Dissipation Dynamics of Granular Gases

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The dynamics of inelastically colliding gases is investigated using event driven numerical simulations and scaling arguments. Two distinct regimes of behavior are found. For times less than a typical scale, the dissipation is dominated by particle collisions. The second stage is marked by the formation of multiparticle aggregates. These clusters are no longer transparent to each other and the dynamics reduce to that of a sticky gas. The characteristic time scales and temperature laws are explained using heuristic arguments.

Inelastic Gases

- **Definition:** Ensemble of inelastically colliding particles.

- **Inelastic collisions:** Conserve momentum. Relative velocity is reduced by a factor $0 \leq r \leq 1$, the restitution coefficient. Alternatively, $r = 1 - 2\epsilon$ the inelasticity degree $0 \leq \epsilon \leq 1/2$.

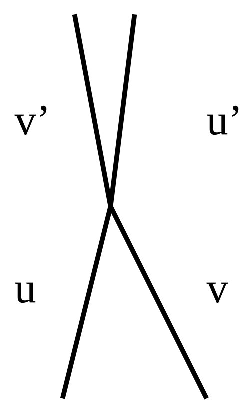


Fig. 1. Illustration of a collision (spacetime diagram).

- **Change in velocity $\Rightarrow$ energy loss:**

$$\Delta v' = r \Delta v \qquad \text{Collision Rule } (\Delta v = v - u)$$

$$v' = v - \epsilon \Delta v$$

$$\Delta E = E - E' = -\epsilon(1-\epsilon)(\Delta v)^2 \qquad \text{Energy Dissipation}$$

Applications

- **Granular materials:** Powders, Sand ($r \cong 0.8 \sim 0.9$).

- **Astrophysics:** Large scale structure formation in the universe.

Problem Setup

- **Space:** N identical equispaced particles on 1-d ring.
- **Initial velocity:** Random (flat) initial velocity distribution with characteristic velocity $= v_0$.
- **Dimensionless variables:** rescale space $x \to x/x_0$ and time $t \to t/t_0$ with $t_0 = x_0/v_0$. Typical initial velocity, spacing $= 1$.

Goals

- **Cooling kinetics:** "Granular Temperature"$= \langle v^2(t) \rangle$ vs. t.
- **Characteristic** time, length scales.

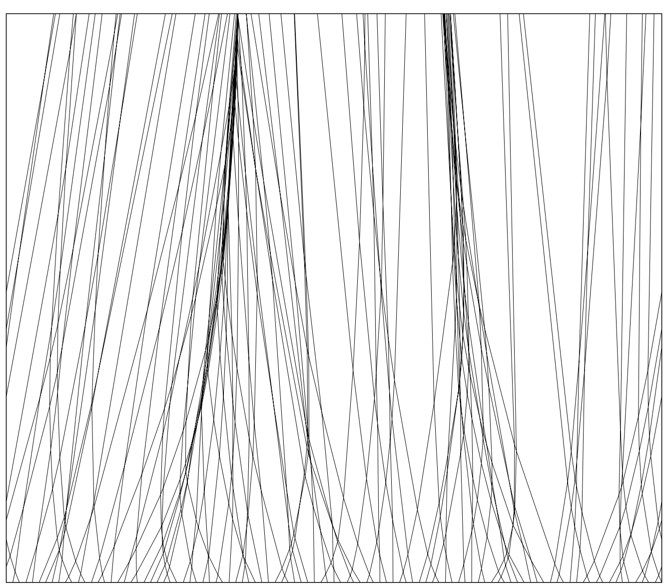


Fig. 2. Spacetime diagram (x vs. t): N=100 particles, $r = 0.75$.

Early Behavior: The Gas Regime

- **Energy dissipation in each collision** $\propto -\epsilon(\Delta v)^2$.

- **Collision frequency** $\sim \Delta x/\Delta v \sim (\Delta v)^{-1}$.

