propto \exp(\pi^2/12p)$$

$$\boxed{\textbf{Non-Maxwellian}}$$

Still, Maxwellians may resurface

- Steady state equation

$$\ln(1+Dk^2)+\ln P(k)-\ln P(\epsilon k)+\ln P((1-\epsilon)k) = 0$$

- Cumulant expansion

$$\ln P(k) = \sum_{n=1} n^{-1}(-Dk^2)^n \psi_n$$

- Rewrite $\ln(1+Dk^2) = -\sum_n n^{-1}(-Dk^2)^n$

$$\sum_{n=1} n^{-1}(-Dk^2)^n[1+\psi_n(1-\epsilon^{2n}-(1-\epsilon)^{2n})] = 0$$

- Fluctuation-dissipation relations

$$\psi_n = [1-(1-\epsilon)^{2n}+\epsilon^{2n}]^{-1}$$

- Small dissipation limit $\epsilon \to 0$

$$P(k) = \exp(-\epsilon^{-1}Dk^2/2) \qquad k \gg \epsilon$$

- Maxwellian for range of velocities

$$P(v) \approx \exp(-\epsilon v^2/D) \qquad v \ll \epsilon^{-1}$$

The small dissipation limit $\epsilon \to 0$

- Maxwell model

$$P(v) \sim \begin{cases} \exp(-\epsilon^{-1} v^2/D) & v \ll \epsilon^{-1} \\ \exp(-|v|/\sqrt{D}) & v \gg \epsilon^{-1} \end{cases}$$

- Boltzmann equation

$$P(v) \sim \begin{cases} \exp(-\epsilon^{a} v^3) & v \ll \epsilon^{-b} \\ \exp(-|v|^{3/2}) & v \gg \epsilon^{-b} \end{cases}$$

- **Limits $v \to \infty$, $\epsilon \to 0$ do not commute!**
- $\epsilon \to 0$ is singular

$$-\epsilon(1-\epsilon)x f'(x) + f(x) = f(\epsilon x) f((1-\epsilon)x)$$

- **Small-ϵ Expansions may not be useful!**

Velocity Moments

- The moments

$$M_n(t) = \int dv v^n P(v,t)$$

- Closed evolution equations

$$\frac{d}{dt}M_n + \lambda_n M_n = \sum_{m=1}^{n-1} \binom{n}{m} \epsilon^m (1-\epsilon)^{n-m} M_m M_{n-m}$$

- Eigenvalues

$$\lambda_n = 1 - \epsilon^n - (1-\epsilon)^n$$

- Asymptotic behavior $\lambda_n > \lambda_m + \lambda_{n-m}$

$$M_n \sim \exp(-\lambda_n t)$$

- Multiscaling

$$M_n / M_2^{n/2} \to \infty \qquad t \to \infty$$

algebraic tails causes multiscaling

Velocity Autocorrelations

- The velocity autocorrelation function

$$A(t_w, t) = \langle \mathbf{v}(t_w) \cdot \mathbf{v}(t) \rangle$$

- Linear evolution equation

$$T^{-1/2} \frac{d}{dt} A(t_w, t) = -(1-\epsilon) A(t_w, t)$$

- Nonuniversal ϵ-dependent decay

$$A(t_w, t) = A_0 [1 + t_w/t_0]^{-2+1/\epsilon} [1 + t/t_0]^{-1/\epsilon}$$

- Memory of initial velocity

$$A(t) \equiv A(0, t) \sim t^{-1/\epsilon}$$

- Logarithmic spreading ("self-diffusion")

$$\langle |\mathbf{x}(t) - \mathbf{x}(0)|^2 \rangle \sim \sqrt{\ln t}$$

$$\boxed{\textbf{Memory/Aging - } A(t_w, t) \neq f(t - t_w)}$$

Higher Dimensions

- **Inelastic collisions** $r = 1 - 2\epsilon$

$$\mathbf{v}_{1,2} = \mathbf{u}_{1,2} \mp (1-\epsilon)\,(\mathbf{g}\cdot\mathbf{n})\,\mathbf{n}$$

- **Boltzmann equation** (collision rate=1)

$$\frac{\partial P(\mathbf{v},t)}{\partial t} = \int d\mathbf{n} \int d\mathbf{u}_1 \int d\mathbf{u}_2\, P(\mathbf{u}_1,t)\, P(\mathbf{u}_2,t) \times \Big\{ \delta\left(\mathbf{v}-\mathbf{v}_1\right) - \delta\left(\mathbf{v}-\mathbf{u}_1\right) \Big\}$$

- **Fourier transform** Krupp 1967

$$F(\mathbf{k},t) = \int d\mathbf{v} e^{i\mathbf{k}\cdot\mathbf{v}} P(\mathbf{v},t)$$

