

Creation of photons in an oscillating cavity with two moving mirrors

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We study the creation of photons in a one-dimensional oscillating cavity with two perfectly conducting moving walls. By means of a conformal transformation, we derive a set of generalized Moore's equations whose solution contains the whole information of the radiation field within the cavity. For the case of resonant oscillations we solve these equations using a renormalization-group procedure that appropriately deals with the secular behavior present in a naive perturbative approach. We study the time evolution of the energy density profile and of the number of created photons inside the cavity. [S1050-2947(99)03604-5]

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I. INTRODUCTION

It is well known that in the presence of moving boundaries the vacuum state of the electromagnetic field may not be stable, which results in the generation of real photons. The generated radiation exerts pressure on the moving boundaries which can be looked upon as a dissipative force that opposes itself to the mechanical motion of the boundaries. The generation of photons, which is an amazing demonstration of the existence of quantum vacuum fluctuations of QED, is referred to in the literature as the dynamical Casimir effect [1] or motion-induced radiation [2]. It goes without saying that it would be very nice to have an experimental verification of this prediction. Due to the technical difficulties involved in the detection of the phenomenon, up to now no concrete experiment has been carried out, and there are only a few experimental proposals [3,4]. However, feasible experimental evidence is not out of reach, and therefore it is of interest to explore different theoretical models to describe the process and identify signatures which permit us to distinguish vacuum radiation from spurious effects.

Research in the field has mainly concentrated on one-dimensional models, which are useful for giving an account of the main physical processes participating in the phenomenon (a small number of works deal with more realistic three-dimensional models [4–6]). In this work we will also restrict ourselves to one-dimensional models. Motion-induced effects of vacuum radiation already show up for a single mirror moving with a nonuniform acceleration in vacuum [7]. Since the amount of radiation generated is very small, basically determined by the ratio of the speed of the mirror to the speed of light, much attention has been paid to the study of one-dimensional models for which the effect is enhanced.

A cavity made of two perfectly parallel reflecting mirrors, one of which is motionless and the other oscillating with a mechanical frequency equal to a multiple of the fundamental optical resonance frequency of the static cavity, is a thoroughly studied example where such an enhancement takes

place [8–13]. It is typically considered that the cavity is motionless and that at some instant one mirror starts to oscillate resonantly with a tiny amplitude. For small times after the motion starts, one can make a perturbative expansion of the equations of motion of the field in terms of the small amplitude to find an approximate solution. In this way one can study the structure of the electromagnetic field inside the cavity, which departs from the standard static Casimir profile (which is constant over the whole cavity) and develops a structure of small and broad pulses. The number of motion-induced photons grows quadratically in time, and the spectrum has an inverted parabolic shape with an upper frequency cutoff given by the mechanical frequency, its maximum being at half that value [4,14]. Similar results are found in [2] by means of a scattering approach for the radiation emitted out of a lossy cavity. However, for long times these methods are not valid, and new approximation techniques are required. In [12,15] it is shown that in such a limit the structure of the electromagnetic field is nontrivial, with a number of pulses equal to the mechanical resonant frequency, whose width decreases exponentially and whose height increases exponentially with time, in such a way that the total energy within the cavity grows exponentially at the expense of the energy given to the system to keep the mirror moving. Also, the spectrum does not have an upper frequency cutoff. Through a process of frequency up-conversion, the generated photons contain frequencies of higher-order cavity modes and thus exceed the mechanical frequency. The physical mechanism of such an optical pumping into the high-frequency region is the Doppler up-shift of the field upon reflection at the mirrors. Similar conclusions are found for the lossy cavity [16].

The case of cavities with two moving mirrors has also been considered recently. In the small time approximation, both for the ideal cavity [17] and for the lossy one [2], it is found that the number of motion-induced photons grows quadratically in time and that the spectrum is once again parabolic. In the long time approximation, the lossy cavity has been studied with the scattering approach [16]. Just as in the case of a single oscillating mirror, it is found that in this regime there is pulse shaping in the time domain and frequency up-conversion in the spectrum of emitted photons from the cavity. A striking feature of the spectrum is that no

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photons are emitted at frequencies equal to multiple integers of the mechanical frequency.

