Kinetics of Averaging

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Talk, papers available from: http://cnls.lanl.gov/~ebn

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Two parts

1. Averaging velocities and angles
2. Averaging opinions

Themes and concepts

1. Self-similarity, scaling
2. Multi-scaling
3. Cascades
4. Phase transitions
5. Synchronization
6. Bifurcations
7. Pattern Formation
8. Coarsening

- Naturally emerge in various kinetic theories
- Useful in complex and nonequilibrium particle systems

Part I:
Averaging velocities and angles

Plan

I. Averaging velocities

   A. Kinetics of pure averaging

   B. Averaging with forcing: steady-states

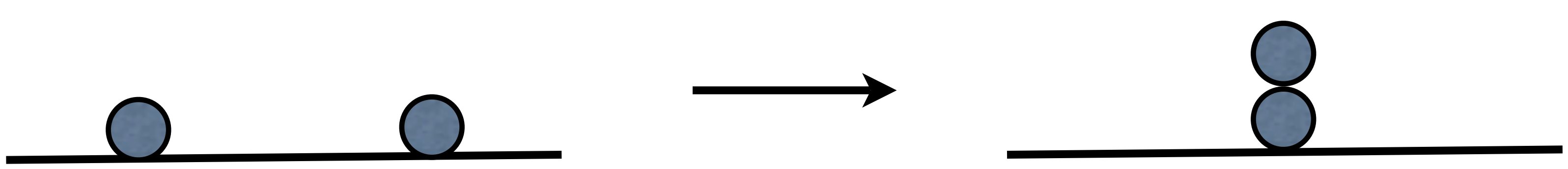
II. Averaging angles

   A. Averaging with forcing: steady states

The basic averaging process

- N identical particles (grains, billiard balls)
- Each particle carries a number (velocity) v_i
- Particles interact in pairs (collision)
- Both particles acquire the average (inelastic)

$$(v_1, v_2) \to \left(\frac{v_1 + v_2}{2}, \frac{v_1 + v_2}{2}\right)$$



Conservation laws & dissipation

- Total number of particles is conserved
- Total momentum is conserved

$$\sum_{i=1}^{N} v_i = \text{constant}$$

- Energy is dissipated in each collision $E_i = \frac{1}{2} v_i^2$

$$\Delta E = \frac{1}{4}(v_1 - v_2)^2$$

We expect the velocities to shrink

Some details

- Dynamic treatment

  Each particle collides once per unit time

- Random interactions

  The two colliding particles are chosen randomly

- Infinite particle limit is implicitly assumed

$$N \to \infty$$

- Process is galilean invariant $v \to v + v_0$

  Set average velocity to zero $\langle x \rangle = 0$

The temperature

- Definition

$$T = \langle v^2 \rangle$$

- Time evolution = exponential decay

$$\frac{dT}{dt} = -\lambda T \qquad \Longrightarrow \qquad T = T_0\, e^{-\lambda t} \qquad \lambda = \frac{1}{2}$$

- All energy is eventually dissipated
- Trivial steady-state

$$P(v) \to \delta(v)$$

The moments

- Kinetic theory

$$\frac{\partial P(v,t)}{\partial t} = \iint dv_1 dv_2 P(v_1,t)P(v_2,t)\left[\underbrace{\delta\left(v - \frac{v_1+v_2}{2}\right)}_{\text{gain}} - \underbrace{\delta(v-v_1)}_{\text{loss}}\right]$$

- Moments of the distribution

$$M_n = \int dv\, v^n P(v,t)$$

$$\begin{aligned} M_0 &= 1 \\ M_{2n+1} &= 0 \end{aligned}$$

- Closed nonlinear recursion equations

$$\frac{dM_n}{dt} + \lambda_n M_n = 2^{-n}\sum_{m=2}^{n-2}\binom{n}{m} M_m M_{n-m}$$

- Asymptotic decay

$$\lambda_n < \lambda_m + \lambda_{n-m}$$

$$M_n \sim e^{-\lambda_n t} \qquad \text{with} \qquad \lambda_n = 1 - 2^{-(n-1)}$$

Multiscaling

- Nonlinear spectrum of decay constants

$$\lambda_n = 1 - 2^{-(n-1)}$$

- Spectrum is concave, saturates

$$\lambda_n < \lambda_m + \lambda_{n-m}$$

- Each moment has a distinct behavior

$$\frac{M_n}{M_m M_{n-m}} \to \infty \qquad \text{as} \qquad t \to \infty$$

Multiscaling Asymptotic Behavior

The Fourier transform

- The Fourier transform $F(k) = \int dv\, e^{ikv} P(v,t)$
- Obeys closed, nonlinear, nonlocal equation

$$\frac{\partial F(k)}{\partial t} + F(k) = F^2(k/2)$$

- Scaling behavior, scale set by second moment

$$F(k,t) \to f\left(ke^{-\lambda t}\right) \qquad \lambda = \frac{\lambda_2}{2} = \frac{1}{4}$$

- Nonlinear differential equation

$$-\lambda\, z\, f'(z) + f(z) = f^2(z/2) \qquad \begin{aligned} f(0) &= 1 \\ f'(0) &= 0 \end{aligned}$$

- Exact solution

$$f(z) = (1+|z|)e^{-|z|}$$

Closure: derivation

- The Fourier transform

$$F(k) = \int dv\, e^{ikv} P(v,t)$$

- The kinetic theory

$$\frac{\partial P(v,t)}{\partial t} + P(v,t) = \iint dv_1 dv_2 P(v_1,t)P(v_2,t)\delta\left(v - \frac{v_1+v_2}{2}\right)$$

- Fourier transform of the gain term

$$\begin{aligned}
& \int dv\, e^{ikv} \iint dv_1 dv_2 P(v_1,t)P(v_2,t)\delta\left(v - \frac{v_1+v_2}{2}\right) \\
= & \iint dv_1 dv_2 P(v_1,t)P(v_2,t) \int dv e^{ikv}\delta\left(v - \frac{v_1+v_2}{2}\right) \\
= & \iint dv_1 dv_2 P(v_1,t)P(v_2,t) e^{ik\frac{v_1+v_2}{2}} \\
= & \int dv_1 P(v_1,t) e^{ik\frac{v_1}{2}} \int dv_2 P(v_2,t) e^{ik\frac{v_2}{2}} \\
= & F(k/2)F(k/2)
\end{aligned}$$

- Closed equation for Fourier Transform

$$\frac{\partial F(k)}{\partial t} + F(k) = F^2(k/2)$$

Fourier transform generates the moments

- The Fourier transform $F(k) = \int dv\, e^{ikv} P(v,t)$
- Is the generating function of the moments $M_n = \int dv\, v^n P(v)$

$$
\begin{aligned}
F(k) &= \int dv e^{ikv} P(v) \\
&= \int dv \left[1 + ikv + \frac{1}{2!}(ikv)^2 + \frac{(ikv)^3}{3!} + \cdots\right] P(v) \\
&= \int dv P(v) + ik \int dv\, v P(v) + \frac{(ik)^2}{2!} \int dv\, v^2 P(v) + \frac{(ik)^3}{3!} \int dv\, v^3 P(v) + \cdots \\
&= M_0 + ikM_1 + \frac{(ik)^2}{2!} M_2 + \frac{(ik)^3}{3!} M_3 + \cdots \\
&= M_0 - \frac{k^2}{2!} M_2 + \frac{k^4}{4!} M_4 + \cdots
\end{aligned}
$$

- Closed equation for Fourier transform

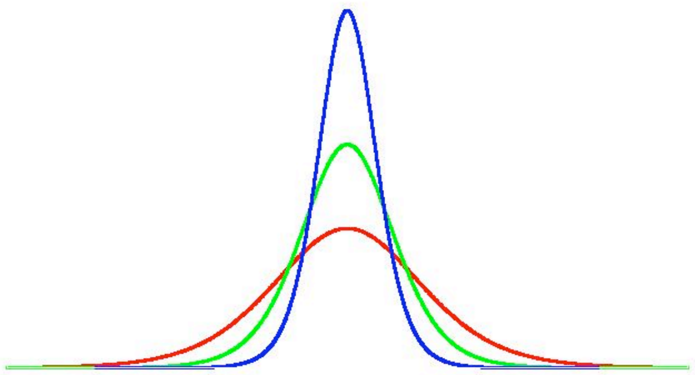
$$\frac{\partial F(k)}{\partial t} + F(k) = F^2(k/2)$$

- Generates closed equations for the moments

$$\frac{dM_2}{dt} = -\frac{M_2}{2}$$

The velocity distribution

- Self-similar form



$$P(v,t) \to e^{\lambda t} p\left(v e^{\lambda t}\right)$$

- Obtained by inverse Fourier transform

$$p(w) = \frac{2}{\pi} \frac{1}{\left(1+w^2\right)^2}$$

Baldassari 02

- Power-law tail

$$p(w) \sim w^{-4}$$

1. Temperature is the characteristic velocity scale
2. Multiscaling is consequence of diverging moments of the power-law similarity function

Stationary Solutions

- Stationary solutions do exist!

$$F(k) = F^2(k/2)$$

- Family of exponential solutions

$$F(k) = \exp(-kv_0)$$

- Lorentz/Cauchy distribution

$$P(v) = \frac{1}{\pi v_0} \frac{1}{1 + (v/v_0)^2}$$

How is a stationary solution consistent with dissipation?

Extreme Statistics

- Large velocities, cascade process

$$v \to \left(\frac{v}{2}, \frac{v}{2}\right)$$

$$(v_1, v_2) \to \left(\frac{v_1 + v_2}{2}, \frac{v_1 + v_2}{2}\right)$$

- Linear evolution equation

$$\frac{\partial P(v)}{\partial t} = 4P\left(\frac{v}{2}\right) - P(v)$$

- Steady-state: power-law distribution

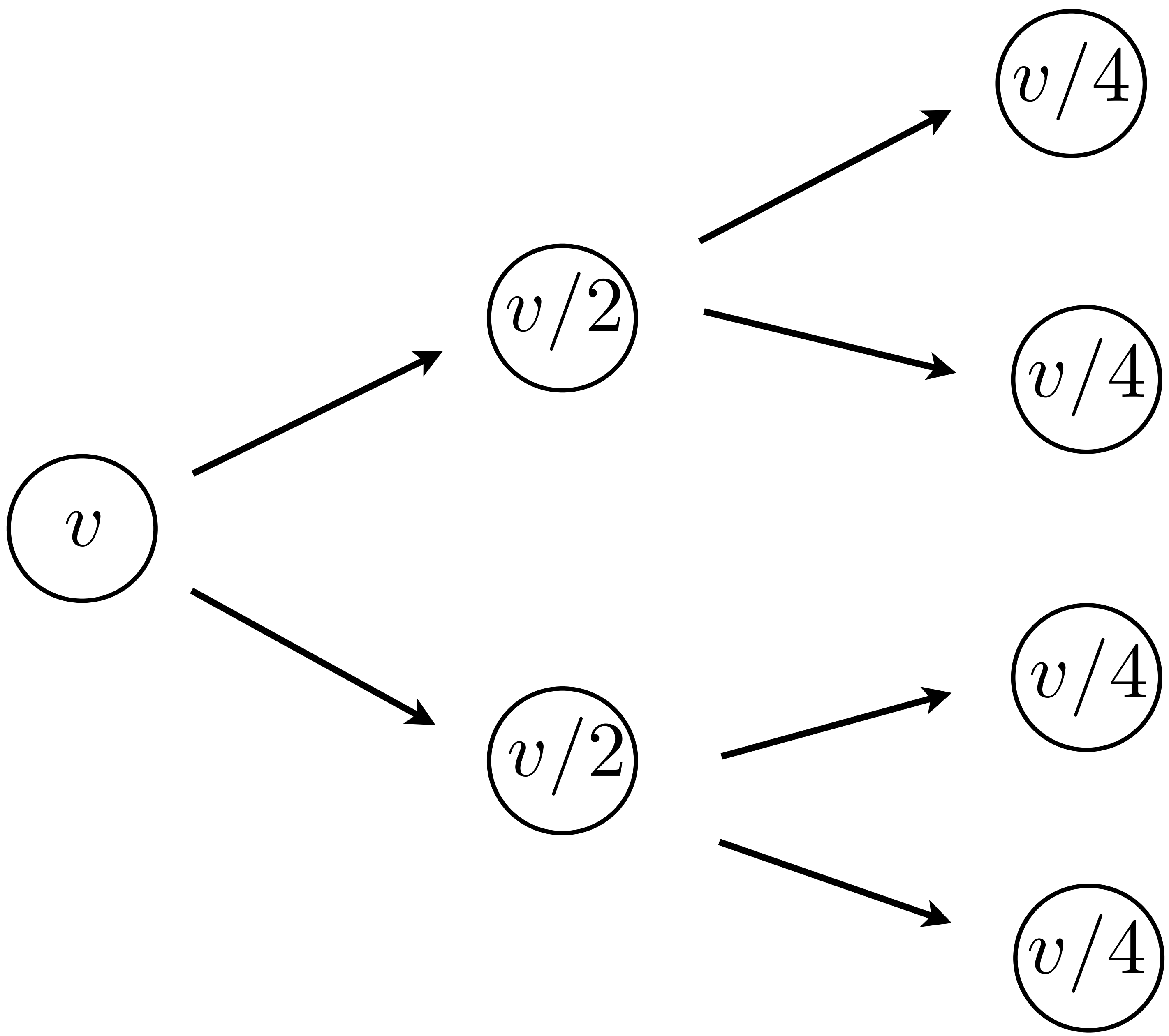
$$P(v) \sim v^{-2}$$

$$4P\left(\frac{v}{2}\right) = P(v)$$

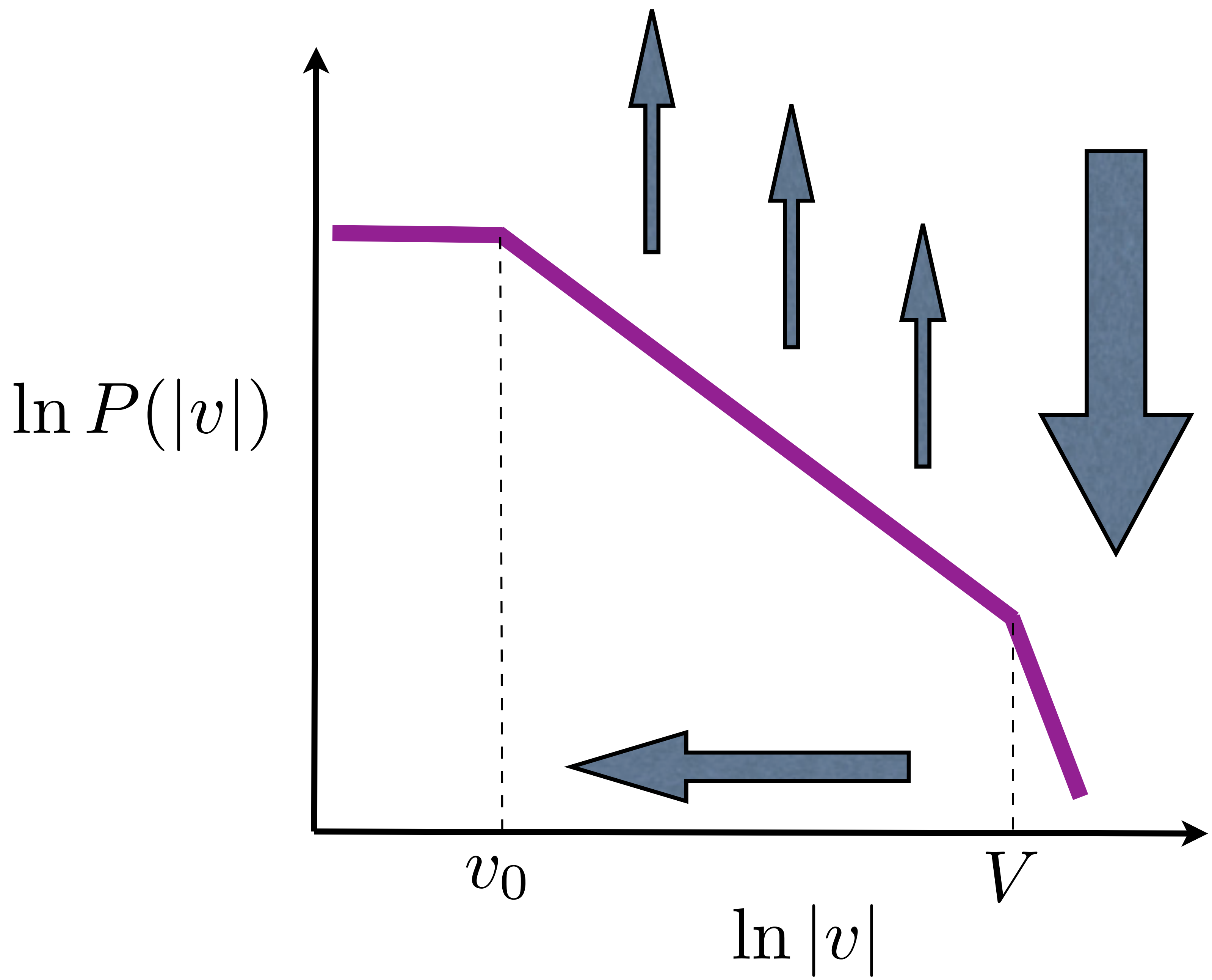
- Divergent energy, divergent dissipation rate