- **Closed equations** $\mathbf{q} = (1-\epsilon)\mathbf{k}\cdot\mathbf{n}\,\mathbf{n}$

$$\frac{\partial}{\partial t} F(\mathbf{k},t) + F(\mathbf{k},t) = \int d\mathbf{n}\, F\left[\mathbf{k}-\mathbf{q},t\right] F\left[\mathbf{q},t\right],$$

$\boxed{\textbf{Theory is analytically tractable}}$

Scaling, Nontrivial Exponents

- **Freely cooling case**

$$T = \langle v^2 \rangle = T_0 \exp(-\lambda t) \qquad \lambda = 2\epsilon(1-\epsilon)/d$$

- **Governing equation** $x = k^2 T$

$$-\lambda\, x\, \Phi'(x) + \Phi(x) = \int d\mathbf{n}\, \Phi(x\xi)\, \Phi(x\eta)$$

$$\xi = 1 - (1-\epsilon^2)\cos^2\theta, \qquad \eta = (1-\epsilon)^2 \cos^2\theta$$

- **Power-law tails**

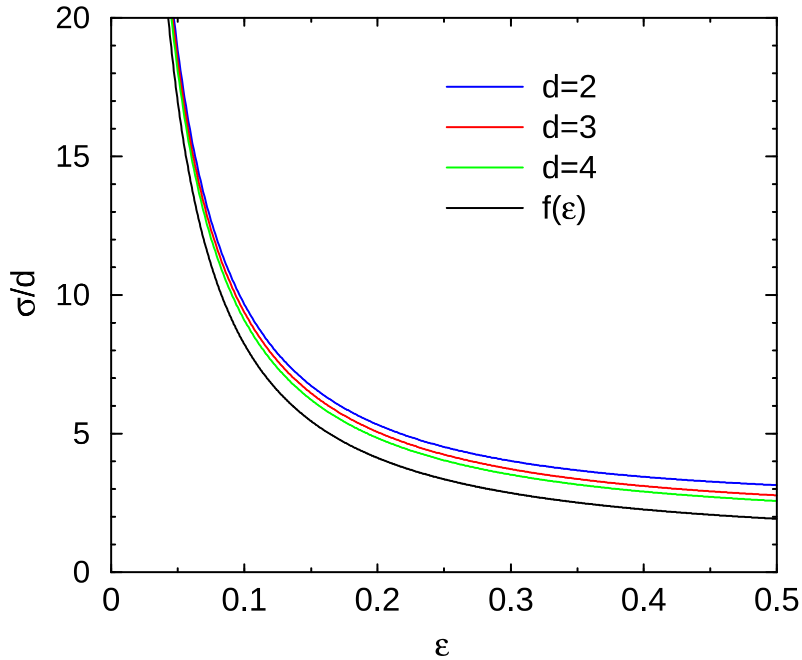
$$\Phi(v) \sim v^{-\sigma}, \qquad v \to \infty.$$

- **Exact solution for the exponent** σ

$$1-\epsilon(1-\epsilon)\,\frac{\sigma-d}{d} = {}_2F_1\left[\frac{d-\sigma}{2}, \frac{1}{2}; \frac{d}{2}; 1-\epsilon^2\right] + (1-\epsilon)^{\sigma-d}\,\frac{\Gamma\left(\frac{\sigma-d+1}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{\sigma}{2}\right)\Gamma\left(\frac{1}{2}\right)}$$

Nonuniversal tails, exponents depend on ϵ, d

The exponent σ



- **Maxwellian distributions:** $d = \infty$, $\epsilon = 0$
- **Diverges in high dimensions**

$$\sigma \propto d$$

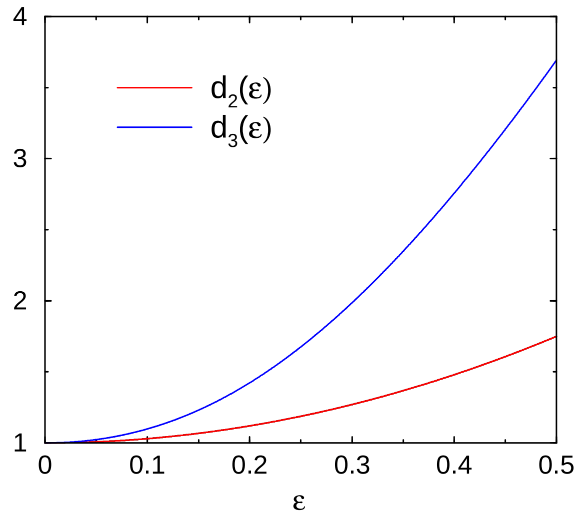
- **Diverges for low dissipation**

$$\sigma \propto \epsilon^{-1}$$

- **In practice, huge**

$$\sigma(d = 3, r = 0.8) \cong 30!$$

Dynamics



- Moments of the velocity distribution

$$M_{2n}(t) = \int d\mathbf{v} |\mathbf{v}|^{2n} P(\mathbf{v}, t)$$

- Multiscaling asymptotic behavior

$$M_n \sim \begin{cases} \exp(-n\lambda_2 t/2) & n < \sigma - 1, \\ \exp(-\lambda_n t) & n > \sigma - 1. \end{cases}$$