- **Assuming a uniform gas state** $v \sim \Delta v \sim T^{1/2}$

$$\frac{dT}{dt} = v\frac{dv}{dt} \propto \frac{\Delta E}{\Delta t} \propto -\epsilon\frac{(\Delta v)^2}{(\Delta v)^{-1}} \propto -\epsilon(\Delta v)^3 \propto -\epsilon T^{3/2}.$$

- **Cooling law:**

$$T(t) = \frac{1}{(1 + A\epsilon t)^2} \sim \begin{cases} 1 & t \ll \epsilon^{-1} \\ \epsilon^{-2}t^{-2} & t \gg \epsilon^{-1} \end{cases}$$

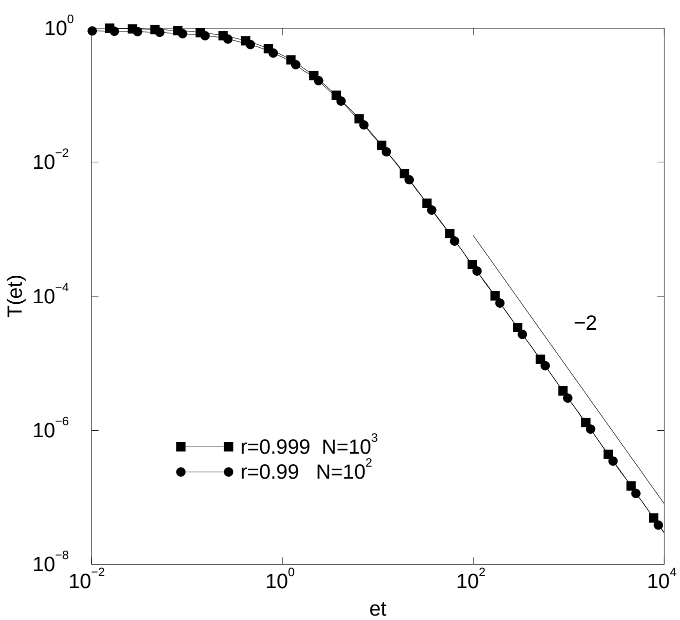


Fig. 3. Temperature decay $T(t)$ vs. ϵt for $\epsilon = 0.05, 0.005$.

$\boxed{\textbf{However, density inhomogeneities eventually develop (Fig. 2)}}$

The Transition: Inelastic Collapse

- **Finite time singularity:** Infinite number of collisions occur in a finite time if the number of particles in the system is sufficiently large $N > N_c(\epsilon)$.

- **Particles no longer transparent:** Aggregate rather than pass through each other.

- **Estimating the critical size:** Imagine a fast ($v_0 = 1$) particle colliding with N slow ($v_1 = 0, \ldots, v_N = 0$) particles

$$(1, 0, \ldots, 0) \to (\epsilon, 1-\epsilon, 0 \ldots, 0)^{\mathrm{N}} \xrightarrow{\text{collisions}} (\epsilon, \ldots, \epsilon(1-\epsilon)^{N-1}, (1-\epsilon)^N)$$

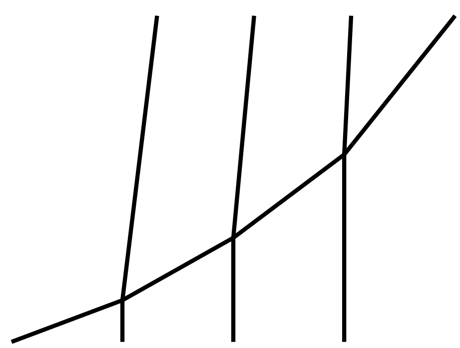


Fig. 4. Fast particle can pass only through sufficiently small clusters.

- **Particle passes through cluster if:** $(1-\epsilon)^N = v > v_1 = \epsilon$

$$N_c(\epsilon) \cong \epsilon^{-1} \ln \epsilon^{-1} \sim \epsilon^{-1}$$

$$\boxed{\textbf{Particles aggregate rather than pass through}}$$

Asymptotic Behavior: The "Sticky Gas" Regime

- **Multiparticle aggregates form:** of typical mass m.

- **Momentum conserved + initially uncorrelated**

$$P_m = \sum_{i=1}^{m} P_i \quad \Rightarrow \quad P \sim m^{1/2}, \quad v \sim m^{-1/2}$$

- **Mass conservation** $\rho =$ density, $c =$ concentration

$$\rho = cm = \text{const} \quad \Rightarrow \quad c \sim m^{-1}$$

- **Dimensional analysis:** t^{-1}=[collision rate]=[cv]=[$m^{3/2}$]

$$m \sim t^{2/3} \qquad T \sim v^2 \sim t^{-2/3}$$

- **Crossover time between two regimes:** $\sim \epsilon^{-3/2}$

  When the typical mass is large enough to reach collapse.

$$m \sim v^{-2} \sim (\epsilon t)^2 \sim \epsilon^{-1} \sim N_c(\epsilon)$$

- **Final state:** 1 aggregate with $m = N$, $v \sim N^{-1/2}$.

$$T(t) \sim \begin{cases} 1 & t \ll \epsilon^{-1}; \\ \epsilon^{-2} t^{-2} & \epsilon^{-1} \ll t \ll \epsilon^{-3/2}; \\ t^{-2/3} & \epsilon^{-3/2} \ll t \ll N^{3/2}; \\ N^{-1} & N^{3/2} \ll t \end{cases}$$

$$\boxed{\textbf{Dynamics independent of } r\textbf{. Effectively, } r = 0}$$

Verification: Numerical Simulations

- **Event-Driven Simulation:** Collision times are sorted.
- **Difficulty with simulation:** Infinite collisions in finite time.
- **Solution:** Elastic collisions ($r = 1$) implemented when relative velocity of colliding particles is below a threshold $dv < \delta << 1$.

