Fragmentation of Random Trees

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Z. Kalay and E. Ben-Naim, J. Phys. A **48**, 045001 (2015)
poster & paper available from: http://cnls.lanl.gov/~ebn

Random Graph Processes, Austin TX, March 11, 2016

Formation of a Random Tree

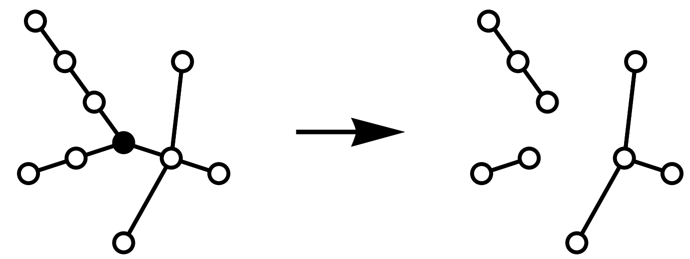
- Start with a single node, the root
- Nodes are added one at a time
- Each new node links to a randomly-selected existing node
- A single connected component with N nodes, N-1 links
- Degree distribution is exponential

$$n_k = 2^{-k}$$

- In-component degree distribution is power-law

$$b_s = \frac{1}{s(s+1)}$$

Fragmentation of a Random Tree



- Nodes are removed one at a time: many previous studies on removal of *links* [Janson, Baur, Bertoin, Kuba]
- When a node is removed, all links associated with it are removed as well
- **Random Forest:** a collection of trees formed by the node removal process
- Degree distribution of individual nodes is known (Moore/Ghosal/Newman PRE 2006)

What is the size distribution of trees in the forest?

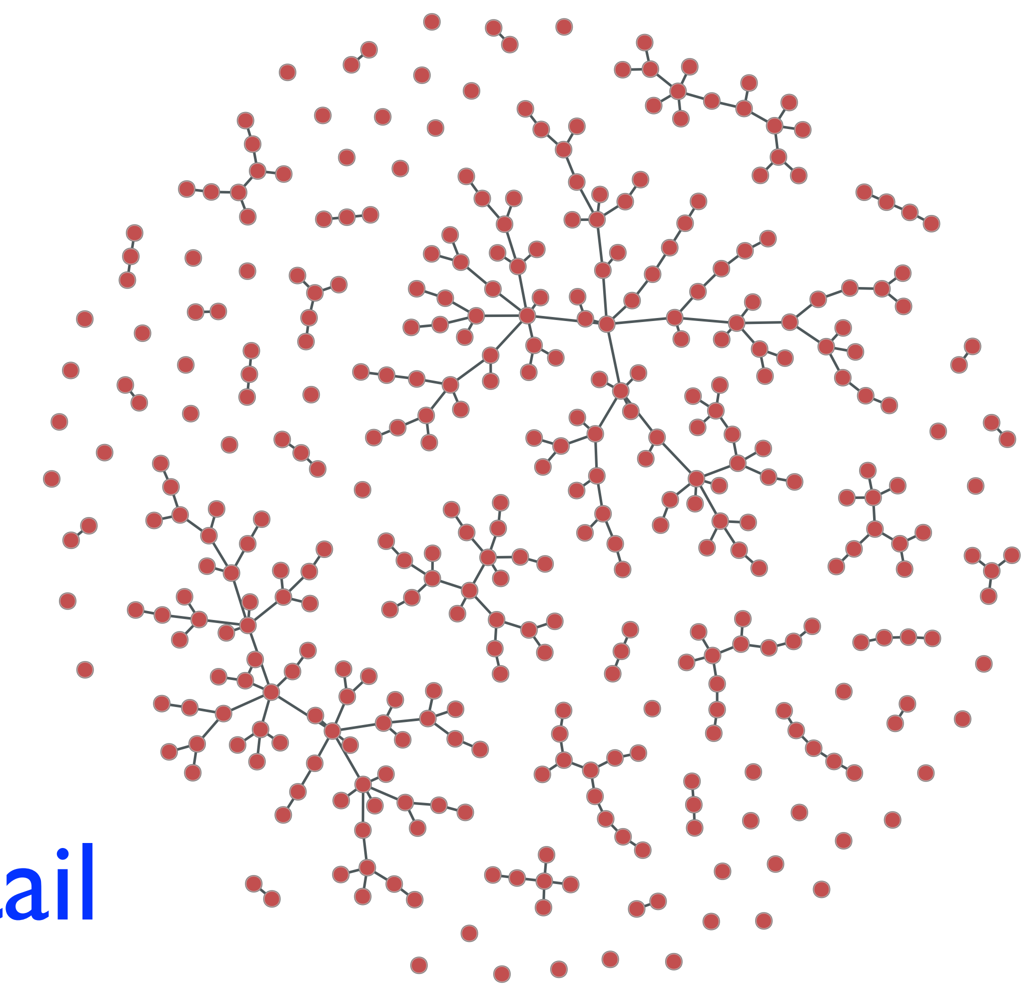
Main Result:
Size Distribution of Trees in Random Forest