Power-law energy distribution $P(E) \sim E^{-3/2}$

Energy cascade



Injection, Cascade, Dissipation



Pure averaging: conclusions

- Moments exhibit multiscaling
- Distribution function is self-similar
- Power-law tail
- Stationary solution with infinite energy exists
- Driven steady-state
- Energy cascade

Averaging with diffusive forcing

Two independent competing processes

1. Averaging (nonlinear)

$$(v_1, v_2) \to \left(\frac{v_1 + v_2}{2}, \frac{v_1 + v_2}{2} \right)$$

2. Random uncorrelated white noise (linear)

$$\frac{dv_j}{dt} = \eta_j(t) \qquad \langle \eta_j(t)\eta_j(t') \rangle = 2D\delta(t - t')$$

- Add diffusion term to equation (Fourier space)

$$(1 + Dk^2)F(k) = F^2(k/2)$$

System reaches a nontrivial steady-state
Energy injection balances dissipation

Infinite product solution

- Solution by iteration

$$F(k) = \frac{1}{1+Dk^2} F^2(k/2) = \frac{1}{1+Dk^2}\,\frac{1}{(1+D(k/2)^2)^2} F^4(k/4) = \cdots$$

- Infinite product solution

$$F(k) = \prod_{i=0}^{\infty} \left[1 + D(k/2^i)^2\right]^{-2^i}$$

- Exponential tail $v \to \infty$

$$P(v) \propto \exp\left(-|v|/\sqrt{D}\right)$$

$$P(k) \propto \frac{1}{1+Dk^2} \propto \frac{1}{k - i/\sqrt{D}}$$

- Also follows from

$$D\frac{\partial^2 P(v)}{\partial v^2} = -P(v)$$

Non-Maxwellian distribution/Overpopulated tails

Cumulant solution

- Steady-state equation

$$F(k)(1 + Dk^2) = F^2(k/2)$$

- Take the logarithm $\psi(k) = \ln F(k)$

$$\psi(k) + \ln(1 + Dk^2) = 2\psi(k/2)$$

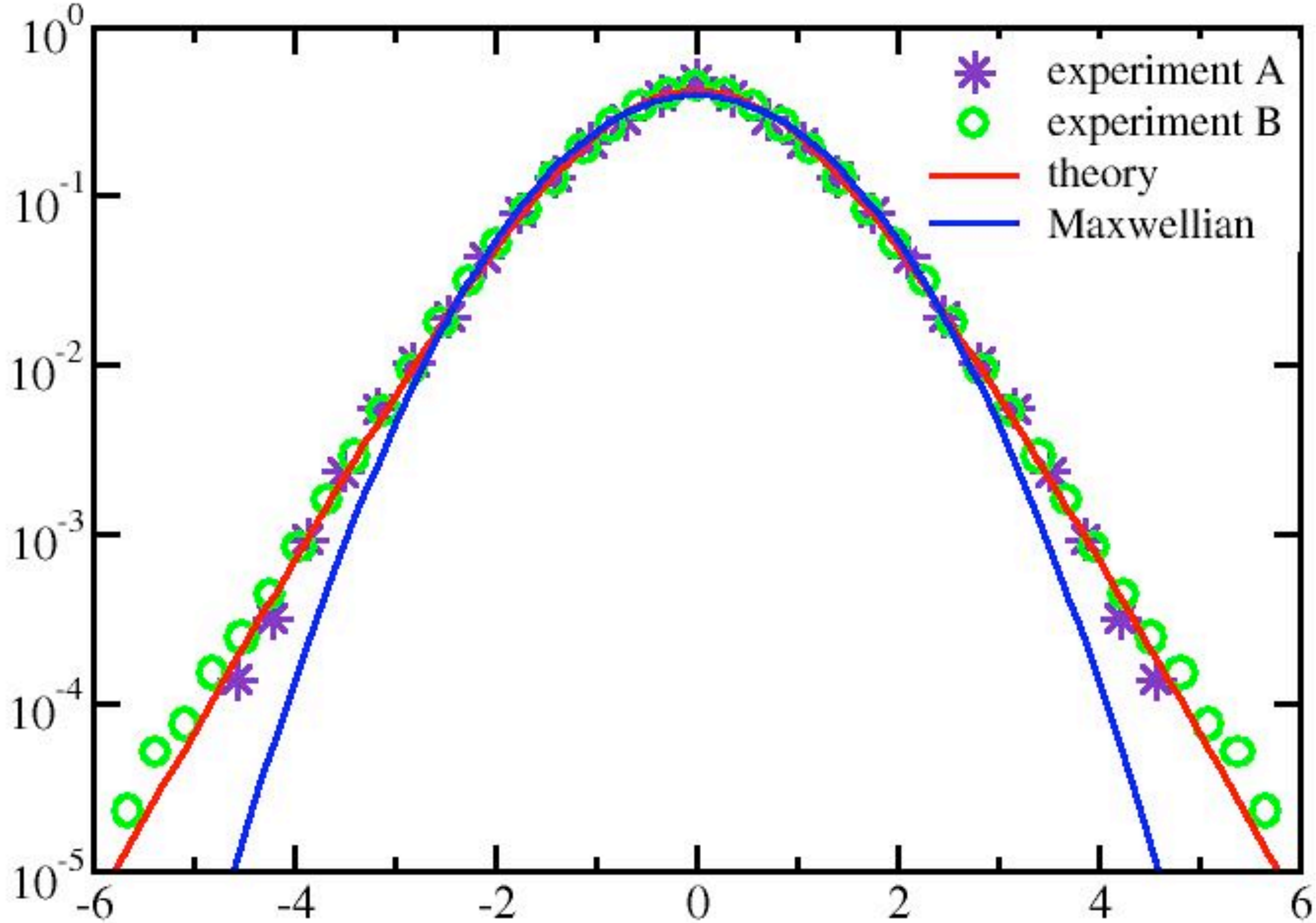
- Cumulant solution

$$F(k) = \exp\left[\sum_{n=1}^{\infty} \psi_n (-Dk^2)^n / n\right]$$

- Generalized fluctuation-dissipation relations

$$\psi_n = \lambda_n^{-1} = \left[1 - 2^{1-n}\right]^{-1}$$

Experiments



"A shaken box of marbles"

Menon 01
Aronson 05

Averaging with forcing: conclusions

- Nonequilibrium steady-states
- Energy pumped and dissipated by different mechanisms
- Overpopulation of high-energy tail with respect to equilibrium distribution

Averaging angles

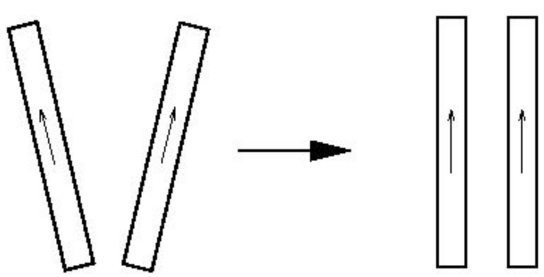
Aronson & Tsimring 05

- Each rod has an orientation

$$0 \leq \theta \leq \pi$$

- Alignment by pairwise interactions (nonlinear)

$$(\theta_1, \theta_2) \rightarrow \begin{cases} \left(\frac{\theta_1+\theta_2}{2}, \frac{\theta_1+\theta_2}{2}\right) & |\theta_1 - \theta_2| < \pi \\ \left(\frac{\theta_1+\theta_2+2\pi}{2}, \frac{\theta_1+\theta_2+2\pi}{2}\right) & |\theta_1 - \theta_2| > \pi \end{cases}$$



- Diffusive wiggling (linear)

$$\frac{d\theta_j}{dt} = \eta_j(t) \qquad \langle \eta_j(t)\eta_j(t')\rangle = 2D\delta(t-t')$$

Relevance

- Biology: molecular motors
- Ecology: flocking
- Granular matter: granular chains and solid rods
- Phase synchronization

Kinetic Theory

- Nonlinear integro-differential equation

$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial \theta^2} + \int_{-\pi}^{\pi} d\phi\, P\left(\theta - \frac{\phi}{2}\right) P\left(\theta + \frac{\phi}{2}\right) - P.$$

- Fourier transform

$$P_k = \langle e^{-ik\theta}\rangle = \int_{-\pi}^{\pi} d\theta e^{-ik\theta} P(\theta) \qquad P(\theta) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} P_k e^{ik\theta}$$

- Closed nonlinear equation

$$(1 + Dk^2)P_k = \sum_{i+j=k} A_{i-j} P_i P_j$$

- Coupling constants

$$A_q = \frac{\sin\frac{\pi\, q}{2}}{\frac{\pi q}{2}} = \begin{cases} 1 & q = 0 \\ 0 & q = 2, 4, \cdots \\ (-1)^{\frac{q-1}{2}} \frac{2}{\pi|q|} \end{cases}$$

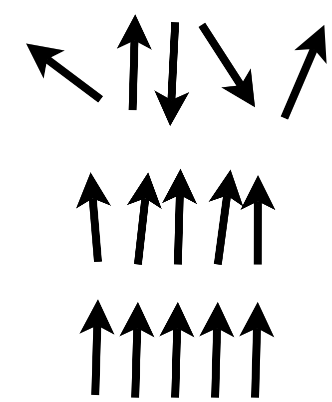
The order parameter

- Lowest order Fourier mode

$$R = |\langle e^{i\theta} \rangle| = |P_{-1}|$$

- Probes state of system

$$R = \begin{cases} 0 & \text{disordered state} \\ 0.4 & \text{partially ordered} \\ 1 & \text{perfectly ordered state} \end{cases}$$



The Fourier equation

- Compact Form

$$P_k = \sum_{i+j=k} G_{i,j} P_i P_j$$

- Transformed coupling constants

$$G_{i,j} = \frac{A_{i-j}}{1+D(i+j)^2-2A_{i+j}}$$

- Properties

$$\begin{aligned} G_{i,j} &= G_{j,i} \\ G_{i,j} &= G_{-i,-j} \\ G_{i,j} &= 0, \qquad \text{for} \qquad |i-j| = 2, 4, \ldots. \end{aligned}$$

Solution

- Repeated iterations (product of three modes)

$$P_k = \sum_{\substack{i+j=k \\ i\neq 0,\, j\neq 0}} \sum_{\substack{l+m=j \\ l\neq 0,\, m\neq 0}} G_{i,j}\, G_{l,m}\, P_i\, P_l\, P_m.$$

- When k=2,4,8,...

$$\begin{aligned} P_2 &= G_{1,1} P_1^2 \\ P_4 &= G_{2,2} P_2^2 = G_{2,2} G_{1,1}^2 P_1^4. \end{aligned}$$

- Generally

$$\begin{aligned} P_3 &= 2G_{1,2} P_1 P_2 + 2G_{-1,4} P_{-1} P_4 + \cdots \\ &= 2G_{1,2} G_{1,1} P_1^3 + 2G_{-1,4} G_{2,2} G_{1,1}^2 P_1^4 P_{-1} \cdots \end{aligned}$$

Partition of Integers

- Diagramatic solution

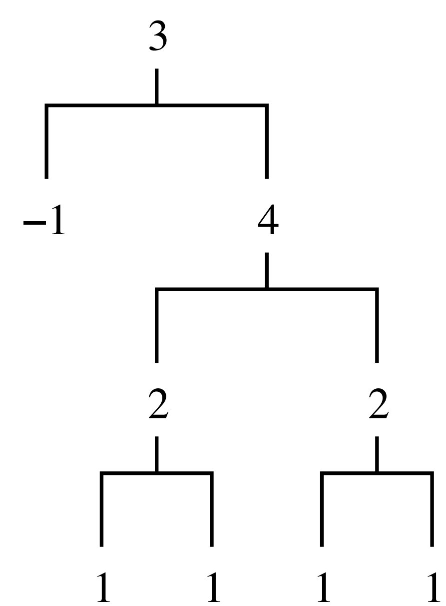
$$P_k = R^k \sum_{n=0}^{\infty} p_{k,n} \, R^{2n}$$

- Partition

$$k = \underbrace{1 + 1 + \cdots + 1 + 1}_{k+n} \underbrace{-1 - \cdots - 1}_{n} .$$

- Partition rules

$$\begin{aligned} k &= i + j \\ i &\neq 0 \\ j &\neq 0 \\ G_{i,j} &\neq 0 \end{aligned}$$



$$p_{3,1} = 2G_{-1,4}G_{2,2}G_{1,1}^2$$

All modes expressed in terms of order parameter

The order parameter

- Diagramatic solution

$$R = R^k \sum_{n=0}^{\infty} p_{1,n} \, R^{2n}$$

- Landau theory

$$R = \frac{C}{D_c - D} R^3 + \cdots$$

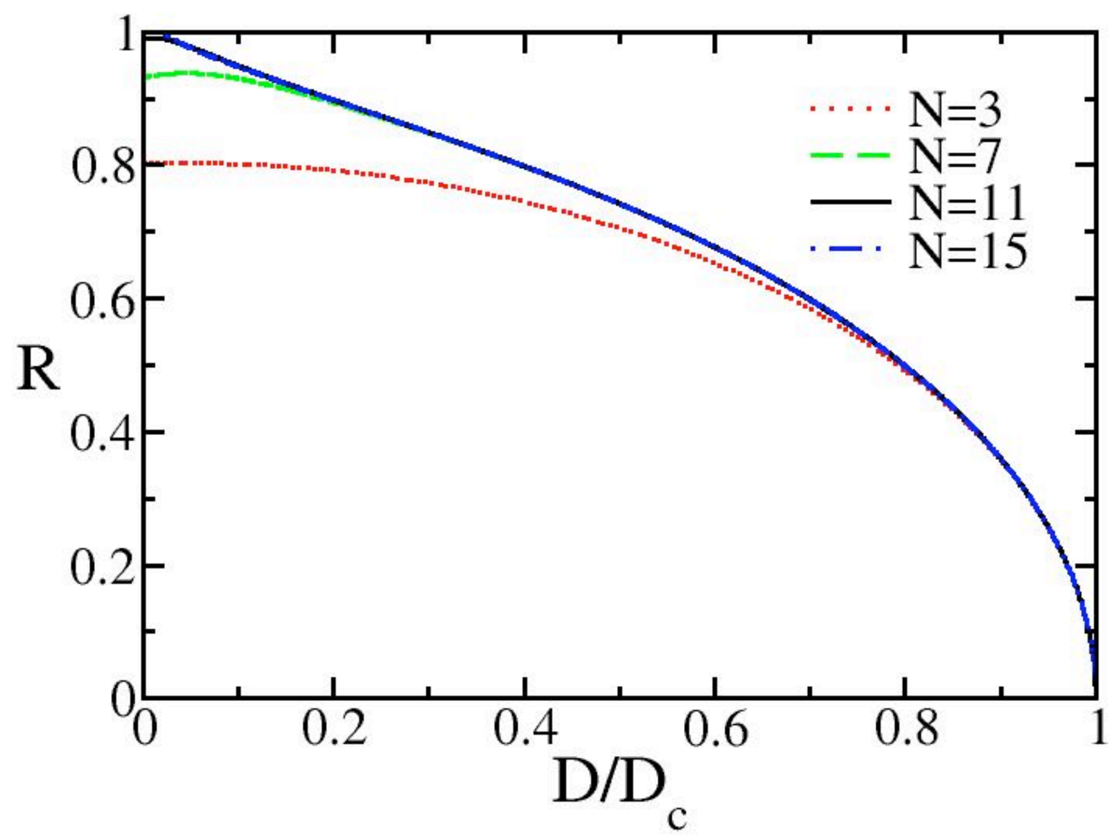
- Critical diffusion constant

$$D_c = \frac{4}{\pi} - 1$$

Closed equation for order parameter

Nonequilibrium phase transition

- Critical diffusion constant $D_c = \dfrac{4}{\pi} - 1$
- Weak diffusion: ordered phase $R > 0$
- Strong diffusion: disordered phase $R = 0$
- Critical behavior $R \sim (D_c - D)^{1/2}$