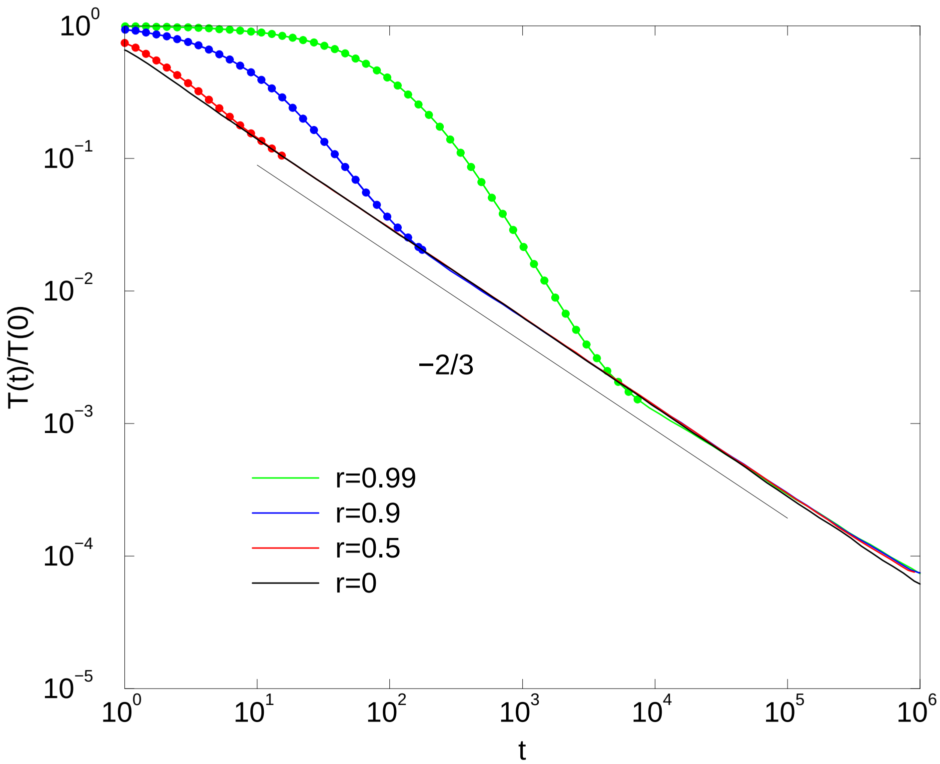


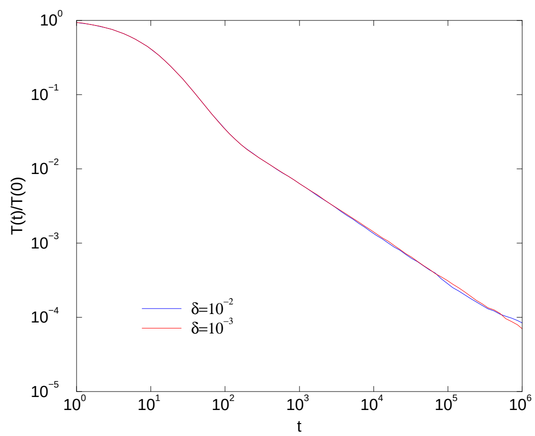
Fig. 5 Temperature vs. time for $r = 0$ (sticky gas), $0.5, 0.9, 0.99$ for $\delta = 10^{-2}$ (lines) and $\delta = 0$ (points). Data represents an average over 10 runs with $N = 10^6$ particles and 10^4 collisions/particle.