distribution of trees of size s is controlled by one parameter: fraction m of remaining nodes*

$$\phi_s = \frac{1-m}{m^2}\frac{\Gamma(s)\Gamma(\frac{1}{m})}{\Gamma(s+1+\frac{1}{m})}$$

size distribution has a power-law tail

$$\phi_s \sim s^{-1-\frac{1}{m}} \qquad \text{for} \qquad s \gg 1$$



*exact result, valid in the infinite N limit

Removal of a Single Node

- Remove a single, randomly-chosen, node from a random tree with N nodes
- Let $P_{s,N}$ be the average number of trees with size s
- Two "conservation" laws

$$\sum_s P_{s,N} = \frac{2(N-1)}{N} \qquad \text{and} \qquad \sum_s s\, P_{s,N} = N-1$$

tree with N nodes has N-1 links
every link connects two nodes

removal of a single node
reduces total size by 1

- Recursion equation (add node to original random tree)

$$P_{s,N+1} = \frac{N}{N+1}\left(\frac{s-1}{N}P_{s-1,N} + \frac{N-s}{N}P_{s,N}\right) + \frac{1}{N+1}\left(\delta_{s,1} + \delta_{s,N}\right)$$

existing trees
grow in size due to new node

new trees
attributed to new node

Size Distribution of Trees

- Manual iteration of recursion equation gives

$$P_{s,2} = \left(\tfrac{1}{1\cdot 2} + \tfrac{1}{1\cdot 2}\right) \delta_{s,1}$$
$$P_{s,3} = \left(\tfrac{1}{1\cdot 2} + \tfrac{1}{2\cdot 3}\right) (\delta_{s,1} + \delta_{s,2})$$
$$P_{s,4} = \left(\tfrac{1}{1\cdot 2} + \tfrac{1}{3\cdot 4}\right) (\delta_{s,1} + \delta_{s,3}) + \left(\tfrac{1}{2\cdot 3} + \tfrac{1}{2\cdot 3}\right) \delta_{s,2}$$
$$P_{s,5} = \left(\tfrac{1}{1\cdot 2} + \tfrac{1}{4\cdot 5}\right) (\delta_{s,1} + \delta_{s,4}) + \left(\tfrac{1}{2\cdot 3} + \tfrac{1}{3\cdot 4}\right) (\delta_{s,2} + \delta_{s,3})$$

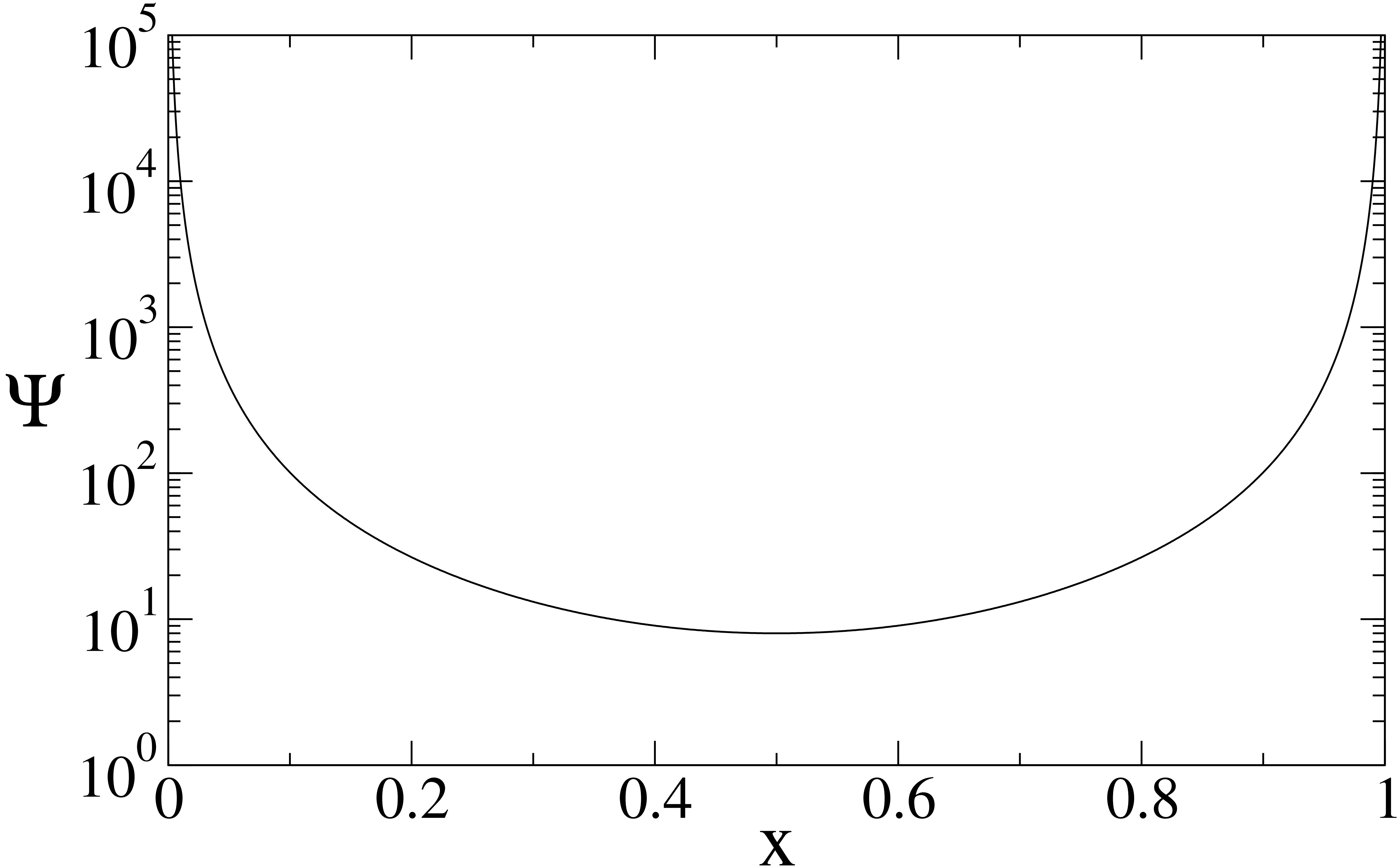
- By induction: incredibly simple distribution