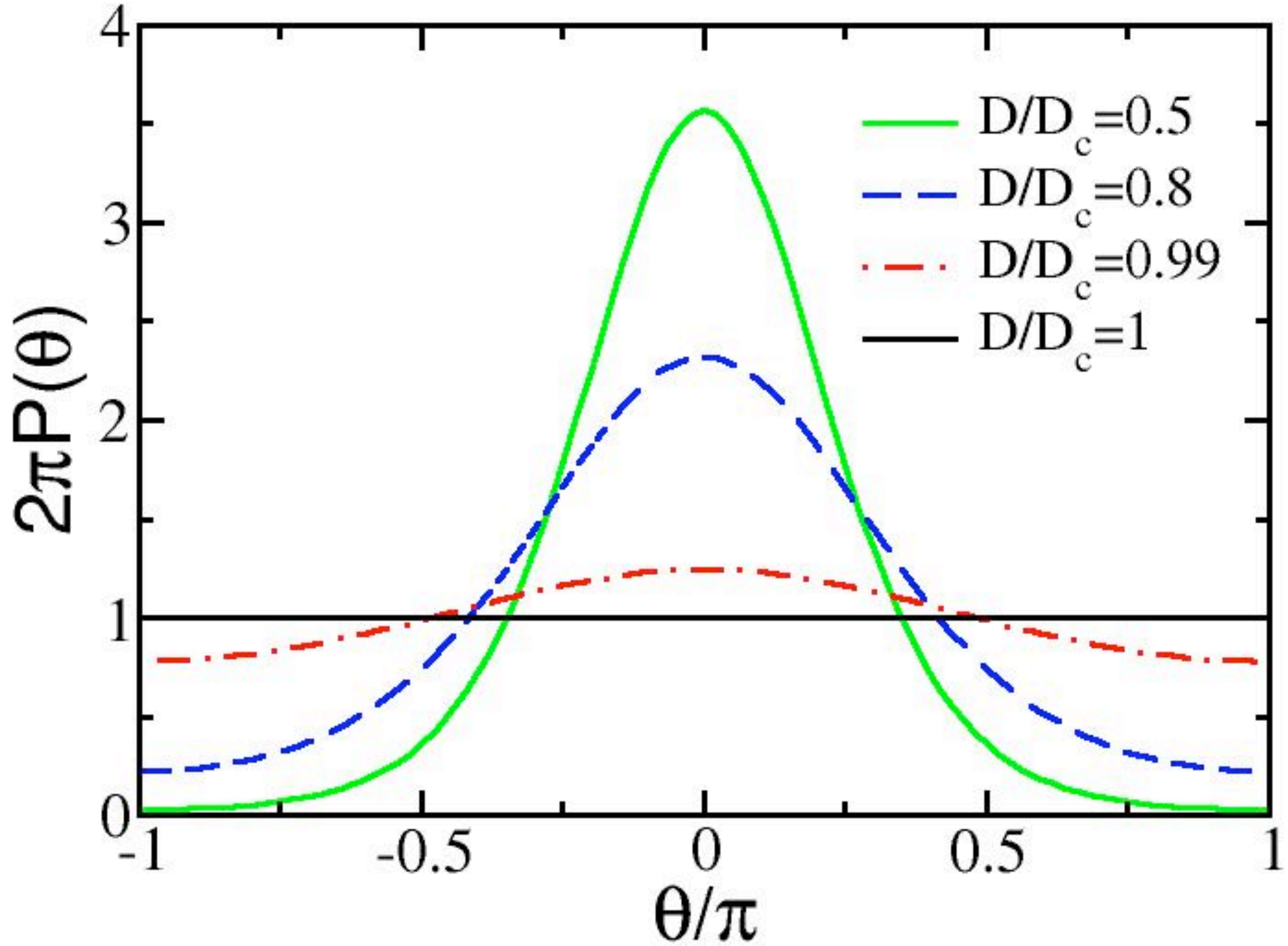


Distribution of orientation

- Fourier modes decay exponentially with R

$$P_k \sim R^k$$

- Small number of modes sufficient



$$P(\theta) = \frac{1}{2\pi} + \frac{1}{\pi} R\cos\theta + \frac{1}{\pi} G_{1,1} R^2 \cos(2\theta) + \frac{2}{\pi} G_{1,2} G_{1,1} R^3 \cos(3\theta) + \cdots$$

Arbitrary alignment rates

- Kinetic theory: arbitrary alignment rates

$$0 = D\frac{d^2P}{d\theta^2} + \int_{-\pi}^{\pi} d\phi\, K(\phi) P\left(\theta - \frac{\phi}{2}\right) P\left(\theta + \frac{\phi}{2}\right) - P(\theta)\int_{-\pi}^{\pi} d\phi\, K(\phi) P(\theta + \phi)$$

- Fourier transform of alignment rate

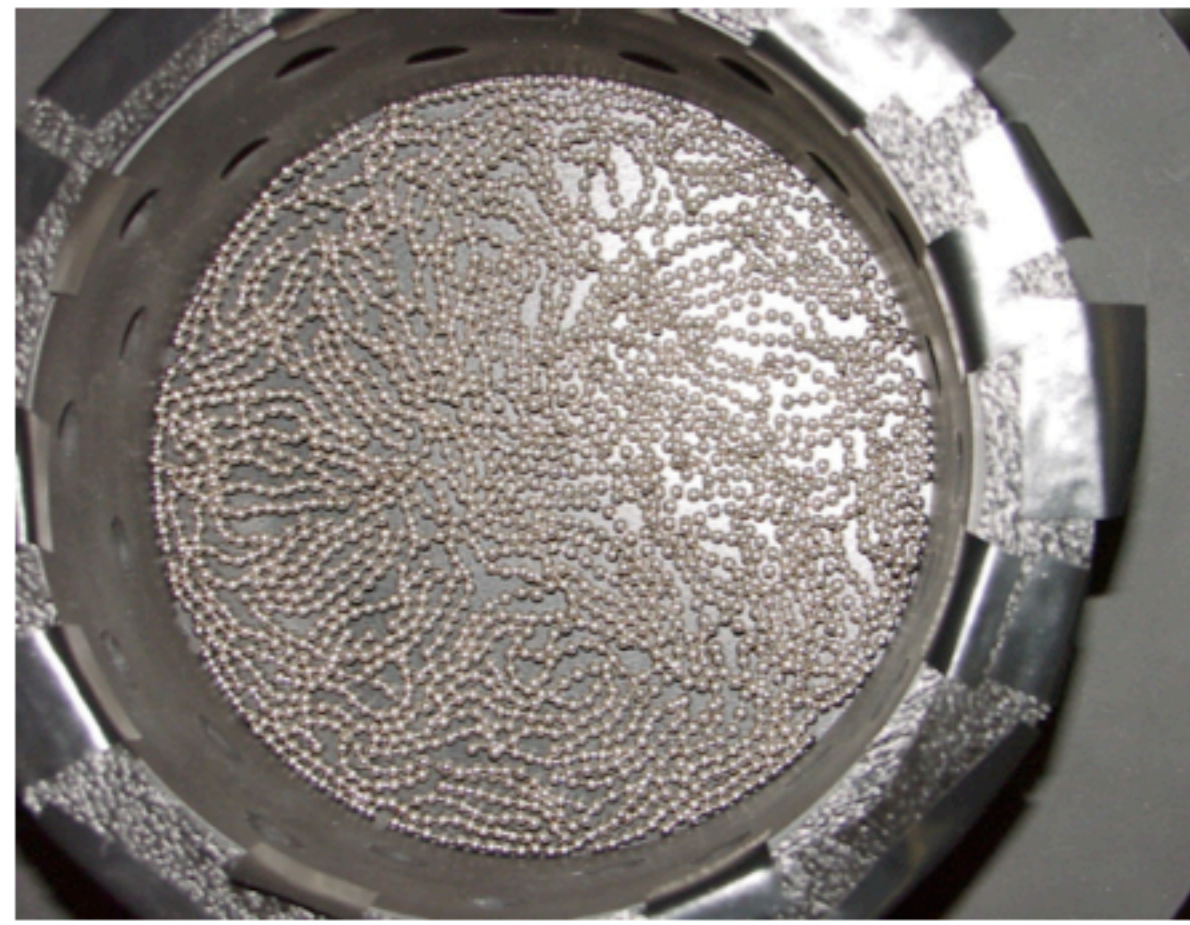
$$A_q = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\, e^{iq\phi/2} K(\phi)$$

- Recover same Fourier equation using

$$G_{i,j} = \frac{1}{2}\frac{A_{i-j} + A_{j-i} - A_{2i} - A_{2j}}{1 + D(i+j)^2 - 2A_{i+j}}$$

When Fourier spectrum is discrete: exact solution is possible for arbitrary alignment rates

Experiments



“A shaken dish of toothpicks”

Averaging angles: conclusions

- Nonequilibrium phase transition
- Weak noise: ordered phase (nematic)
- Strong noise: disordered phase (isotropic)
- Solution relates to iterated partition of integers
- KInetic theory of synchronization
- Only when Fourier spectrum is discrete: exact solution possible for arbitrary averaging rates

Publications

1. E. Ben-Naim and P.L. Krapivsky,
   Phys. Rev. E **61**, R5 (2000).

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3. E. Ben-Naim and P.L. Krapivsky,
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5. K. Kohlstedt, A. Znezhkov, M. Sapozhinsky, I. Aranson, J. Olafsen, E. Ben-Naim
   Phys. Rev. Lett. 95, 068001 (2005).

6. E. Ben-Naim and P.L. Krapivsky,
   Phys. Rev. E 73, 031109 (2006).

Part 2:
Averaging Opinions

Plan

I. Restricted averaging as a compromise process

   A. Continuous opinions

   B. Discrete opinions

II. Restricted averaging with noise

   A. Single-party dynamics

   B. Two-party dynamics

   C. Multi-party dynamics

1. Restricted averaging

The compromise process

- Opinion measured by a continuum variable

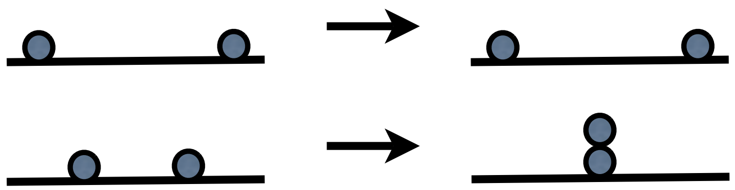
$$-\Delta < x < \Delta$$

1. **Compromise**: reached by pairwise interactions

$$(x_1, x_2) \rightarrow \left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2} \right)$$

2. **Conviction**: restricted interaction range

$$|x_1 - x_2| < 1$$



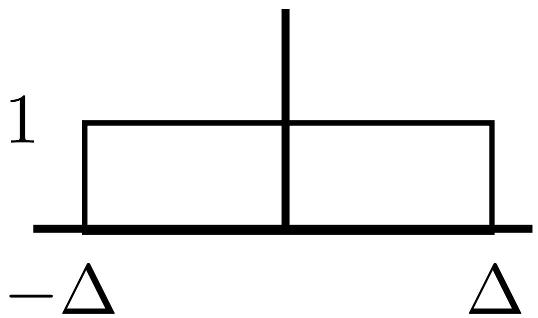
- **Restricted averaging process**
- One parameter model
- Mimics competition between compromise and conviction

Deffuant & Weisbuch (2000)

Problem set-up

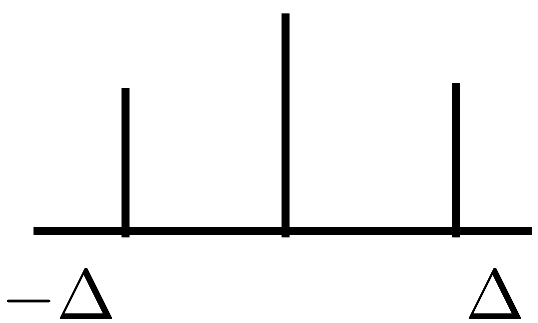
- Given uniform initial (un-normalized) distribution

$$P_0(x) = \begin{cases} 1 & |x| < \Delta \\ 0 & |x| > \Delta \end{cases}$$



- Find final distribution

$$P_\infty(x) = ?$$



- Multitude of final steady-states

$$P_0(x) = \sum_{i=1}^{N} m_i \, \delta(x - x_i) \qquad |x_i - x_j| > 1$$

- Dynamics selects one (deterministically!)

