Multiscaling in Inelastic Collisions

Eli Ben-Naim† and Paul Krapivsky‡

†*Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545*

‡*Department of Physics, Boston University, Boston, MA 02215*

We study relaxation properties of two-body inelastic collisions processes on the mean-field level. We show that this process exhibits multiscaling asymptotic behavior as the underlying distribution is characterized by an infinite set of nontrivial exponents. These nonequilibrium relaxation time scales are found to be closely related to steady state cumulants of the velocity distribution in the presence of noise. This behavior can be viewed as generalized fluctuation-dissipation relations.

E. Ben-Naim and P. L. Krapivsky, *Phys. Rev. E* **61**, R5 (2000).

Inelastic Collisions

$$\begin{pmatrix} u_1 \\ \\ u_2 \end{pmatrix} \longrightarrow \begin{pmatrix} v_1 \\ \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} v_1 \\ \\ v_2 \end{pmatrix} = \begin{pmatrix} \gamma & 1-\gamma \\ \\ 1-\gamma & \gamma \end{pmatrix} \begin{pmatrix} u_1 \\ \\ u_2 \end{pmatrix}.$$

- Restitution coefficient $r = 1 - 2\gamma$
- Energy Dissipation

$$\Delta E \propto \gamma(1-\gamma)(u_1 - u_2)^2/4$$

Motivation

- Granular gases: interparticle collisions are dissipative [1-2]
- Traffic flows: headway distribution [3]
- Combinatorial exchange processes [4]
- Exchange processes: wealth [5], opinion/voting

Mean-Field Model

- Infinite particle system
- Identical particles
- Two-body inelastic collision
- **Velocity independent collision rates [6]**

Problem Set-Up

- Momentum is conserved $\langle v \rangle = \text{const.}$
- System is Galilean invariant (under $v \to v - v_0$ transformation)
- Work in center of mass reference frame, $\langle v \rangle = 0$
- Velocity distribution function $= P(v,t)$, $\int dv P(v,t) = 1$
- Arbitrary initial conditions $P(v, t = 0) = P_0(v)$

Questions

- Kinetics of approach to trivial final state $P(v,t) = \delta(v)$
- Statistics of steady state when coupled to heat bath

Kinetic Theory

- Formal Boltzmann equation

$$\frac{\partial P(v,t)}{\partial t} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} du_1\, du_2\, P(u_1,t)P(u_2,t)$$
$$\times\left[\delta(v-\gamma u_1-(1-\gamma)u_2)-\delta(v-u_2)\right].$$

- Compact Boltzmann equation

$$\frac{\partial P(v,t)}{\partial t} + P(v,t) = \frac{1}{1-\gamma}\int_{-\infty}^{\infty} du P(u,t) P\left(\frac{v-\gamma u}{1-\gamma},t\right).$$

- Evolution equations for Fourier transform

$$\hat{P}(k,t) = \int dv\, e^{ikv}\, P(v,t)$$

$$\frac{\partial}{\partial t}\,\hat{P}(k,t) + \hat{P}(k,t) = \hat{P}[\gamma k,t]\,\hat{P}[(1-\gamma)k,t].$$

- Evolution equations are exact

- Nonlinear and nonlocal structure

$\boxed{\textbf{Simple structure, yet intractable}}$

Asymptotic Analysis - Moments

- Moments of the velocity distribution

$$M_n(t) = \int dv\, v^n P(v,t)$$

- Closed hierarchy of evolution equations for the moments

$$\dot{M}_n + a_n M_n = \sum_{m=2}^{n-2} \binom{n}{m} \gamma^m (1-\gamma)^{n-m} M_m M_{n-m}$$

- Hierarchy is solved recursively. Leading asymptotic behavior can be evaluated (using the conservation law $a_0 = 0$ and the lemma $a_n < a_m + a_{n-m}$ when $1 < m < n-1$)

$$M_n(t) \sim e^{-a_n t} \qquad \text{when} \qquad t \to \infty$$

- Decay coefficients $a_n \equiv a_n(\gamma) = 1 - (1-\gamma)^n - \gamma^n$

- Every moment decays in a different fashion!