$$P_{s,N} = \frac{1}{s(s+1)} + \frac{1}{(N-s)(N+1-s)}$$

- Scaling form

$$P_{s,N} \simeq \frac{1}{N^2}\Psi\left(\frac{s}{N}\right) \qquad \Psi(x) = \frac{1}{x^2} + \frac{1}{(1-x)^2}$$

The Scaling Function



Iterative Removal of Nodes

- Remove randomly-selected nodes, one at a time
- Key observation: all trees in the random forest are <u>statistically equivalent</u> to a random tree!
- Treat the number of removed nodes as time t
- Let $F_{s,N}(t)$ be the average number of trees with size s at time t
- A single conservation law

$$\sum_s s\, F_{s,N}(t) = N - t$$

- Recursion equation (represents removal of one node)

$$F_s(t+1) = F_s(t) - \underbrace{s f_s(t)}_{\text{loss of trees}} + \underbrace{\sum_{l>s} l\, f_l(t)\, P_{s,l}}_{\text{gain of trees}} \quad \text{with} \quad f_s(t) = \frac{F_s(t)}{\sum_s s\, F_s(t)}$$

<u>loss of trees</u>
loss rate =
tree size

<u>gain of trees</u>
by fragmentation
of larger ones

normalized
tree-size distribution

Rate Equation Approach

- Take the infinite tree-size limit: $N \to \infty$
- Treat time as continuous variable
- Recursion equation becomes a differential equation

$$\frac{dF_s}{dt} = -s f_s + \sum_{l>s} l\, f_l\, P_{s,l}$$

- Use limiting size distribution, fraction of remaining nodes

$$\phi_s(m) = \lim_{\substack{N\to\infty \\ t\to\infty}} \frac{F_{s,N}\,(t)}{\sum_s s\, F_{s,N}(t)} \qquad \text{and} \qquad m = \frac{N-t}{N}$$

- Problem reduces to the differential equation

$$(\alpha-1)\frac{d\phi_s}{d\alpha} = (1-s)\phi_s + \sum_{l>s}\left[\frac{l\,\phi_l}{s(s+1)} + \frac{l\,\phi_l}{(l-s)(l+1-s)}\right] \qquad \alpha = 1 + \frac{1}{m}$$

fragmentation kernel = size distribution, single node removal

The Size Distribution

- *Miraculously*, exact solution of the rate equation feasible

$$\phi_s = \frac{1-m}{m^2}\frac{\Gamma(s)\Gamma(\frac{1}{m})}{\Gamma(s+1+\frac{1}{m})}$$

