Scaling and Multiscaling in Fragmentation

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Fragmentation underlies many physical systems

Physical phenomena	Fragmented quantity
Turbulence	Energy, Momentum
Elementary Particles Collisions	Energy
Shattering, Grinding	Mass
Meteor impact	Mass
Surface Adsorption	Area
Membrane Wrinkling	Area

I. Uniform Fragmentation in one dimension

Fig. 1: The random fragmentation process in one dimension.

$P(x,t)$ — The distribution of fragments of length (mass) x evolves according to the rate equation

$$\frac{\partial P(x,t)}{\partial t} = -xP(x,t) + 2\int_x^\infty dy P(y,t).$$

The loss rate is proportional to the interval length since the fragmentation is uniform, while the gain rate reflects the fact that two interval are gained in each event. The asymptotic length distribution

$$P(x,t) \sim t^2 e^{-xt},$$

is characterized by a single scale $\langle x \rangle \sim t^{-1}$. The average moments satisfy $\langle x^n \rangle \sim \langle x \rangle^n$, and the length distribution can be written as

$$P(x,t) = \frac{1}{\langle x \rangle^2}\Phi\left(\frac{x}{\langle x \rangle}\right).$$

Simple scaling distribution

II. Fragmentation in two dimensions

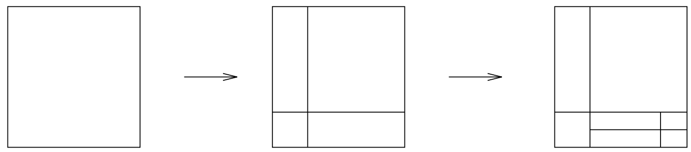

Fig. 2: Points deposited with unit rate on the unit square resulting in four rectangles.

$P(x_1, x_2; t)$ — the distribution of rectangles of size $x_1 \times x_2$.

$M(n_1, n_2; t) = \int\int dx_1 dx_2 x_1^{n_1-1} x_2^{n_2-1} P(x_1, x_2; t)$ — its moments.

Both satisfy *linear* rate equations

$$\frac{\partial P(x_1, x_2; t)}{\partial t} = -x_1 x_2 P(x_1, x_2; t) + 4 \int_{x_1}^{\infty} \int_{x_2}^{\infty} dy_1 dy_2 P(y_1, y_2; t)$$

$$\frac{\partial M(n_1, n_2; t)}{\partial t} = \left(\frac{4}{n_1 n_2} - 1\right) M(n_1 + 1, n_2 + 1; t)$$

Infinitely many conserved integrals

In addition to conservation of the total area ($M(2, 2; t) = \text{const.}$), all moments $M(n_1, n_2; t)$ with $n_1 n_2 = 4$ are time independent. These integrals imply that no ordinary scaling solution is possible for the size distribution.

Exact Solution for the moments

$$\langle x_1^{n_1} x_2^{n_2} \rangle = \frac{M(n_1+1, n_2+1; t)}{M(1,1;t)} \sim t^{-\alpha(n_1,n_2)}$$

$$\alpha(n_1, n_2) = \frac{n_1 + n_2 + 4 - \sqrt{(n_1 - n_2)^2 + 16}}{2}$$

The total number of fragments, $N = M(1,1;t) = 1+3t$ implies that the average area of a fragment is $\langle A \rangle = \langle x_1 x_2 \rangle \sim t^{-1}$. If scaling was to hold, the average length $\langle x_1^n \rangle \sim \langle A \rangle^{n/2} \sim t^{-n/2}$. However, we find that $\langle x_1 \rangle \sim t^{-(5-\sqrt{17})/2} \sim t^{-.438}$, $\langle x_1^2 \rangle \sim t^{-(3-\sqrt{5})} \sim t^{-.764}$, *etc.*

Multiple scales characterize the patterns

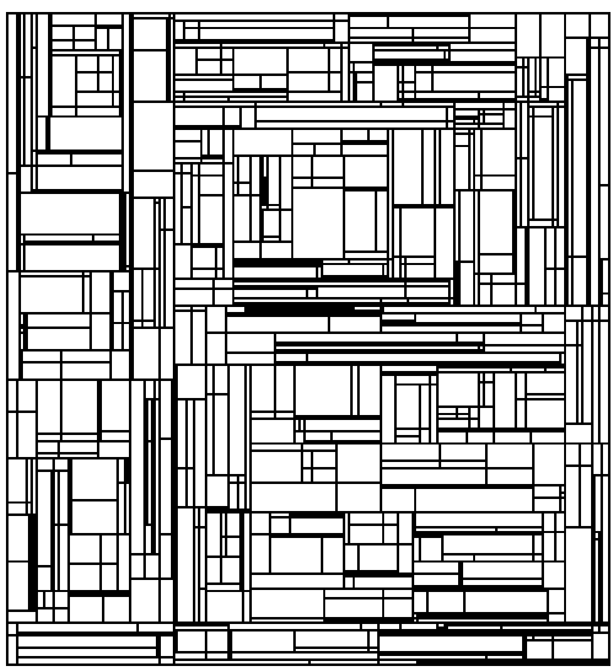

Fig. 2 fragmentation pattern at time $t = 1000$.