Multiple localized clusters

Further details

- Dynamic treatment

  Each individual interacts once per unit time

- Random interactions

  Two interacting individuals are chosen randomly

- Infinite particle limit is implicitly assumed

$$N \to \infty$$

- Process is galilean invariant $x \to x + x_0$

  Set average opinion to zero $\langle x \rangle = 0$

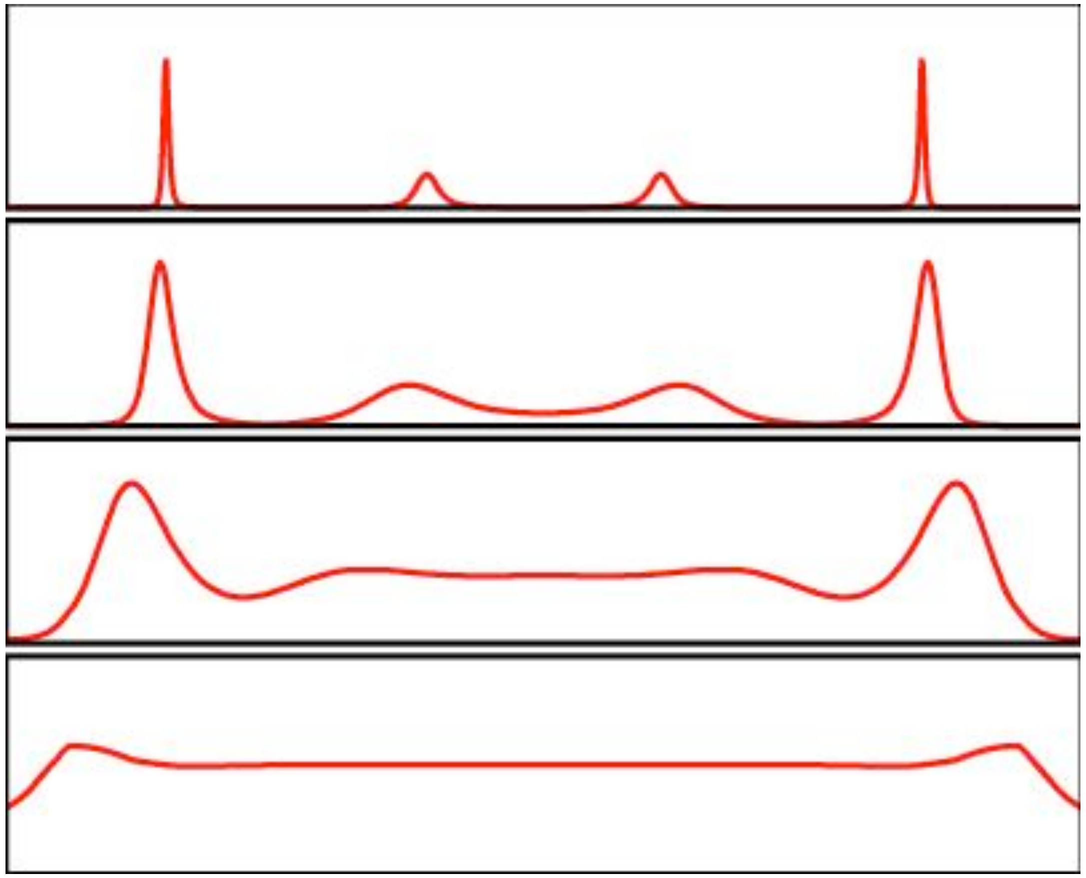
Numerical methods, kinetic theory

- Same master equation, restricted integration

$$\frac{\partial P(x,t)}{\partial t} = \iint_{|x_1 - x_2| < 1} dx_1 dx_2 P(x_1,t)P(x_2,t)\left[\delta\left(x - \frac{x_1+x_2}{2}\right) - \delta(x - x_1)\right]$$

☐ Direct Monte Carlo simulation of stochastic process

☑ Numerical integration of rate equations



Two Conservation Laws

- Total population is conserved

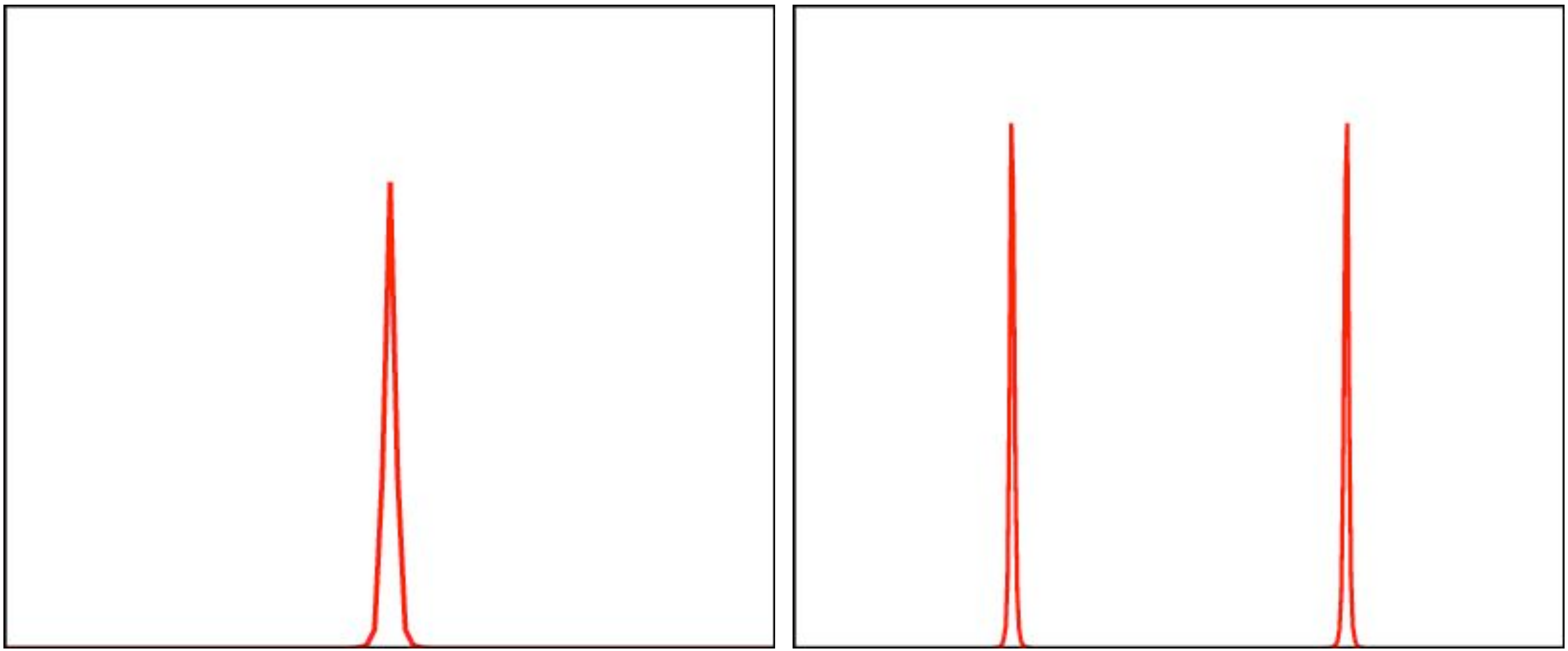
$$\int_{-\Delta}^{\Delta} dx\, P(x) = 2\Delta$$

- Average opinion is conserved

$$\int_{-\Delta}^{\Delta} dx\, x\, P(x) = 0$$

Rise and fall of central party

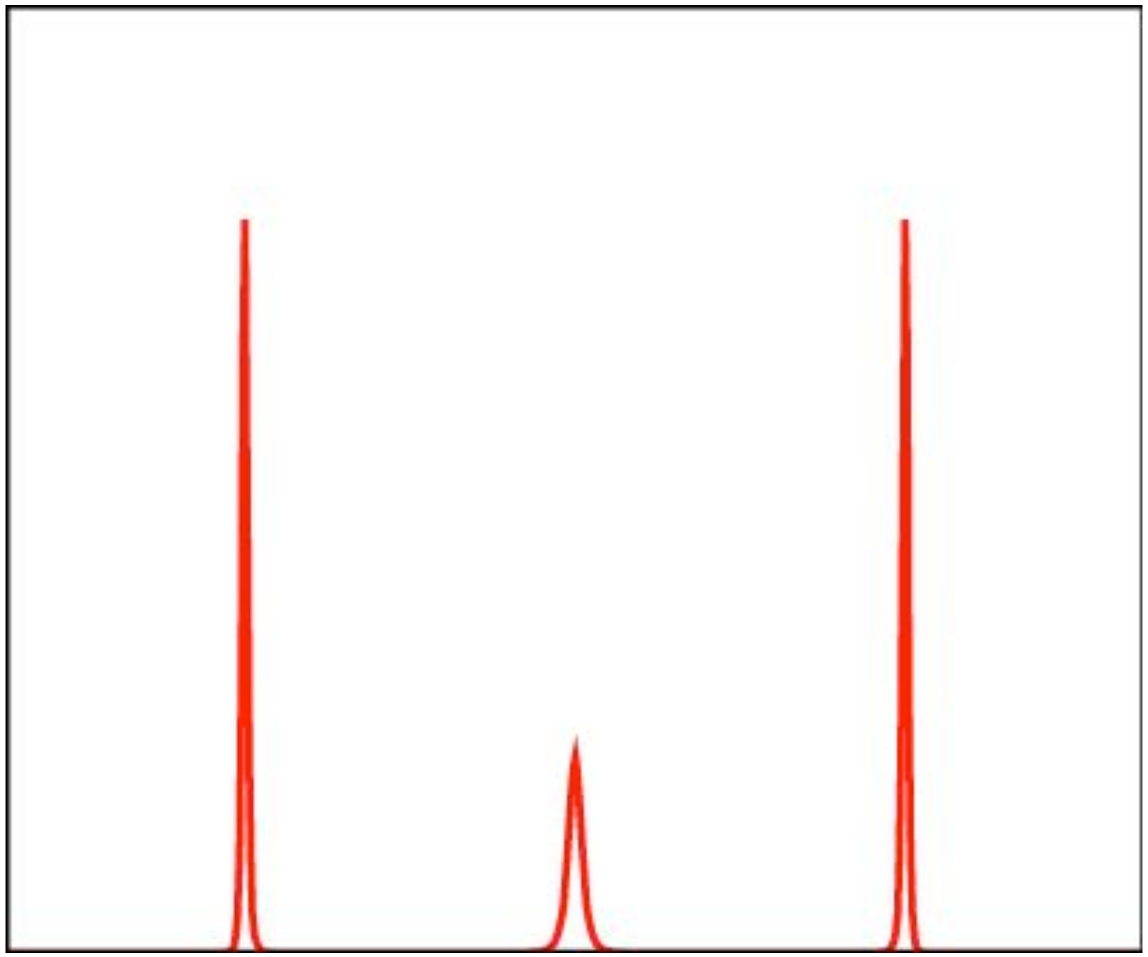
$0 < \Delta < 1.871$ $\qquad$ $1.871 < \Delta < 2.724$



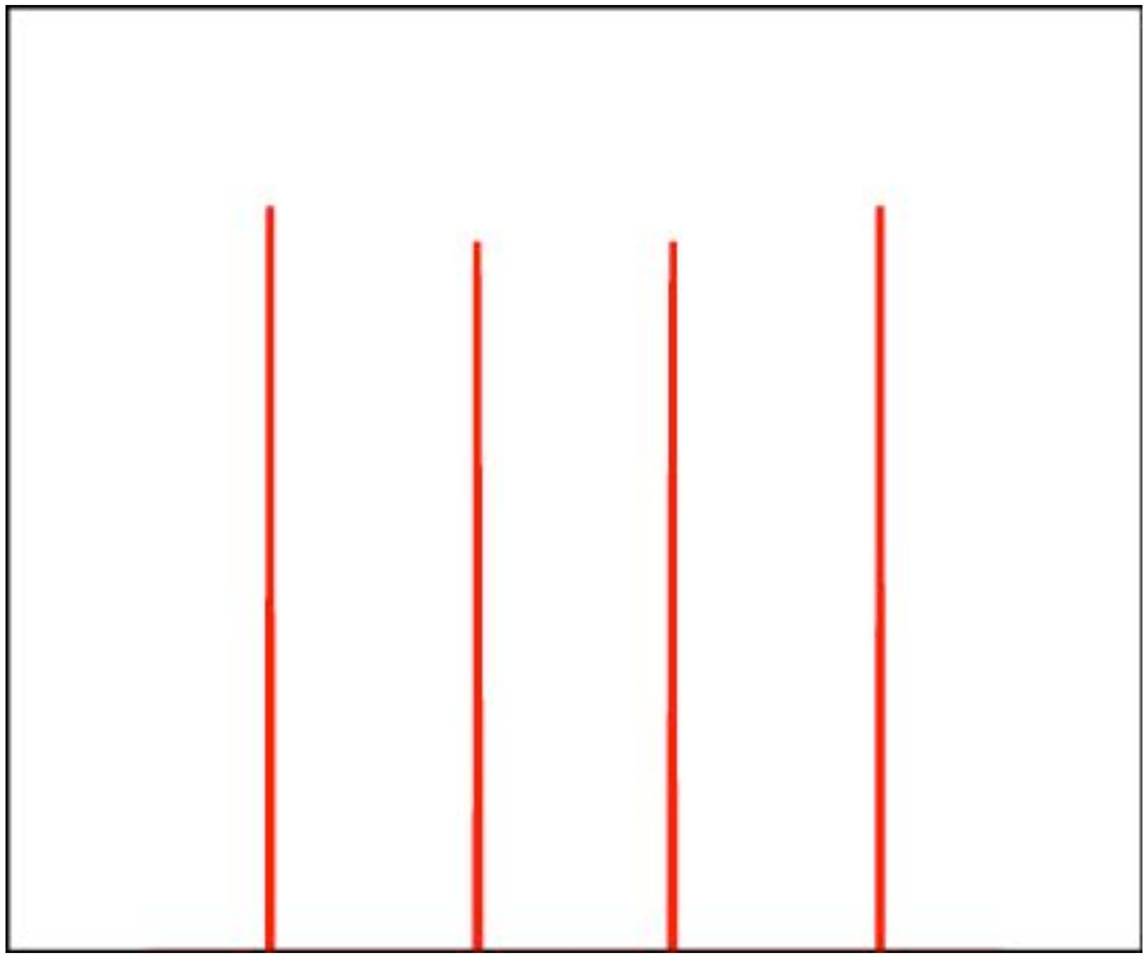
Central party may or may not exist!