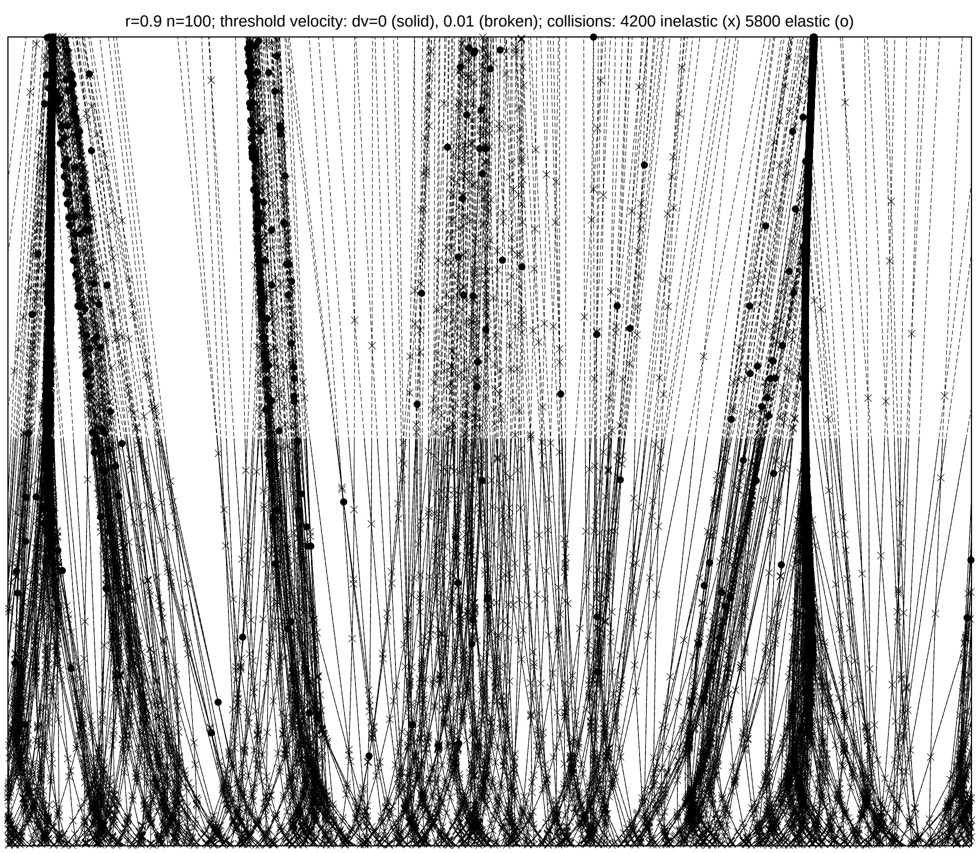
$$\boxed{\textbf{Asymptotic behavior is independent of } r}$$

Validity of Simulation Method

- **Results are valid:** as long as $v \sim t^{-1/3} \gg \delta$, i.e., $t \ll \delta^{-3}$

- **Results independent of threshold:** $N = 10^5$, $r = 0.99$





$\boxed{\textbf{Method faithfully continues past collapse}}$

Continuum Description

- **Sticky Gases** are described by the $\nu \to 0$ limiting behavior of the Nonlinear Diffusion (Burgers) Equation

$$v_t + v v_x = \nu v_{xx} \qquad \text{i.c.} \quad \langle v_0(x) v_0(x') \rangle = \delta(x - x')$$

- **Shocks** correspond to the aggregates. Velocity profile is

$$v(x,t) = \frac{x - q(x,t)}{t} \qquad q(x,t) = \text{lagrangian coordinate}$$

- **Statistics** is identical to sticky gas regime: $T \sim v^2 \sim t^{-2/3}$.