In this work we will consider an ideal cavity with two mirrors oscillating resonantly at the same frequency, and we will allow for different amplitudes and a possible dephasing between the mirrors. To investigate the problem, we will deduce a generalization of Moore's equation [18] to the problem of two moving boundaries, whose solution gives complete information on the electromagnetic field inside the cavity. For the case of resonant harmonic motions, no exact solution exists, and we find an approximate analytic solution based on a renormalization-group (RG) technique. We have already applied this method in [15] for the case of a single oscillating mirror, and just as in that case, the strategy allows us to find a single solution valid for both short and long times. This will allow us to describe precisely the behavior of the energy density and the number of photons for all times.

As we shall see, motion-induced radiation strongly depends on the relation among the amplitudes of oscillation, the frequency, and the dephasing. For some relations among these variables, there is constructive interference and a series of pulses develops within the cavity that grow exponentially in time, and frequency up-conversion takes place. For some other relations, there is destructive interference and hence no vacuum radiation. We also show that our solution is capable of accounting for other physical behaviors, for which the peaks grow quadratically rather than exponentially.

The paper is organized as follows. In Sec. II we will introduce the generalization of Moore's equations for a moving cavity and we will explain how to calculate the energy density and the number of motion-induced photons. In Sec. III the renormalization-group method is described and applied to the problem of harmonically oscillating walls with dephasing. In Sec. IV we study some particular dephasings, which we will call translational and breathing modes. In Sec. V another motion is considered, which has a qualitatively different behavior as compared to those of Sec. IV. Finally in Sec. VI we make our conclusions.

II. GENERALIZED MOORE EQUATIONS

We consider a one-dimensional cavity formed by two perfectly reflecting mirrors, each of which follows a given trajectory, say $L(t)$ for the left mirror and $R(t)$ for the right one. These two trajectories are predetermined (i.e., are given data for the problem) and act as time-dependent boundary conditions for the electromagnetic field inside the cavity. The field equation for the vector potential takes the form of the equation for a massless scalar field $(-\partial_t^2 + \partial_x^2)A(x,t) = 0$,¹ and the boundary conditions are $A(x=L(t),t) = A(x=R(t),t) = 0$ for all times. If we express the field in terms of creation a_k^\dagger and annihilation a_k operators for photons in the form

$$A(x,t) = \sum_{k=1}^{\infty} [a_k \psi_k(x,t) + a_k^\dagger \psi_k^*(x,t)], \quad (1)$$

then the mode functions $\psi_k(x,t)$ must be chosen so as to satisfy the above boundary conditions.

In the case where only one of the walls moves, say the right one, the modes can be written in terms of a function $U(t)$ as

$$\psi_k(x,t) = \frac{i}{\sqrt{4\pi k}} [e^{-ik\pi U(t+x)} - e^{-ik\pi U(t-x)}], \quad (2)$$

and the boundary condition on the right is met² provided that the function U verifies $U(t+R(t)) - U(t-R(t)) = 2$, which is known as Moore's equation [18]. The complete solution to the problem involves finding a solution $U(t)$ in terms of the prescribed motion $R(t)$. Moore's equation can in fact be deduced by means of a conformal transformation from the original space-time coordinates (t,x) to a new set of coordinates (\bar{t},\bar{x}) in which not only the left mirror but also the right one is fixed. This transformation takes the form

$$\bar{t} + \bar{x} = U(t+x), \quad \bar{t} - \bar{x} = U(t-x), \quad (3)$$

which, after mapping the coordinate of the left mirror as $L(t) \equiv 0 \rightarrow \bar{x}_L = 0$ and the right one as $R(t) \rightarrow \bar{x}_R = 1$, leads to Moore's equation.