Resurrection of central party

$2.724 < \Delta < 4.079$

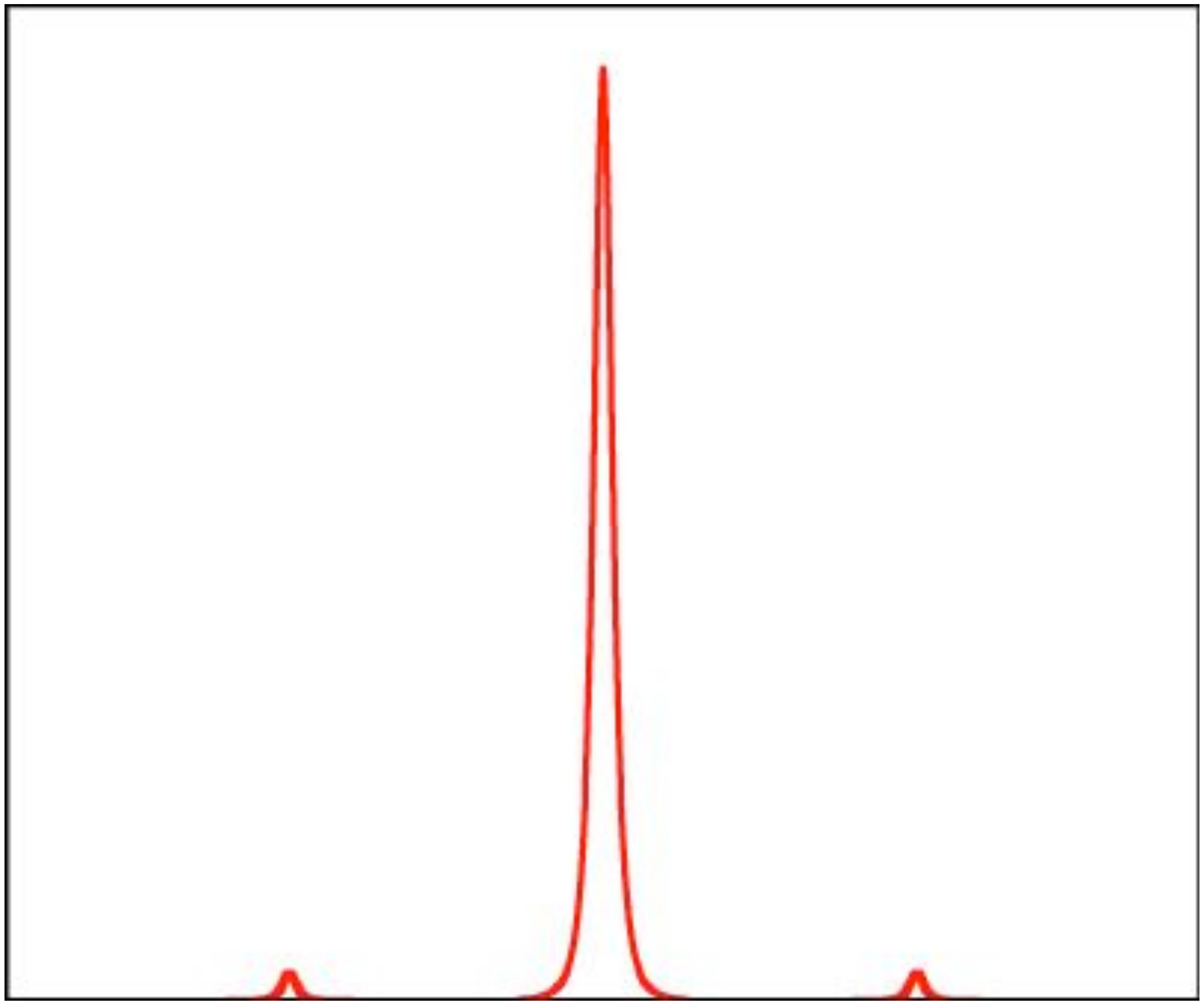


$4.079 < \Delta < 4.956$



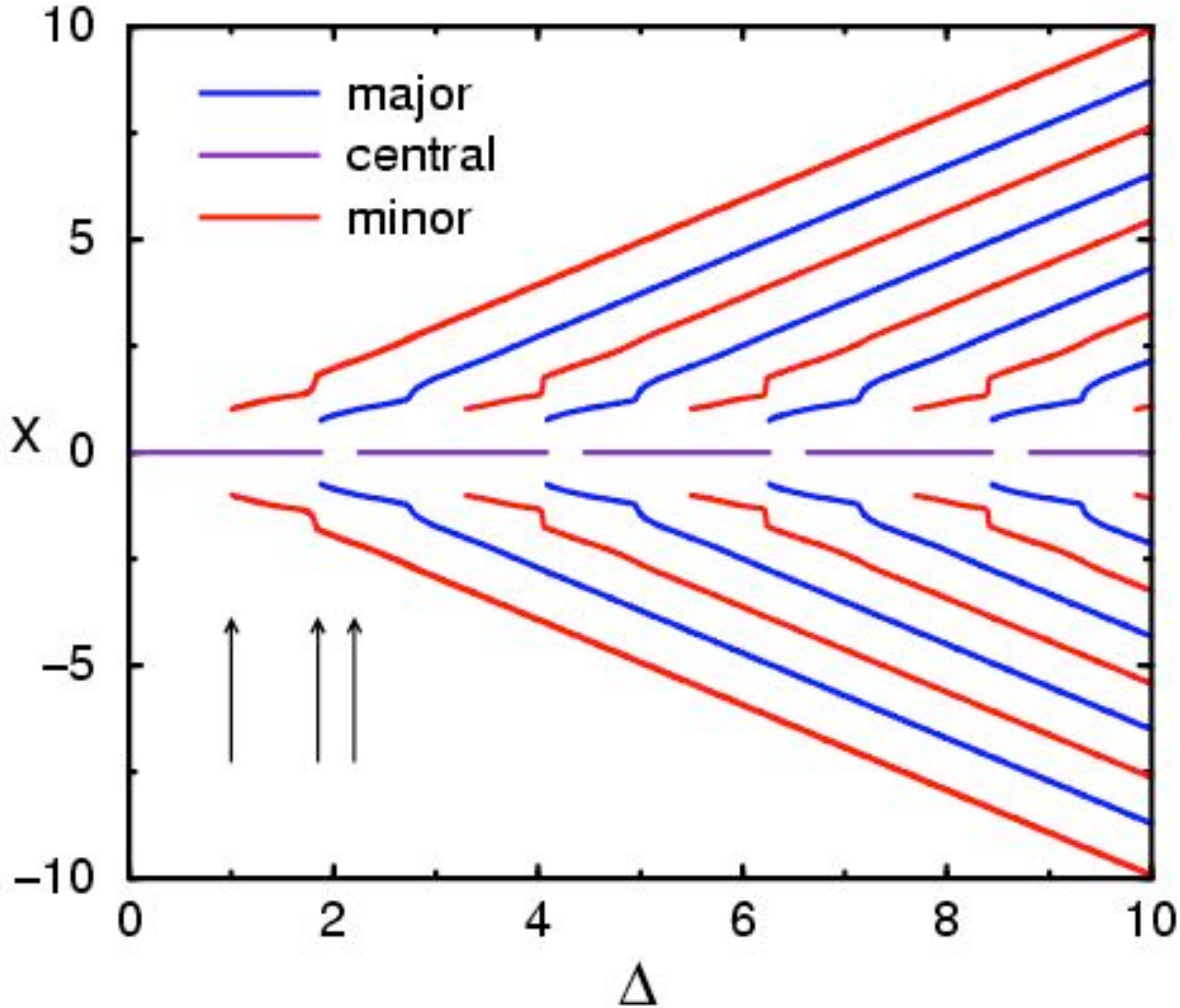
Parties may or may not be equal in size

Emergence of extremists



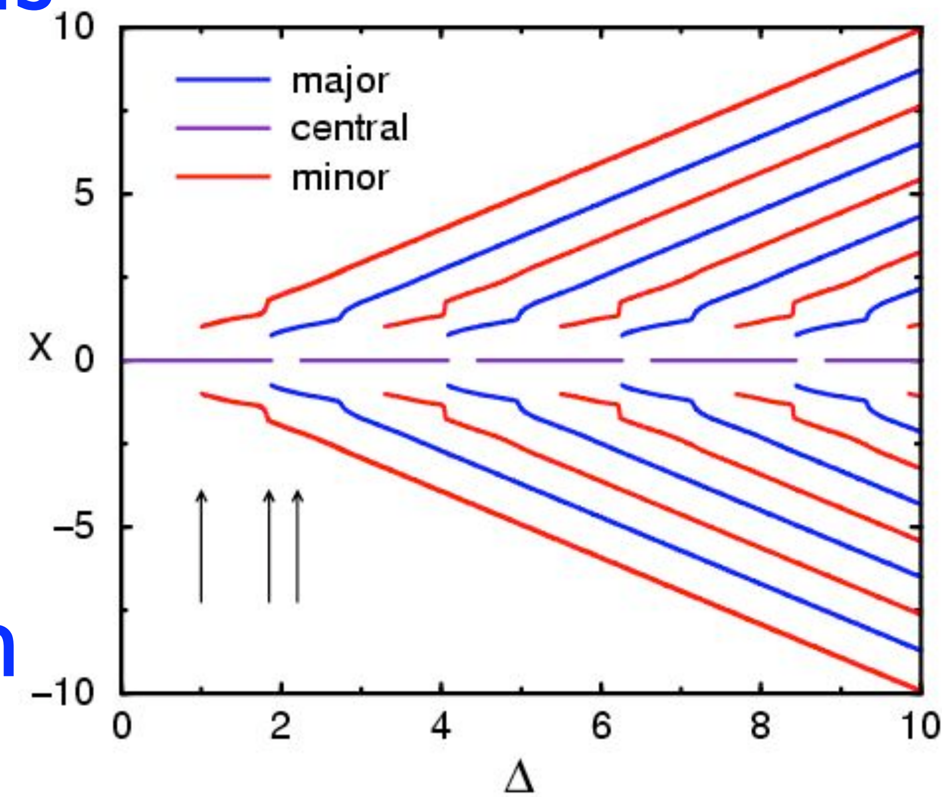
Tiny fringe parties ($m \sim 10^{-3}$)

Bifurcations and Patterns



Self-similar structure, universality

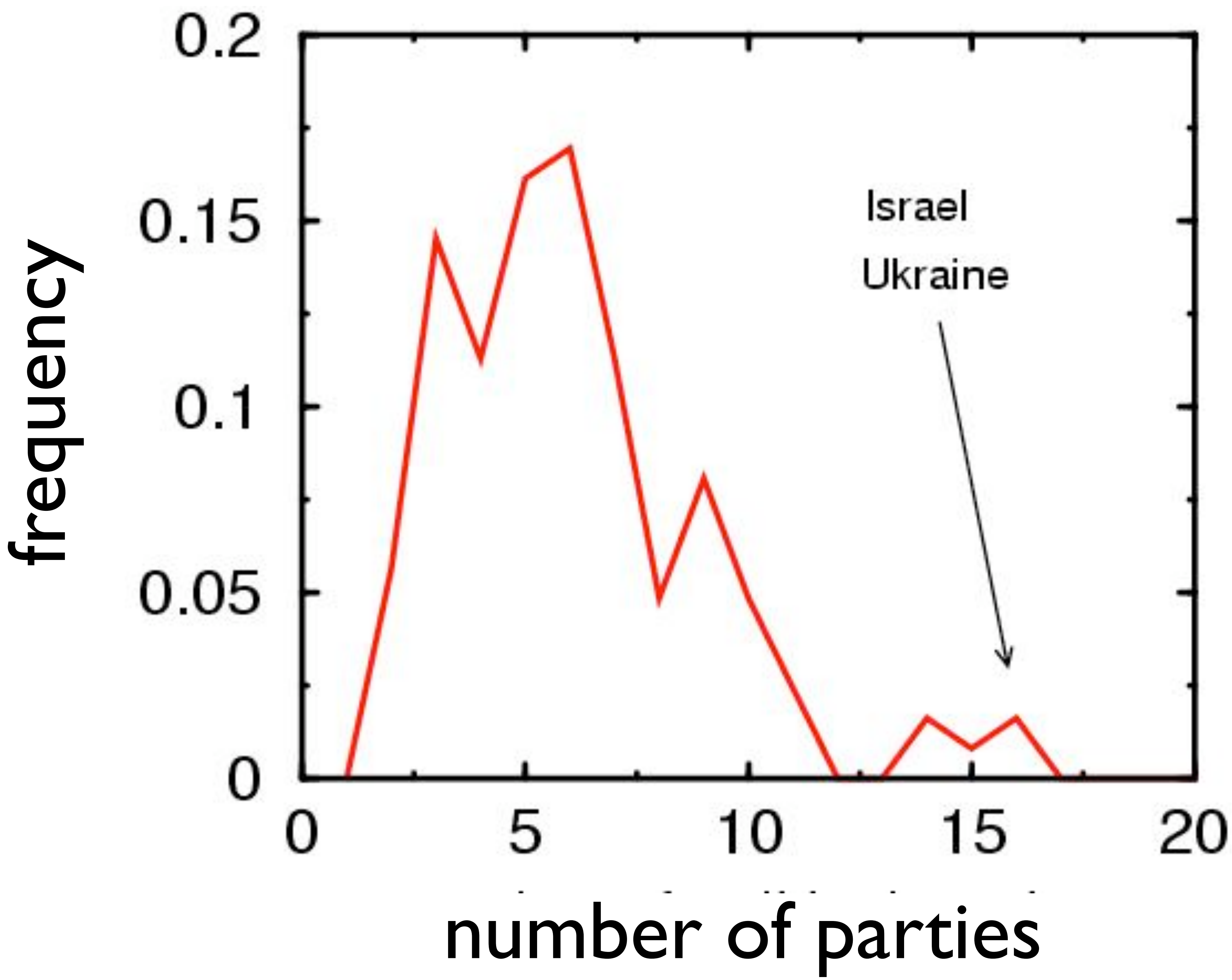
- Periodic sequence of bifurcations
    1. Nucleation of minor cluster branch
    2. Nucleation of major cluster brunch
    3. Nucleation of central cluster



- Alternating major-minor pattern
- Clusters are equally spaced
- Period L gives major cluster mass, separation

$$x(\Delta) = x(\Delta) + L \qquad L = 2.155$$

How many political parties?



- Data: CIA world factbook 2002
- 120 countries with multi-party parliaments
- Average=5.8; Standard deviation=2.9

Cluster mass

- Masses are periodic

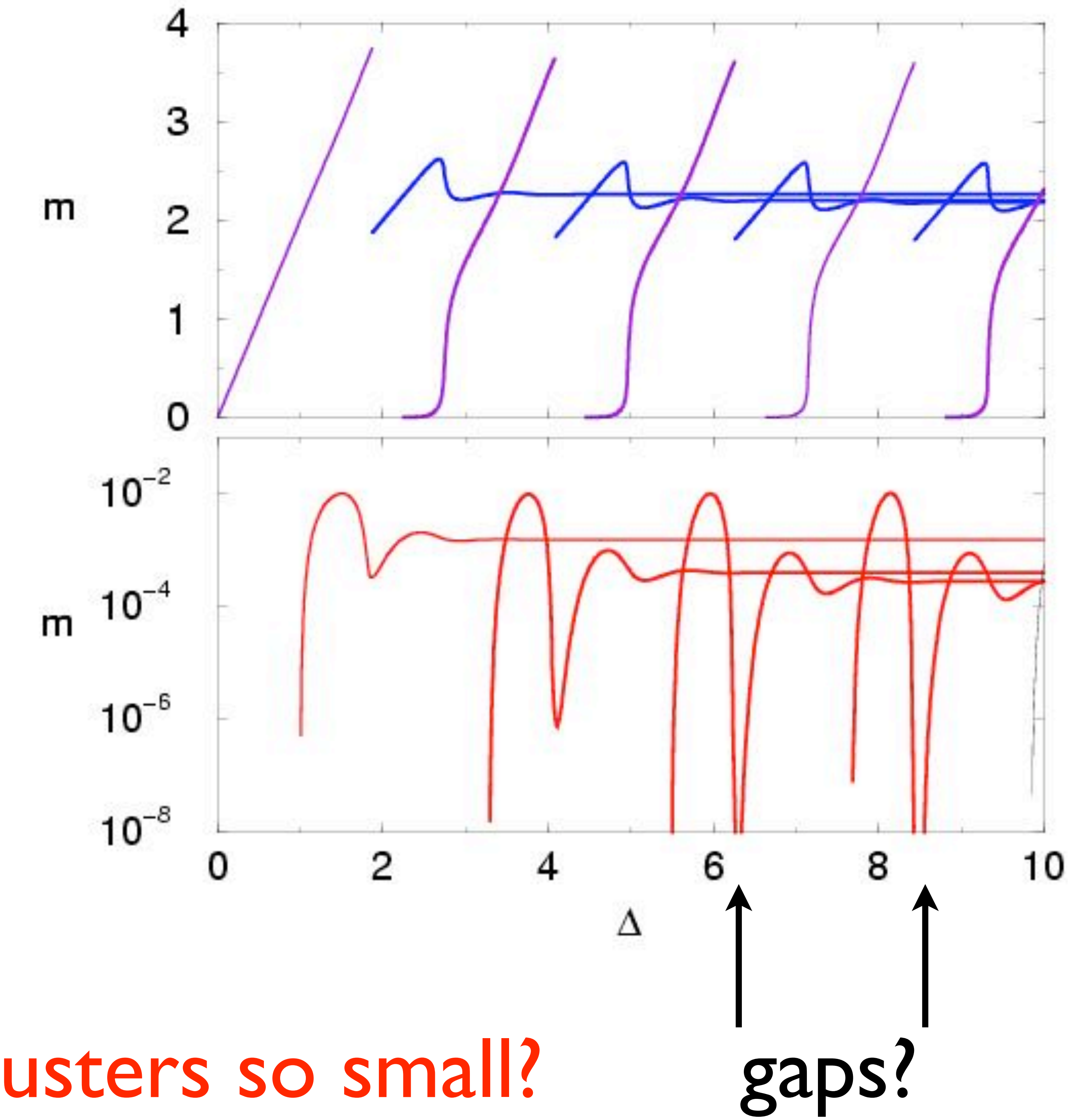
$$m(\Delta) = m(\Delta + L)$$

- Major mass

$$M \to L = 2.155$$

- Minor mass

$$m \to 3 \times 10^{-4}$$



Why are the minor clusters so small?