$$\boxed{\textbf{Multiscaling asymptotic behavior}}$$

Singularities in Compact Distributions

- Starting with compact distributions, a progressively weaker set of singularities (discontinuities in derivatives of growing order) occur at (for example, we take $\gamma = 1/2$ and support in [-1,1])

$$v_n = \frac{1}{2^n} \qquad n = 1, 2, \ldots$$

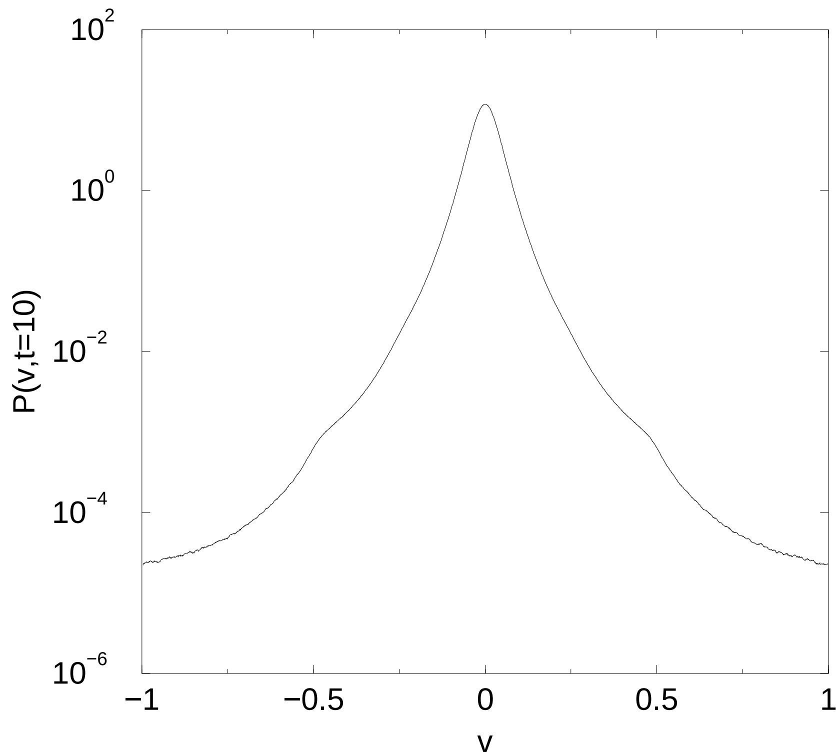

FIG. 1. Development of a singularity for a compact initial distribution. Shown is the probability distribution obtained by simulating the collision process with $\gamma = 1/2$. The data represents an average over 200 independent realization in a system with 10^7 particles, starting from a uniform distribution in the range $[-1, 1]$.

Steady state in presence of energy input

- Add white noise $\frac{dv_j}{dt}|_{\rm heat} = \eta_j \qquad \langle \eta_i(t)\eta_j(t')\rangle = 2D\delta_{ij}\delta(t-t')$

- Steady state equations in Fourier space $\hat{P}_\infty(k) = \hat{P}(k, t=\infty)$

$$(1+Dk^2)\hat{P}_\infty(k) = \hat{P}_\infty[\gamma k]\,\hat{P}_\infty[(1-\gamma)k]$$

- Recursive solution (using $\hat{P}_\infty(0)=1$ and $\hat{P}'_\infty(0)=0$)

$$\hat{P}_\infty(k) = \prod_{i=0}^{\infty}\prod_{j=0}^{i}\left[1+\gamma^{2j}(1-\gamma)^{2(i-j)}Dk^2\right]^{-\binom{i}{j}}.$$

- Explicit solution: simplify by taking logarithm

$$\hat{P}_\infty(k) = \exp\left\{\sum_{n=1}^{\infty}\frac{(-Dk^2)^n}{n a_{2n}(\gamma)}\right\}$$

- Cumulants κ_n related to kinetic decay coefficients a_n

$$\kappa_{2n} = \frac{(2n-1)!D^n}{2a_{2n}}$$

$$\boxed{\textbf{Generalized fluctuation-dissipation relations}}$$

Conclusions

- Multiscaling asymptotic behavior
- Singularities for compact distributions
- Steady-state cumulants in steady state directly related to kinetic decay coefficients

References

1. P. K. Haff, J. Fluid Mech. **134**, 401 (1983).
2. E. Ben-Naim, S. Y. Chen, G. D. Doolen, and S. Redner, *Phys Rev. Lett.* **86**, 4069 (1999).
3. J. Krug and J. Garcia, *cond-mat/9909034*.
4. Z. A. Melzak, *Mathematical Ideas, Modeling and Applications, Volume II* (Wiley, New York, 1976), p. 279.
5. S. Ispolatov, P. L. Krapivsky, and S. Redner, Eur. Phys. J. B **2**, 267 (1998).
6. M. H. Ernst, Phys. Rep. **78**, 1 (1981).