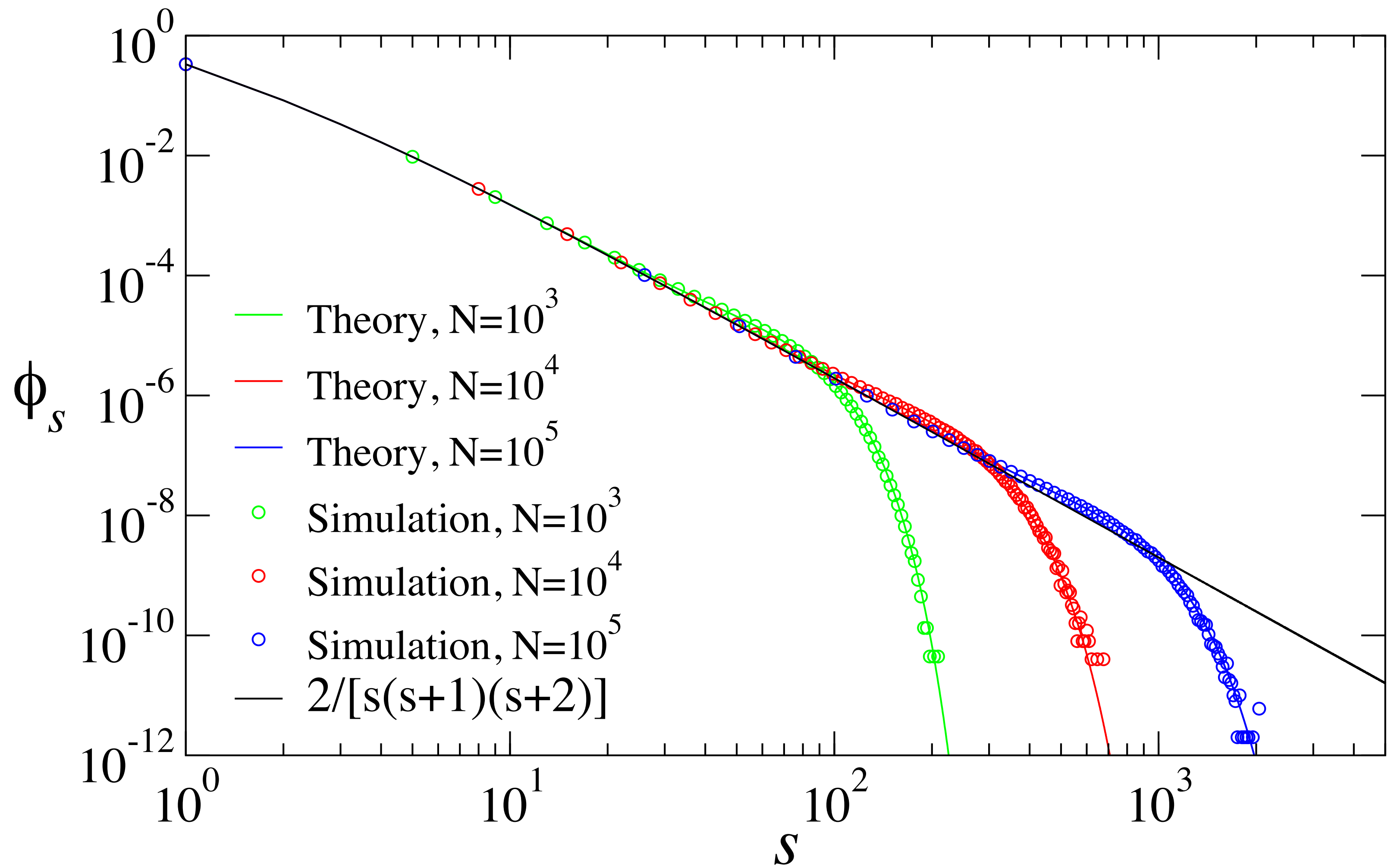
- Power-law tail

$$\phi_s \sim s^{-1-\frac{1}{m}}$$

- Special case

$$m = 1/2$$

$$\phi_s = \frac{2}{s(s+1)(s+2)}$$



Addition and Removal of Nodes

- **Addition**: Nodes are added at constant rate r
- **Removal**: Nodes are removed at constant rate 1
- Outcome: random forest with growing number of nodes
- Straightforward generalization of rate equation

$$\frac{dF_s}{dt} = r\left[(s-1)f_{s-1} - sf_s\right] - sf_s(t) + \sum_{l>s} l\, f_l(t)\, P_{s,l}.$$

- Normalized distribution of tree size decays exponentially

$$\phi_s \sim s^{-r}\left(1-e^{-r}\right)^s$$

Summary

- Studied fragmentation of a random tree into a random forest
- Nodes removed one at a time
- Distribution of tree size becomes universal in the limit of infinitely many nodes
- Distribution of tree size has a power law tail
- Exponent governing the power law depends only on the fraction of remaining nodes
- Rate equation approach is a powerful analysis tool