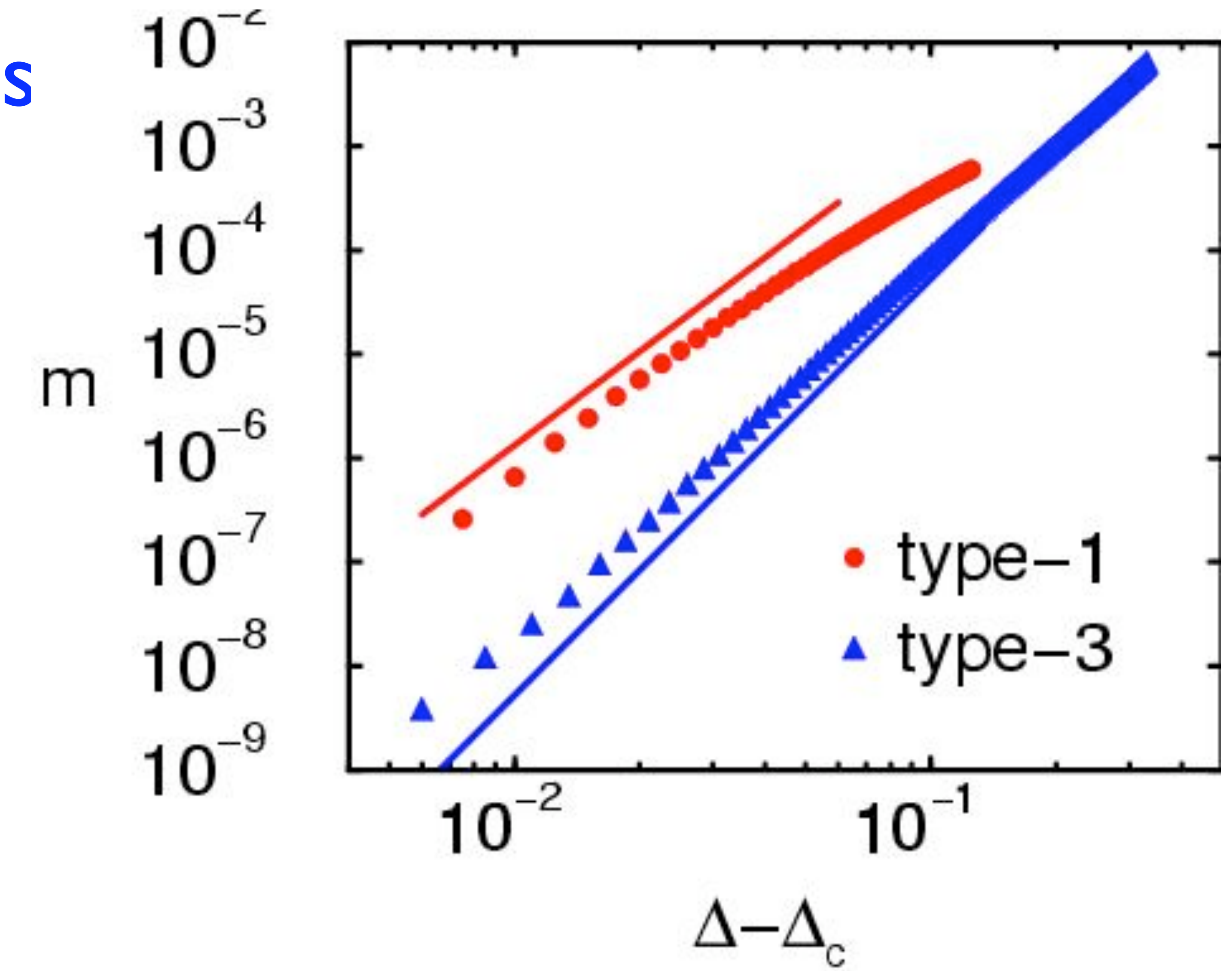
Scaling near bifurcation points

- Minor mass vanishes

$$m \sim (\Delta - \Delta_c)^{\alpha}$$

- Universal exponent

$$\alpha = \begin{cases} 3 & \text{type } 1 \\ 4 & \text{type } 3 \end{cases}$$



L-2 is the small parameter
explains small saturation mass

Consensus = pure averaging

- Integrable for $\Delta < 1/2$

$$\langle x^2(t)\rangle = \langle x^2(0)\rangle\, e^{-\Delta t}$$

- Final state: localized

$$P_\infty(x) = 2\Delta\,\delta(x)$$

- Rate equations in Fourier space

$$P_t(k) + P(k) = P^2(k/2)$$

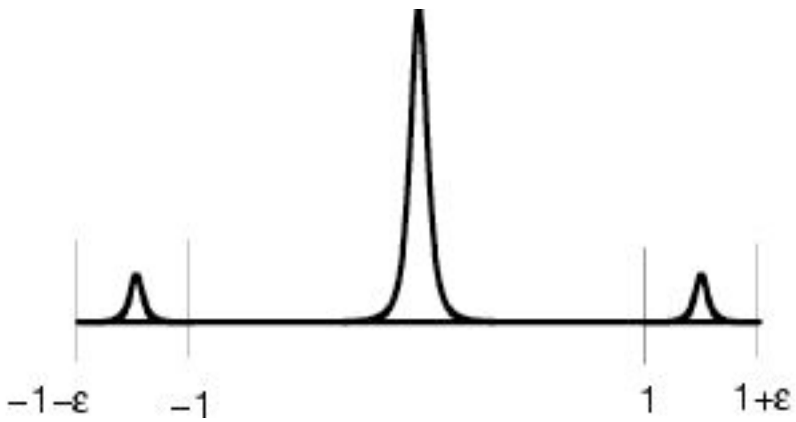
- Self-similar collapse dynamics

$$\Phi(z) \propto \left(1+z^2\right)^{-2} \qquad z = x/\sqrt{\langle x^2\rangle}$$

The Inelastic Maxwell Model, EB & PL Krapivsky, Lecture Notes in Physics **624,** 65 (2003)

Heuristic derivation of exponent

- Perturbation theory $\Delta = 1 + \epsilon$
- Major cluster $x(\infty) = 0$
- Minor cluster $x(\infty) = \pm(1 + \epsilon/2)$



- Rate of transfer from minor cluster to major cluster

$$\frac{dm}{dt} = -m\,M \longrightarrow m \sim \epsilon\, e^{-t}$$

- Process stops when

$$x \sim e^{-t_f/2} \sim \epsilon \qquad \langle x^2 \rangle \sim e^{-t}$$

- Final mass of minor cluster

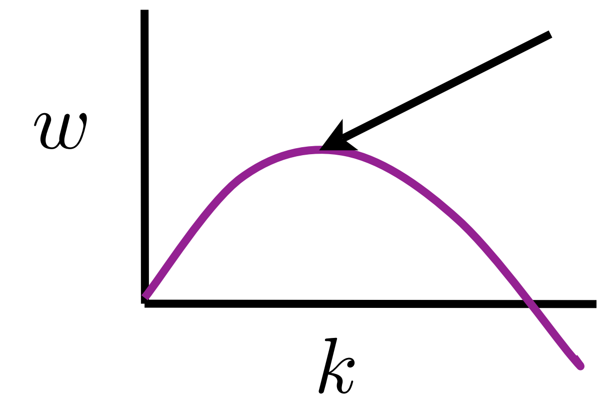
$$m(\infty) \sim m(t_f) \sim \epsilon^3 \qquad \alpha = 3$$

Pattern selection

- Linear stability analysis

$$P - 1 \propto e^{i(kx+wt)} \quad \Longrightarrow \quad w(k) = \frac{8}{k}\sin\frac{k}{2} - \frac{2}{k}\sin k - 2$$

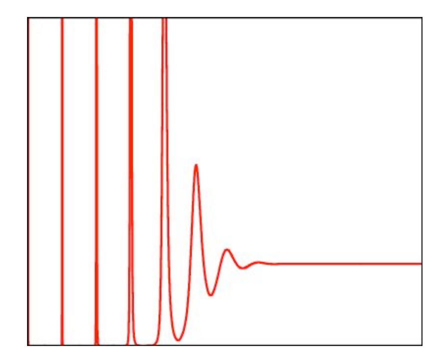
- Fastest growing mode



$$\frac{dw}{dk} \quad \Longrightarrow \quad L = \frac{2\pi}{k} = 2.2515$$

- Traveling wave (FKPP saddle point analysis)

$$\frac{dw}{dk} = \frac{\mathrm{Im}(w)}{\mathrm{Im}(k)} \quad \Longrightarrow \quad L = \frac{2\pi}{k} = 2.0375$$



Patterns induced by wave propagation from boundary
However, emerging period is different

$$2.0375 < L < 2.2515$$

Pattern selection is intrinsically nonlinear

Discrete opinions

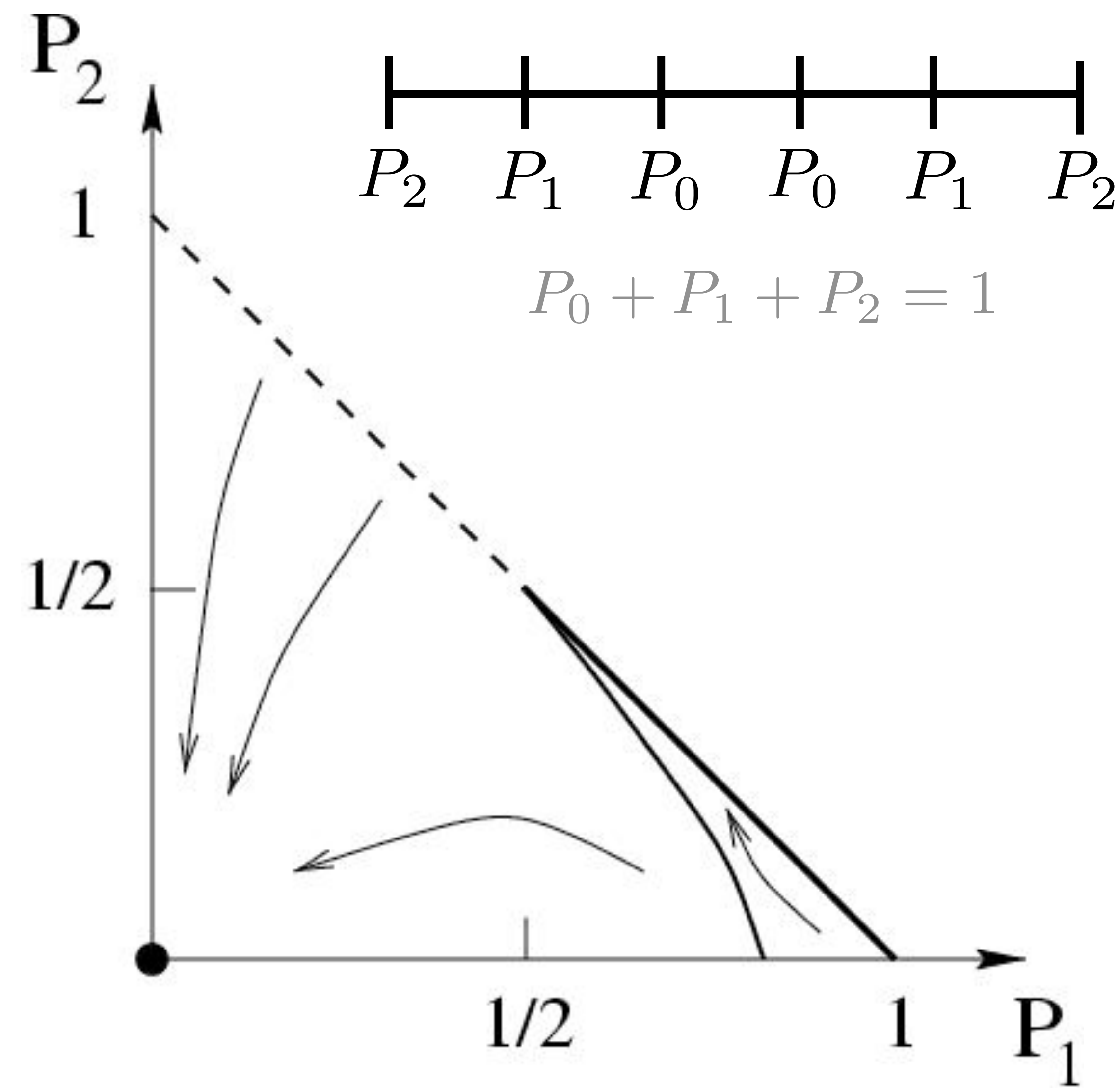
- Compromise process

$$(n-1, n+1) \to (n, n)$$

- Master equation

$$\frac{dP_n}{dt} = 2P_{n-1}P_{n+1} - P_n(P_{n-2} + P_{n+2})$$

- Simplest example: 6 states
- Symmetry + normalization:
- Two-dimensional problem

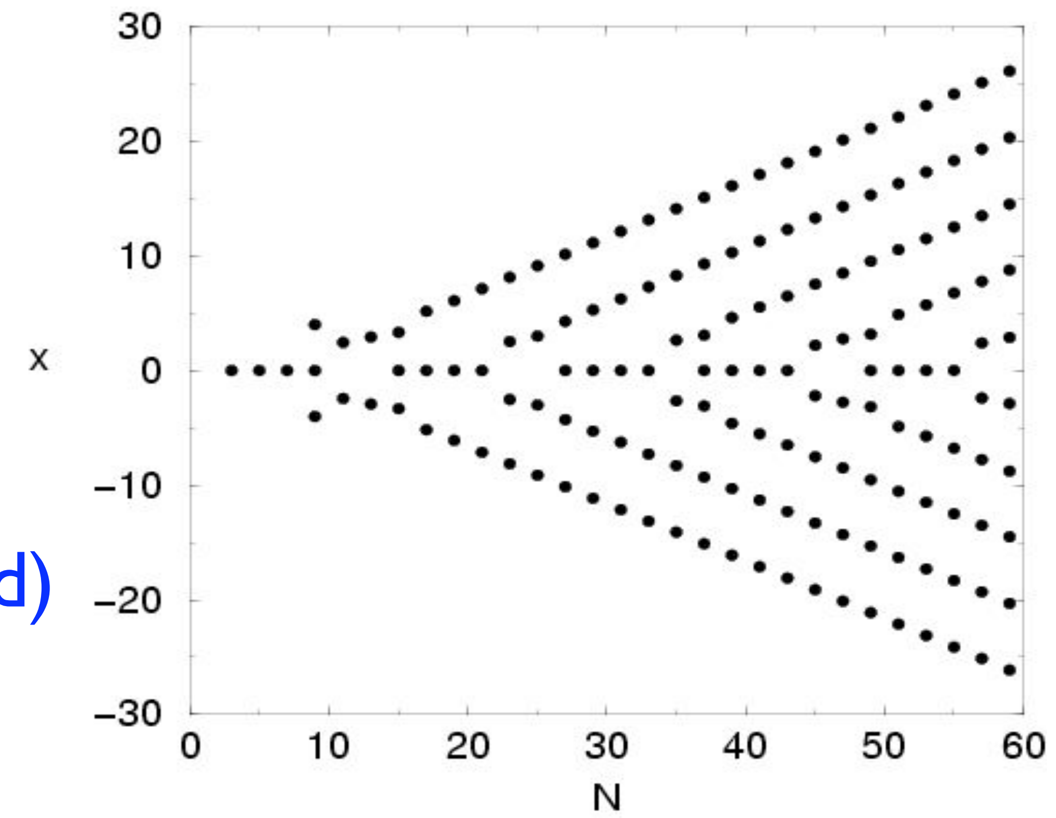


Initial condition determines final state

Isolated fixed points, lines of fixed points

Discrete opinions

- Dissipative system, volume contracts
- Energy (Lyapunov) function exists
- No cycles or strange attractors
- Uniform state is unstable (Cahn-Hilliard)



$$P_i = 1 + \phi_i \qquad \phi_t + (\phi + a\,\phi_{xx} + b\,\phi^2)_{xx}$$

Discrete case yields useful insights

Pattern selection

- Linear stability analysis

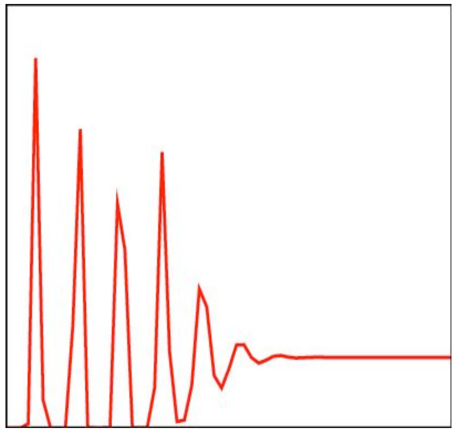
$$P - 1 \propto e^{i(kx+wt)} \longrightarrow w(k) = 4\cos k - 4\cos 2k - 2$$

- Fastest growing mode

$$\frac{dw}{dk} \quad \Longrightarrow \quad L = \frac{2\pi}{k} = 6$$

- Traveling wave (FKPP saddle point analysis)

$$\frac{dw}{dk} = \frac{\mathrm{Im}(w)}{\mathrm{Im}(k)} \quad \Longrightarrow \quad L = \frac{2\pi}{k} = 5.31$$