- Nonlinear multiscaling spectrum (1D):

$$\alpha_n(\epsilon) = \frac{1 - \epsilon^{2n} - (1-\epsilon)^{2n}}{1 - \epsilon^2 - (1-\epsilon)^2}$$

Sufficiently large moments exhibit multiscaling

Velocity Correlations

- Definition (correlation between v_x^2 and v_y^2)

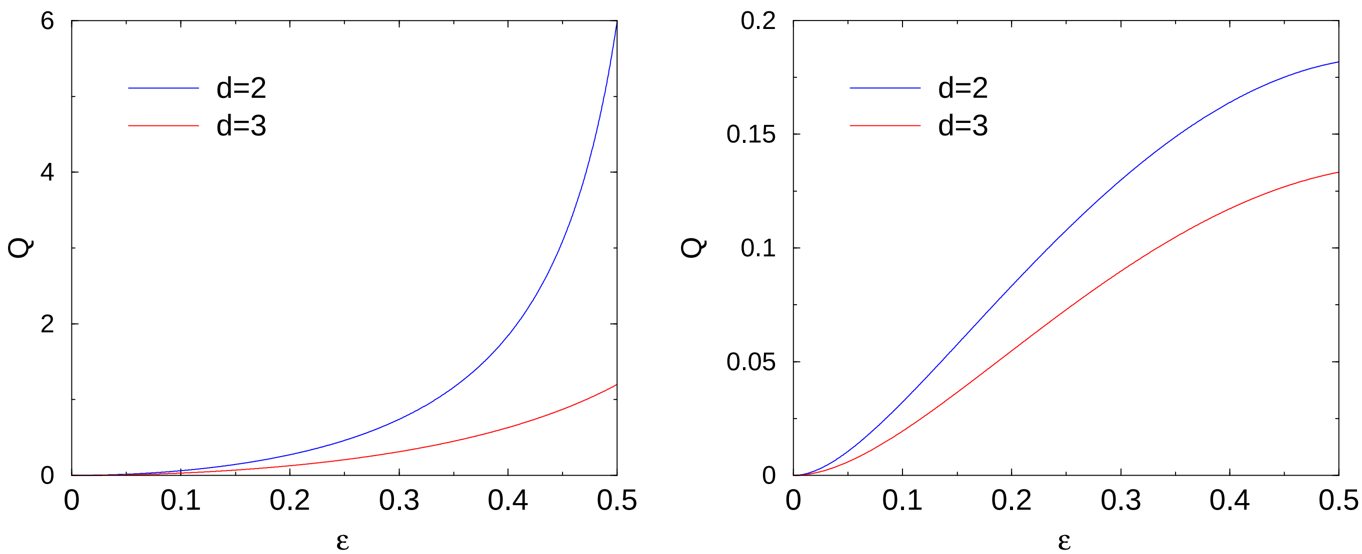
$$Q = \frac{\langle v_x^2 v_y^2 \rangle - \langle v_x^2 \rangle \langle v_y^2 \rangle}{\langle v_x^2 \rangle \langle v_y^2 \rangle}$$

- Unforced case (freely evolving) $P(v) \sim v^{-\sigma}$

$$Q = \frac{6\epsilon^2}{d - (1 + 3\epsilon^2)}$$

- Forced case (white noise) $P(v) \sim e^{-|v|}$

$$Q = \frac{6\epsilon^2(1-\epsilon)}{(d+2)(1+\epsilon) - 3(1-\epsilon)(1+\epsilon^2)}.$$



Correlations diminish with energy input

The "Brazil nut" problem

- Fluid background: mass 1
- Impurity: mass m
- Theory: Lorentz-Boltzmann equation
- Series of transition masses

$$1 < m_1 < m_2 < \cdots < m_\infty$$

- Ratio of moments diverges asymptotically

$$\frac{\langle v_I^{2n}\rangle}{\langle v_F^{2n}\rangle} \sim \begin{cases} c_n & m < m_n; \\ \infty & m > m_n. \end{cases}$$