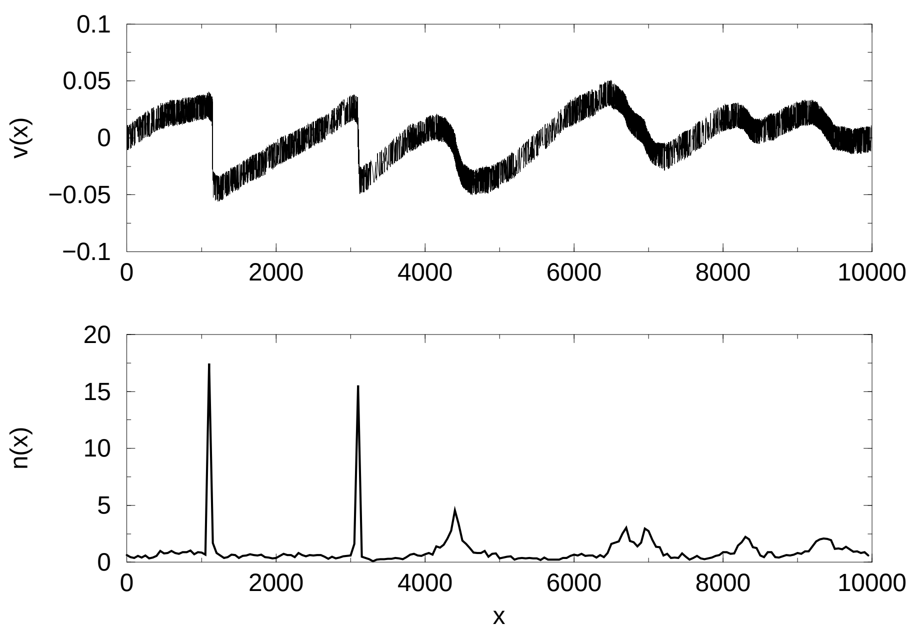


Fig. 8. Shock formation in an inelastic gas. Density and velocity profiles plotted for $t = 2 \times 10^4$, $N = 10^4$, $r = 0.99$. Leftmost shock is a collapse.

Higher Dimensions

- Assuming $r_{\text{eff}} = 0$ then the $\nu \to 0$ limit of the Burgers equation $\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = \nu \nabla^2 \mathbf{v}$ describes the asymptotic behavior.
- $2 \le d \le 4$: $N_c(\epsilon) = \epsilon^{-2d/(4-d)}$ $(d_c = 4)$, Cooling law is

$$T(t) \sim \begin{cases} 1 & t \ll \epsilon^{-1}; \\ \epsilon^{-2} t^{-2} & \epsilon^{-1} \ll t \ll \epsilon^{-4/(4-d)}; \\ t^{-d/2} & \epsilon^{-4/(4-d)} \ll t \ll N^{2/d}; \\ N^{-1} & N^{2/d} \ll t \end{cases}$$

- Similar behavior to simulations of inelastic gases in 2d

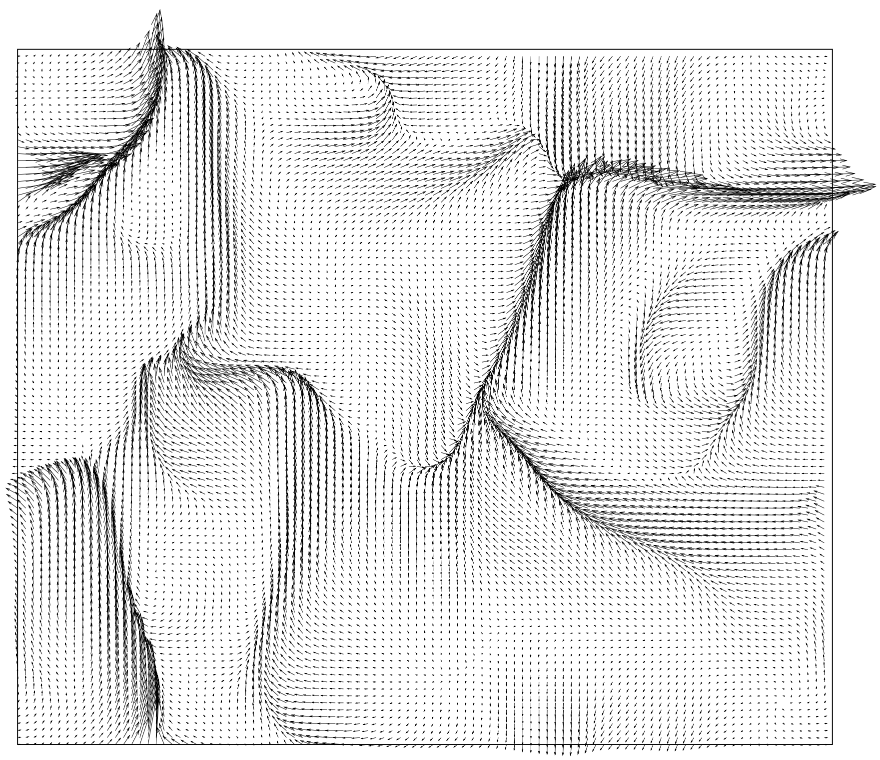


Fig. 9. Velocity field of the 2d Burgers Equation. Results are for $t = 10^3$, 100×100. High gradient regions correspond to high density regions (using the continuity equation $\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0$).

Conclusions

- **Asymptotic behavior of inelastic gases is an aggregation process ("sticky gas").**
- **Behavior can be described by the Burgers Equation.**
- **When $r \to 1$, a diverging time scale characterizes the transition between gas and "sticky gas" regimes.**
- **Only sufficiently small systems $N \ll \epsilon^{-1}$ remain in the gas regime.**

References

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