Again, linear stability gives useful upper and lower bounds

$$5.31 < L < 6 \qquad \text{while} \quad L_{\text{select}} = 5.67$$

Pattern selection is intrinsically nonlinear

I. Restricted averaging: conclusions

- Clusters form via bifurcations
- Periodic structure
- Alternating major-minor pattern
- Central party does not always exist
- Power-law behavior near transitions
- Nonlinear pattern selection

I. Outlook

- Pattern selection criteria
- Gaps
- Role of initial conditions, classification
- Role of spatial dimension, correlations
- Disorder, inhomogeneities
- Tiling/Packing in 2D
- Discord dynamics (seceder model, Halpin-Heally 03)

Many open questions

11. Restricted averaging with noise

Diffusion (noise)

- **Diffusion:** Individuals change opinion spontaneously

$$n \xrightarrow{D} n \pm 1$$

or

- Adds noise (“temperature”)
- Linear process: no interaction
- Mimics unstable, varying opinion
- Influence of environment, news, editorials, events

Rate equations

- **Compromise:** reached through pairwise interactions

$$(n-1, n+1) \to (n, n)$$

- Conserved quantities: total population, average opinion
- Probability distribution $P_n(t)$
- Kinetic theory: nonlinear rate equations

$$\frac{dP_n}{dt} = 2P_{n-1}P_{n+1} - P_n(P_{n-2} + P_{n+2}) + D(P_{n-1} + P_{n+1} - 2P_n)$$

☐ Direct Monte Carlo simulations of stochastic process

☑ Numerical integration of rate equations

Single-party dynamics

- Initial condition: large isolated party

$$P_n(0) = m(\delta_{n,0} + \delta_{n,-1})$$

- Steady-state: compromise and diffusion balance

$$DP_n = P_{n-1}P_{n+1}$$

- Core of party: localized to a few opinion states

$$P_0 = m \qquad P_1 = D \qquad P_2 = D^2 m^{-1}$$

- Compromise negligible for n>2

Party has a well defined core

The tail

- Diffusion dominates outside the core

$$\frac{dP_n}{dt} = D(P_{n-1} + P_{n+1} - 2P_n) \qquad P \ll D$$

- Standard problem of diffusion with source

$$P_n \sim m^{-1}\Psi(n\,t^{-1/2})$$

- Tail mass

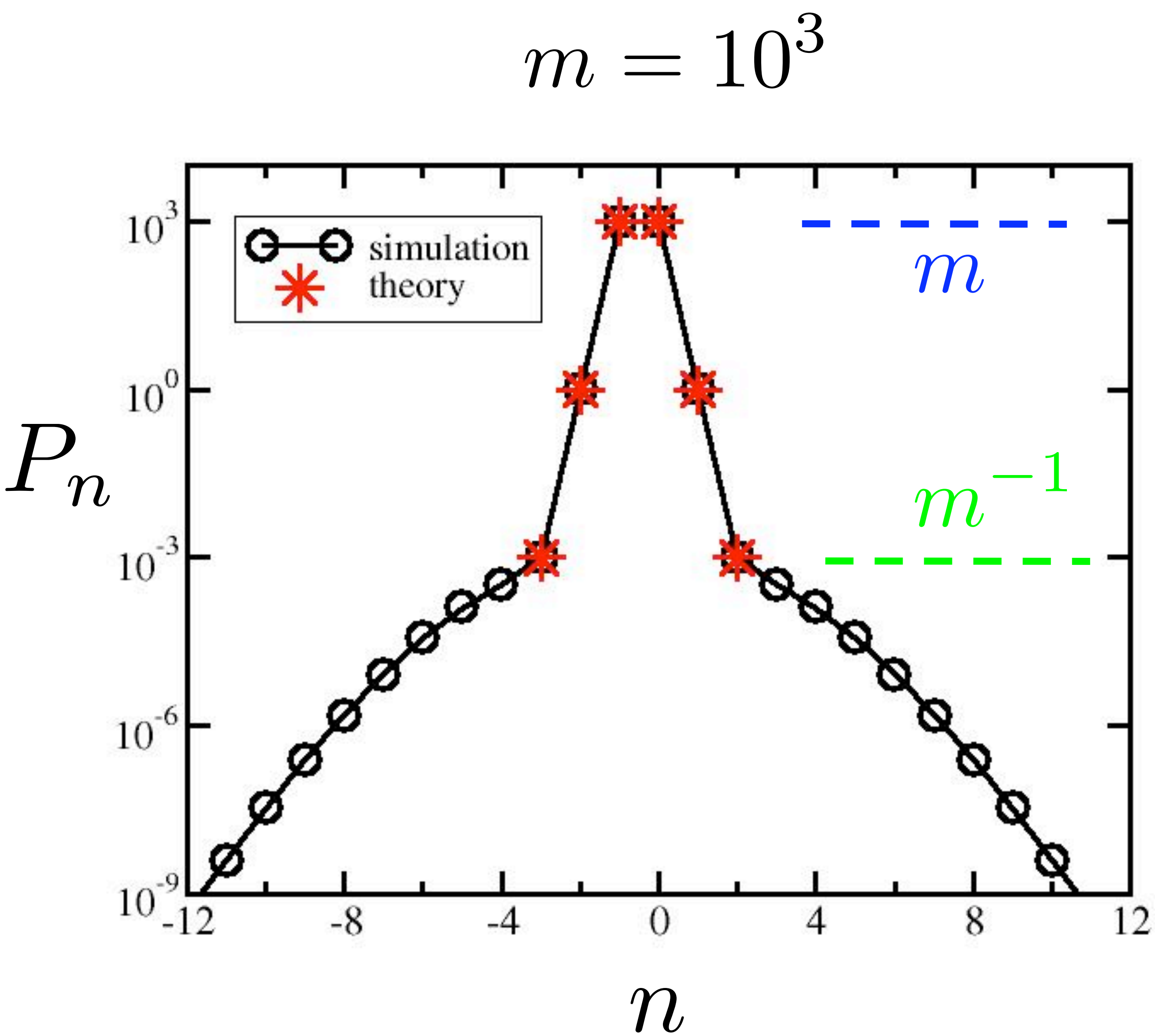
$$M_{\text{tail}} \sim m^{-1} t^{1/2}$$

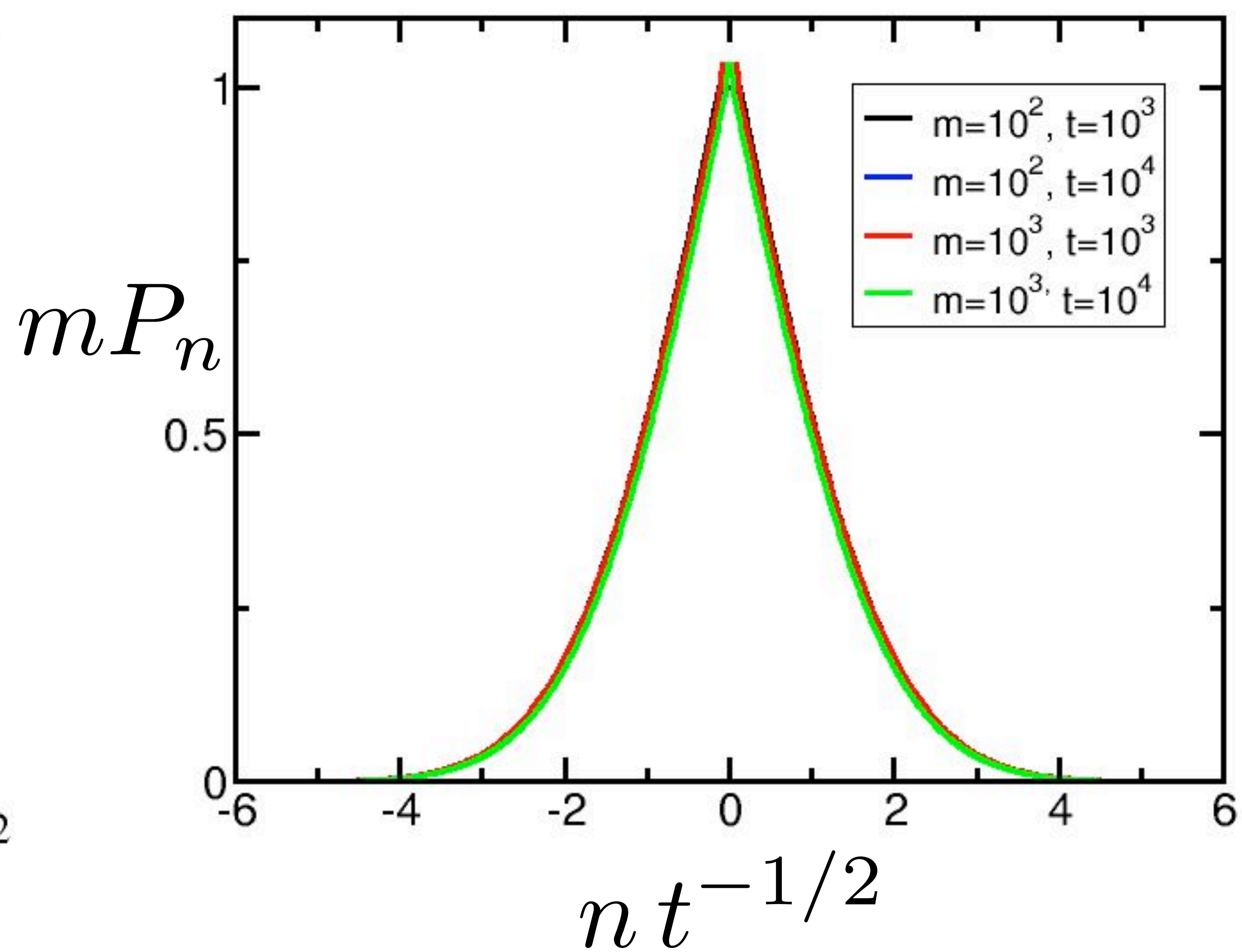
- Party dissolves when

$$M_{\text{tail}} \sim m \qquad \Longrightarrow \qquad \tau \sim m^4$$

Party lifetime grows dramatically with its size

Core versus tail





Party height=m

Party depth~m^{-1}

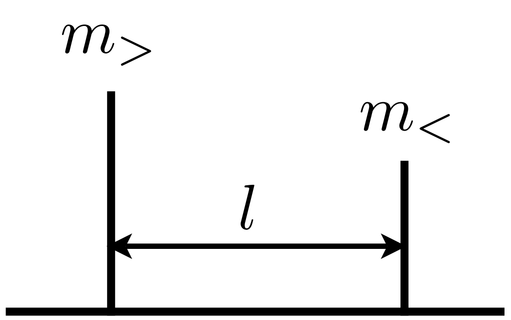
Self-similar shape

Gaussian tail

Qualitative features

- Exists in a quasi-steady state
- Tight core localized to a few sites
- Random opinion changes of members do not affect party position
- Party lifetime grows very fast with size
- Ultimate fate of a party: demise
- Its remnant: a diffusive cloud
- Depth inversely proportional to size, the larger the party the more stable

Two party dynamics



- Initial condition: two large isolated parties

$$P_n(0) = m_> \left(\delta_{n,0} + \delta_{n,-1}\right) + m_< \left(\delta_{n,l} + \delta_{n,l+1}\right)$$

- Interaction between parties mediated by diffusion

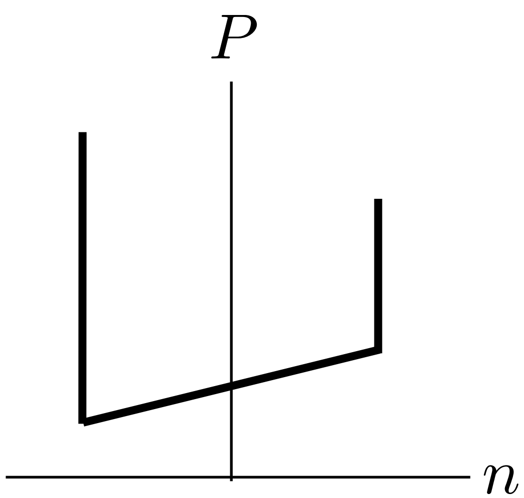
$$0 = P_{n-1} + P_{n+1} - 2P_n$$

- Boundary conditions set by parties depths

$$P_0 = \frac{1}{m_>} \qquad P_l = \frac{1}{m_<}$$

- Steady state: linear profile

$$P_n = \frac{1}{m_<} + \left(\frac{1}{m_<} - \frac{1}{m_>}\right)\frac{n}{l}$$



Merger

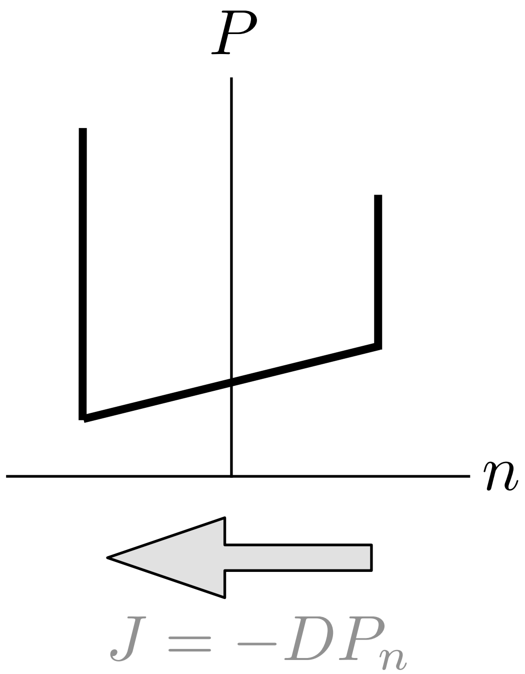
- Steady flux from small party to larger one

$$J \sim \frac{1}{l}\left(\frac{1}{m_<} - \frac{1}{m_>}\right) \sim \frac{1}{lm_<}$$

- Merger time

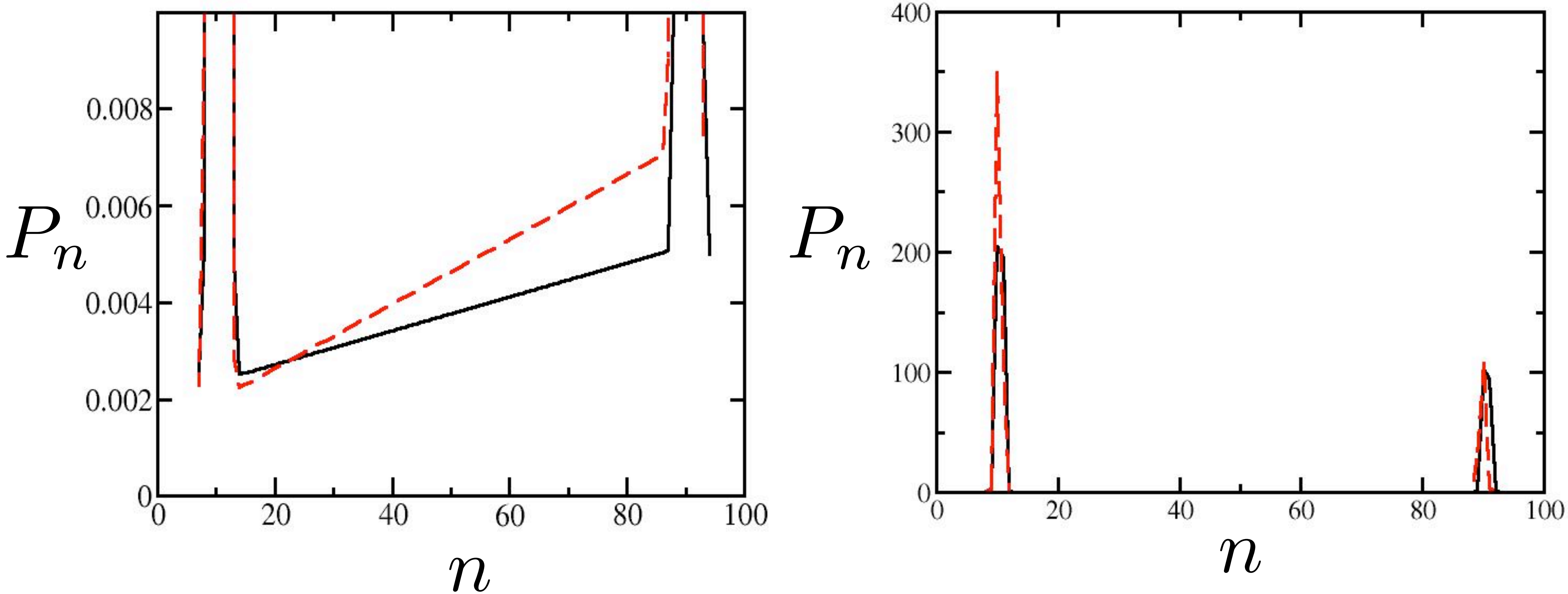
$$T \sim \frac{m_<}{J} \sim lm_<^2$$

- Lifetime grows with separation (“niche”)
- Outcome of interaction is deterministic
- Larger party position remains fixed throughout merger process