- Light impurity: moderate violation of equipartition, impurity mimics the fluid
- Heavy impurity: extreme violation of equipartition, impurity sees a static fluid

series of phase transitions

Conclusions (Maxwell specific)

- Power-law high energy tails
- Non-universal exponents
- Multiscaling of the moments, Temperature insufficient to characterize large moments

Generic features

- Overpopulated tails
- Energy input diminishes correlations, tails
- Multiple asymptotics in $\epsilon \to 0$ limit
- Logarithmic self-diffusion
- Correlations between velocity components
- Spatial correlations
- Algebraic autocorrelations, aging

Outlook

- Polydisperse media: impurities, mixtures
- Lattice gases: correlations
- Hydrodynamics
- Shear flows, Shocks
- Opinion dynamics
- Economics

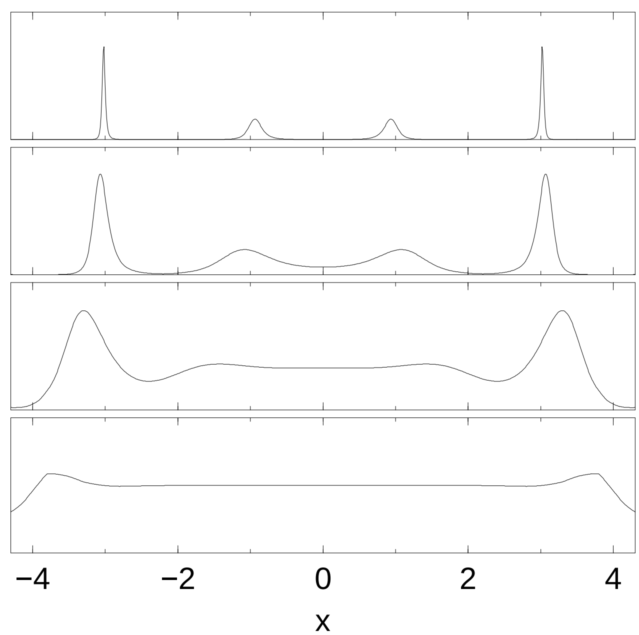
The Compromise Model

- Opinion $-\Delta < x < \Delta$
- Reach compromise in pairs

Weisbuch 2001

$$(x_1, x_2) \to \left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}\right)$$

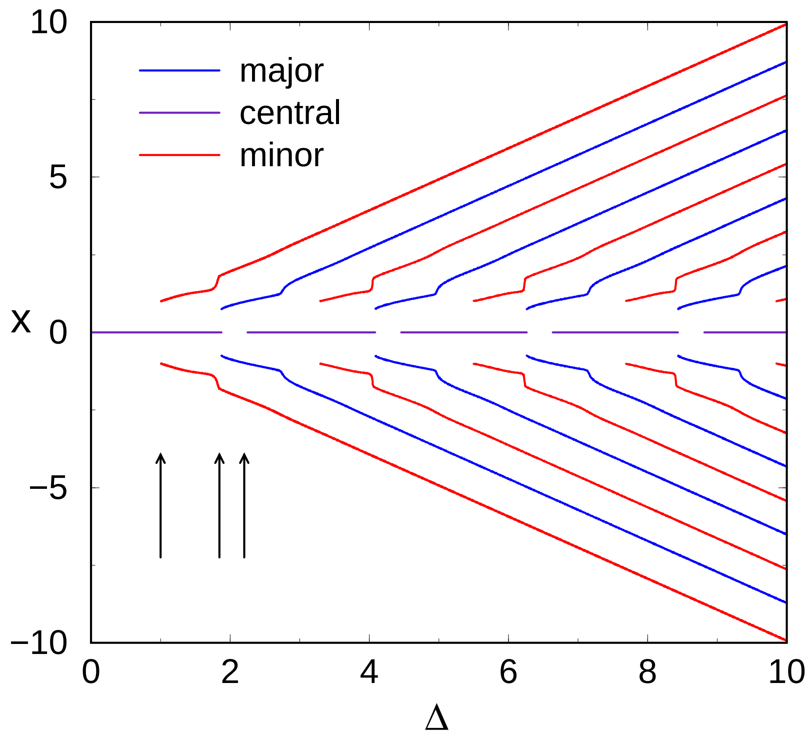
- As long as we are close $|x_1 - x_2| < 1$



$$P_\infty(x) = \sum_i m_i \, \delta(x - x_i)$$

Final State: localized clusters

Bifurcations and Patterns



- Periodic bifurcations

$$x(\Delta) = x(\Delta + L)$$

- Alternating major-minor pattern
- Critical behavior

$$m \sim (\Delta - \Delta_c)^{\alpha} \qquad \alpha = 3 \text{ or } 4.$$

$\boxed{\textbf{Self-similar structure}}$