$\boxed{\text{The aspect ratio diverges}}$

$$\langle (x_1/x_2)^n \rangle \sim t^{\sqrt{n^2+4}-1} \qquad |n| < 1$$

$\boxed{\text{Nevertheless, the area distribution scales}}$

$P(A, t)$ — the distribution of rectangles of area $A = x_1 x_2$ can be written as a scaling function, $P(A, t) \sim \langle A \rangle^{-2} \Phi(z)$, with the scaling variable $z = A/\langle A \rangle$. Therefore, the moments obey ordinary scaling, $\langle A^n \rangle \sim \langle A \rangle^n \sim t^{-n}$. The scaling distribution exhibits a weak logarithmic singularity in the limit of small fragments

$$\Phi(z) \sim \begin{cases} \ln(1/z) & z \ll 1; \\ z^{-2} e^{-z} & z \gg 1. \end{cases}$$

III. Isotropic fragmentation

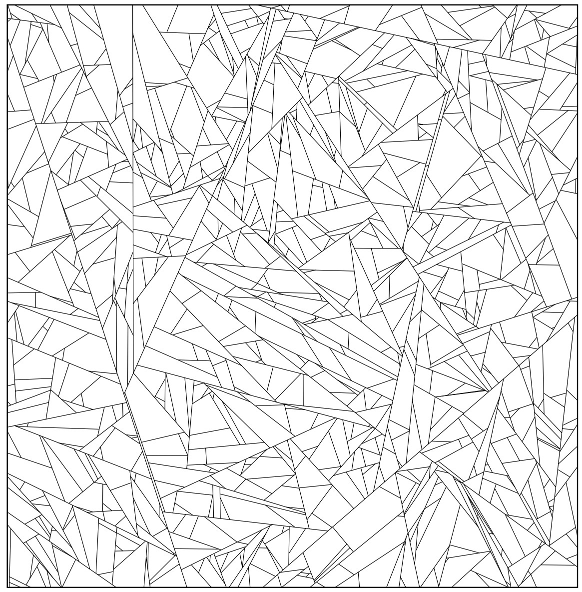

Fig. 4 A realization of the Isotropic case with 1000 fragments.

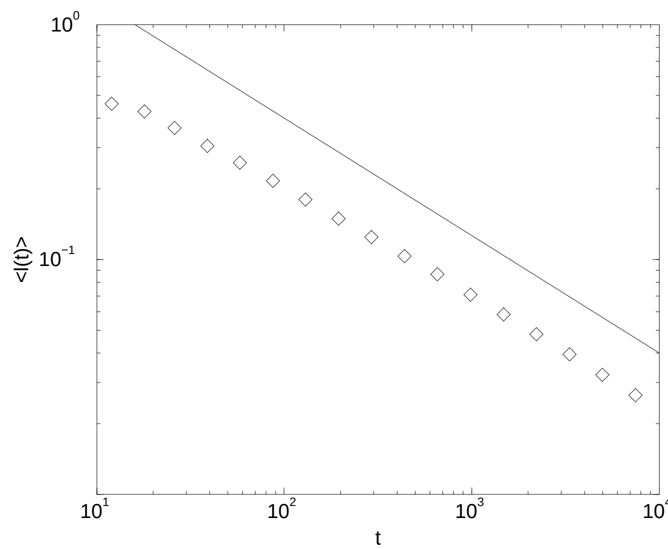

Fig. 4 The average fragment length vs. time (the reference line has $-1/2$ slope).

When the "cutting" angle is random, numerical simulations suggest that scaling is restored, $\langle l^n \rangle \sim \langle l \rangle^n \sim t^{-n/2}$.

Generalizations and Applications

- **Higher dimensions**: The moment method is applicable in arbitrary dimension. The behavior is similar to the 2D case — Volume exhibits scaling. Length, surface area, *etc.* shows multiscaling.

- **Stochastic fractals**: Many fractal structures are actually generated by fragmentation processes. By cutting the unit interval into three pieces at random points, one obtains the stochastic counterparts of the Cantor set. Fragmentations technique can be used here as well to quantify many statistical properties.

References

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