Small party absorbed by larger one

Merger: numerical results



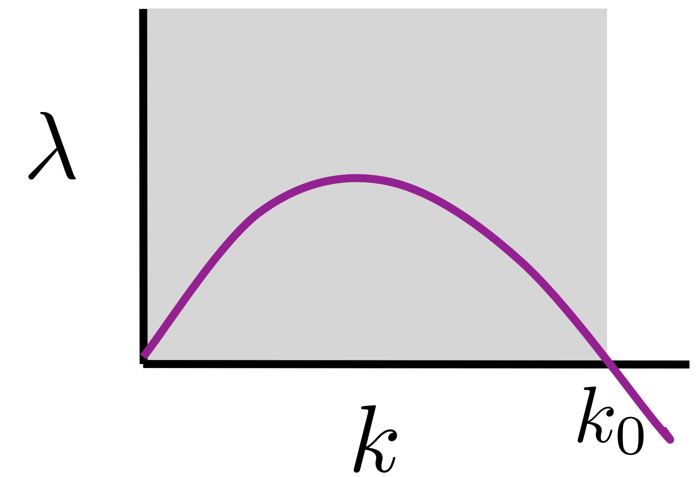
Multiple party dynamics

- Initial condition: large isolated party

$$P_n(0) = \text{randomly chosen number in } [1-\epsilon : 1+\epsilon]$$

- Linear stability analysis

$$P_n - 1 \sim e^{ikn+\lambda t}$$



- Growth rate of perturbations

$$\lambda(k) = (4\cos k - 4\cos 2k - 2) - 2D(1-\cos 2k)$$

- Long wavelength perturbations unstable

$$k < k_0 \qquad \cos k_0 = D/2$$

P=1 stable only for strong diffusion $D>D_c=2$

Strong noise ($D>D_c$)

- Regardless of initial conditions

$$P_n \to \langle P_n(0) \rangle$$

- Relaxation time

$$\lambda \approx (D_c - D)k^2 \quad \Longrightarrow \quad \tau \sim (D - D_c)^{-2}$$

No parties, disorganized political system

Weak noise ($D<D_c$): Coarsening

- Smaller parties merge into large parties
- Party size grows indefinitely
- Assume a self-similar process, size scale m
- Conservation of populations implies separation

$$l \sim m$$

- Use merger time to estimate size scale

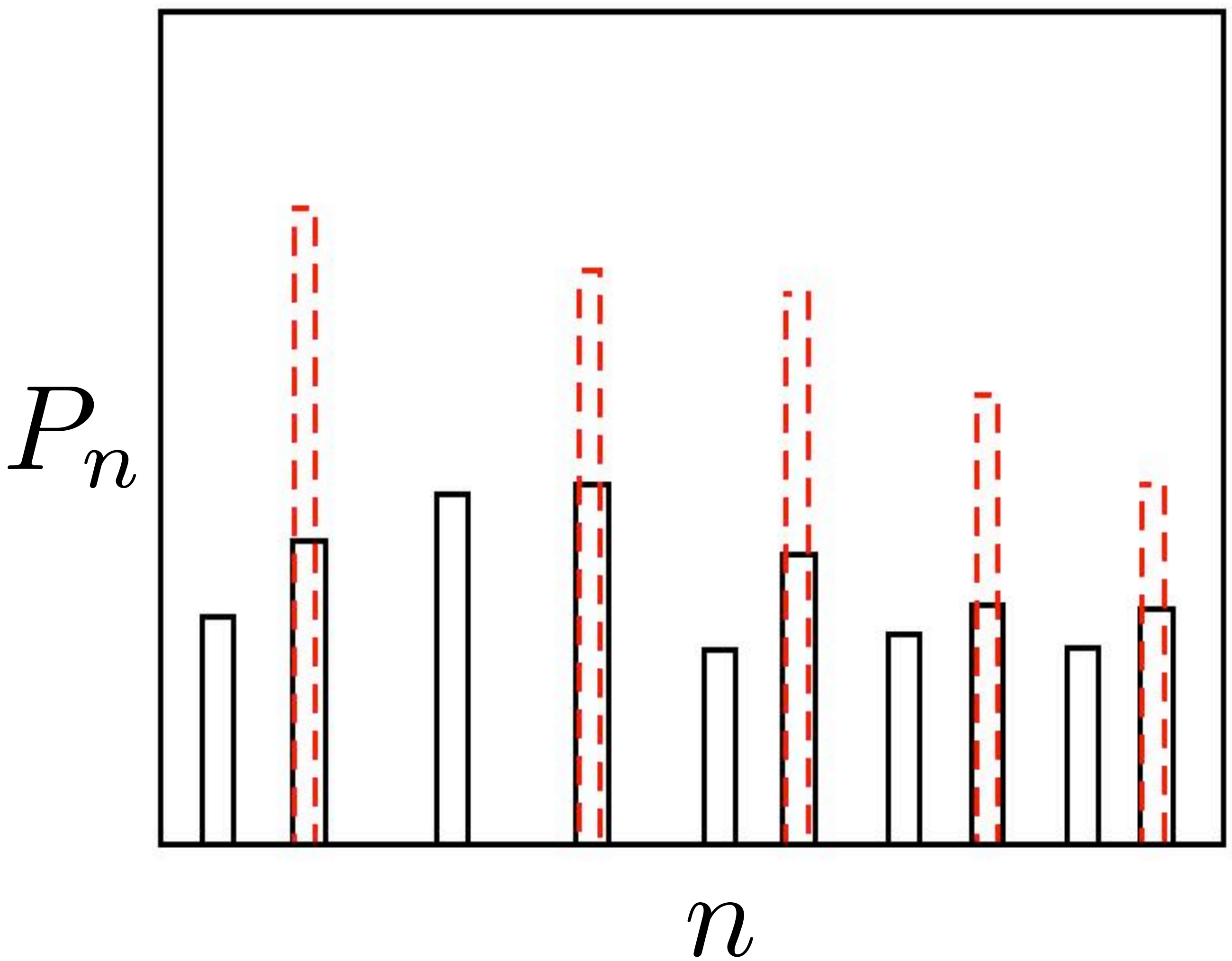
$$t \sim lm^2 \sim m^3 \qquad \Longrightarrow \qquad m \sim t^{1/3}$$

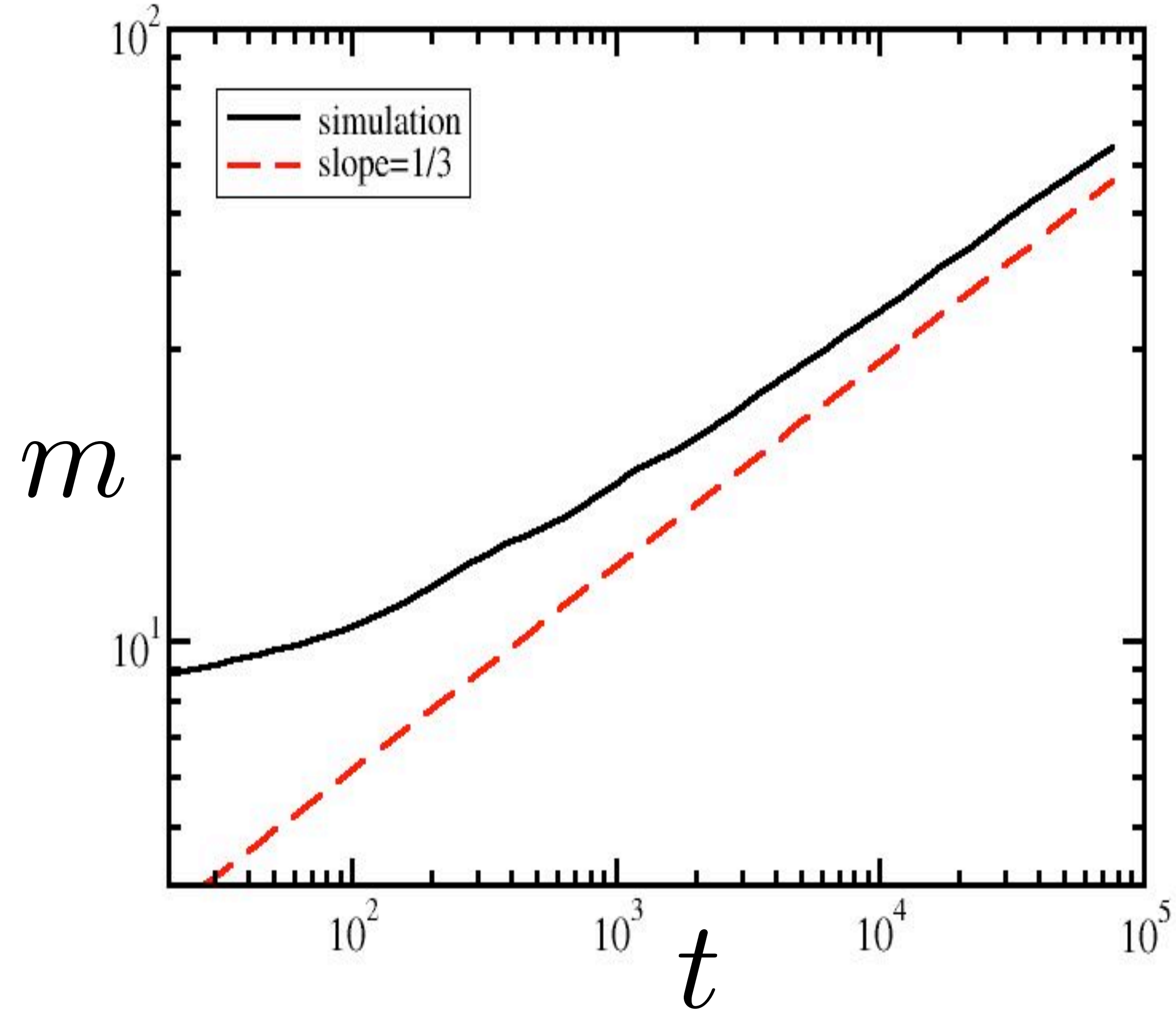
- Self-similar size distribution

$$P_m \sim t^{-1/3} F(m\, t^{-1/3})$$

Lifshitz-Slyozov coarsening

Coarsening: numerical results





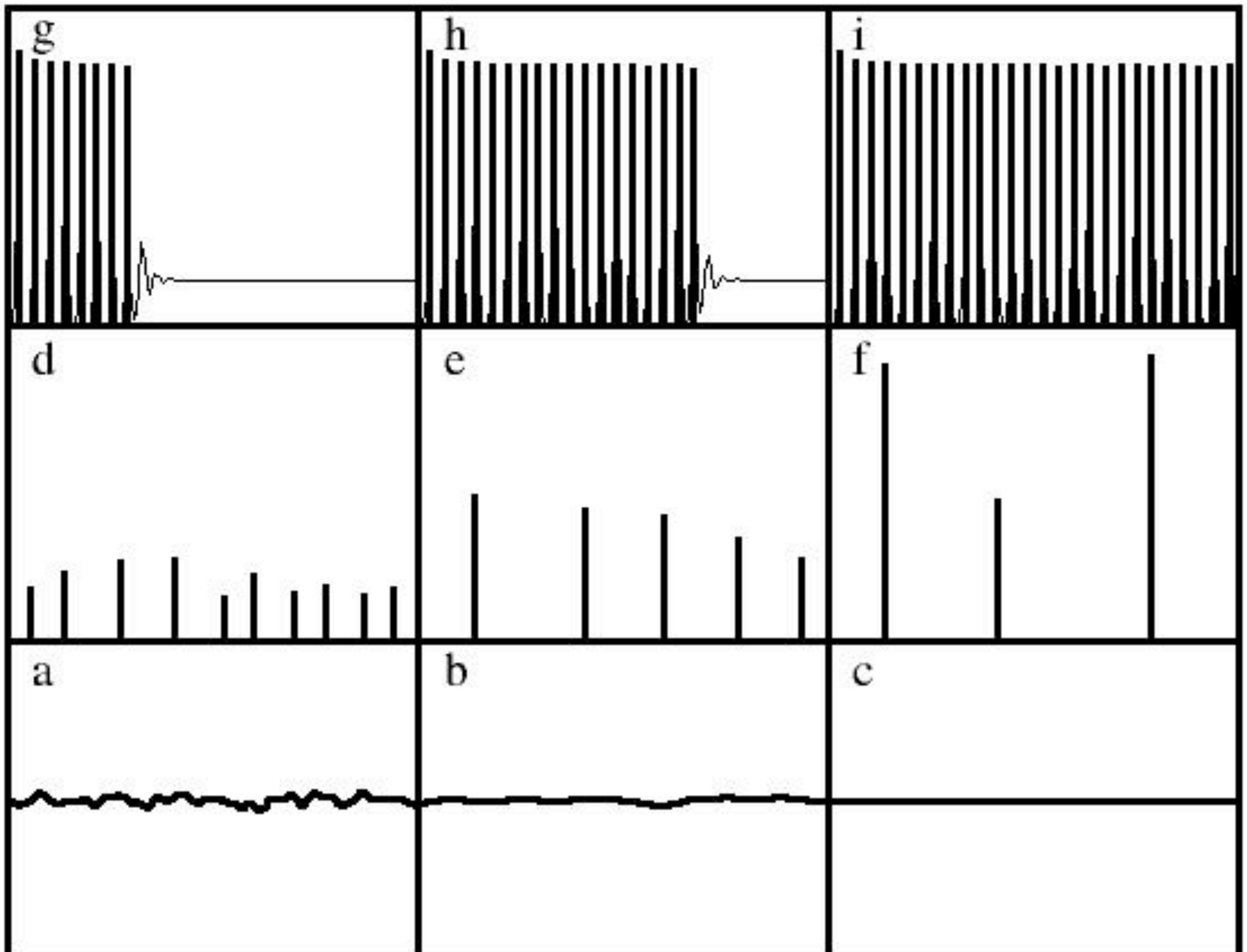
- Parties are static throughout process
- A small party with a large niche may still outlast a larger neighbor!

Three scenarios

$D = 0$

$D < D_c$

$D > D_c$



early intermediate late

11. Restricted averaging with noise: conclusions

- Isolated parties
  - Tight, immobile core and diffusive tail
  - Lifetime grows fast with size
- Interaction between two parties
  - Large party grows at expense of small one
  - Deterministic outcome, steady flux
- Multiple parties
  - Strong noise: disorganized political system, no parties
  - Weak noise: parties form, coarsening mosaic
  - No noise: stable parties, pattern formation

Publications

1. E. Ben-Naim, P.L. Krapivsky, and S. Redner, Physica D **183**, 190 (2003).
2. E. Ben-Naim, Europhys. Lett. **69**, 671 (2005).

"I can calculate the motions of heavenly bodies, but not the madness of people."

Isaac Newton