Let us now consider the more general case in which both mirrors move. Evidently, we can also make a similar conformal transformation, but now we need two functions U instead of one. Defining the transformation as

$$\bar{t} + \bar{x} = G(t+x), \quad \bar{t} - \bar{x} = F(t-x), \quad (4)$$

and mapping $L(t)$ and $R(t)$ as before, we obtain a set of generalized Moore's equations

$$\begin{aligned} G(t+L(t)) - F(t-L(t)) &= 0, \\ G(t+R(t)) - F(t-R(t)) &= 2, \end{aligned} \quad (5)$$

which, when solved for given motions for the mirrors, allows us to find the solution for the modes inside the cavity. Indeed, the modes can be cast in the form

$$\psi_k(x,t) = \frac{i}{\sqrt{4\pi k}} [e^{-ik\pi G(t+x)} - e^{-ik\pi F(t-x)}], \quad (6)$$

and they satisfy both the field equation and the boundary conditions.

We shall be interested in studying the space-time profile of the energy density of the field between the moving walls

$$\langle T_{00}(x,t) \rangle = \frac{1}{2} \left[\left\langle \left(\frac{\partial A(x,t)}{\partial t} \right)^2 \right\rangle + \left\langle \left(\frac{\partial A(x,t)}{\partial x} \right)^2 \right\rangle \right], \quad (7)$$

where the expectation values are taken with respect to the vacuum state. As is well known, this quantity is divergent and a regularization method is needed to get meaningful results. Using the point-splitting method and introducing ad-

¹The speed of light is set to unity.

²The boundary condition on the left (fixed) mirror is automatically fulfilled by this form for the modes.

vanced $u = t + x$ and retarded $v = t - x$ coordinates, the energy density can be rewritten in terms of the functions G and F in the following way [7]:

$$\langle T_{00}(u, v) \rangle = \frac{\pi}{4} \sum_{k=1}^{\infty} k \{ G'(u) G'(u + i\epsilon) e^{-ik\pi[G(u) - G(u + i\epsilon)]} + F'(v) F'(v + i\epsilon) e^{-ik\pi[F(v) - F(v + i\epsilon)]} \}, \quad (8)$$

with $\epsilon \rightarrow 0^+$. From here it is straightforward to get the renormalized version, $\langle T_{00}(x, t) \rangle_{\text{ren}} = -f_G(t + x) - f_F(t - x)$, where

$$f_G = \frac{1}{24\pi} \left[\frac{G'''}{G'} - \frac{3}{2} \left(\frac{G''}{G} \right)^2 + \frac{\pi^2}{2} (G')^2 \right], \quad (9)$$

$$f_F = \frac{1}{24\pi} \left[\frac{F'''}{F'} - \frac{3}{2} \left(\frac{F''}{F} \right)^2 + \frac{\pi^2}{2} (F')^2 \right].$$

Suppose that for $t < 0$ both walls were at rest separated by a distance Λ and that the field was in its vacuum state. The solution of the generalized Moore equations is simply $G(t) = F(t) = t/\Lambda$ and the mode functions ψ_k correspond to positive frequency modes. If at $t = 0$ the boundaries begin to move, it is well known that for some types of motion the field does not remain in vacuum, but photons are produced through nonadiabatic processes. A consistent calculation of the number of created photons through motion-induced radiation requires having a well-defined vacuum state in the future. To this end we consider that at time $t = T$ both walls come to rest [$L(t) = 0$ and $R(t) = \Lambda$ for $t \geq T$]. The evolved old vacuum state does not coincide with the vacuum in the future, but rather it contains a number of photons, which can be calculated by means of the Bogoliubov coefficients $\beta_{nm} = (\psi_n^*, \psi_m^{(0)})^*$, where $\psi_m^{(0)}(x, t) = (\pi m)^{-1/2} \sin(m\pi x/\Lambda) e^{-i\pi m t/\Lambda}$ is the mode function for the static problem,³ and $\psi_n(x, t)$ is the mode function which solves the nonstationary problem for $t > 0$ and coincides with $\psi_n^{(0)}(x, t)$ for $t < 0$. Writing the mode functions in terms of the functions G and F , integrating by parts, and using the set of Moore's equations to drop the surface terms, we get the following relation for the Bogoliubov coefficient:

$$\beta_{nm}(t, T) = \frac{1}{2} \sqrt{\frac{m}{n}} \left\{ \int_{t/\Lambda-1}^{t/\Lambda} dx \exp\{-i\pi[nF(\Lambda x) + mx]\} + \int_{t/\Lambda}^{t/\Lambda+1} dx \exp\{-i\pi[nG(\Lambda x) + mx]\} \right\} \quad (10)$$

for times $t > T$. The number of created photons inside the cavity after the stopping time T in the n th mode is given by $N_n(T) = \sum_m |\beta_{nm}(t, T)|^2$ (the dependence of the Bogoliubov coefficient on t is just a phase) and summing over n we get the total amount of motion-induced photons.

III. RENORMALIZATION-GROUP METHOD FOR MOORE'S EQUATIONS

For resonantly harmonic motions it is not possible to find an exact solution to Eqs. (5) and approximation methods are compelling. A naive approach is to make perturbations in the amplitude of the oscillation, but it turns out that the strategy is ill-fated, because of the appearance of secular terms proportional to the time which, after a short period, make the approximation break down. In [15] we have applied a method inspired in the renormalization group to treat these singular perturbations for the case of a one-dimensional cavity with one oscillating mirror. The method has a wide range of applications in different fields, especially for studying ordinary differential equation problems involving boundary layers, multiple scales, etc. [19]. In the following we shall extend the method for the case of a one-dimensional cavity whose mirrors oscillate in resonance with the cavity. More specifically, we consider that for $t < 0$ the two mirrors are motionless and separated by a distance Λ , and that at $t = 0$ they start to move as

$$L(t) = \epsilon A_L \sin\left(\frac{q\pi t}{\Lambda}\right) \equiv 0 + \epsilon \delta L(t),$$

$$R(t) = \Lambda - \epsilon A_R \sin(\phi) + \epsilon A_R \sin\left(\frac{q\pi t}{\Lambda} + \phi\right) \equiv \Lambda + \epsilon \delta R(t), \quad (11)$$

for the left and right mirrors, respectively. Here ϕ is a possible dephasing angle, A_L and A_R are amplitudes of oscillation, and $\epsilon \ll 1$ is a small parameter.

Let us first start with the perturbative approximation. We expand both unknown functions $G(t)$ and $F(t)$ in terms of the small parameter ϵ and retain first-order terms only, $G(t) = G_0(t) + \epsilon G_1(t)$ and $F(t) = F_0(t) + \epsilon F_1(t)$. Equating terms of the same order in the set of generalized Moore equations, we get for the zeroth-order part

$$G_0(t) - F_0(t) = 0, \quad (12)$$

$$G_0(t + \Lambda) - F_0(t - \Lambda) = 2, \quad (13)$$

and for the first-order part

$$G_1(t) - F_1(t) = -\theta(t) \delta L(t) [G_0'(t) + F_0'(t)], \quad (14)$$

$$G_1(t + \Lambda) - F_1(t - \Lambda) = -\theta(t) \delta R(t) \times [G_0'(t + \Lambda) + F_0'(t - \Lambda)]. \quad (15)$$

The general solution to Eqs. (12) and (13) is

$$G_0(t) = F_0(t) = c + \frac{t}{\Lambda} + \sum_{n \geq 1} \left[A_n \sin\left(\frac{n\pi t}{\Lambda}\right) + B_n \cos\left(\frac{n\pi t}{\Lambda}\right) \right], \quad (16)$$

where c , A_n , and B_n are constants determined by the boundary conditions of the problem. These are obtained from the fact that the modes $\psi_k(x, t)$ must be positive frequency modes for $t < 0$, which implies that $G(t) = t/\Lambda$ for $-\infty < t$

³The inner product is the usual for the Klein-Gordon equation, namely $(\psi, \xi) = -i \int_{L(t)}^{R(t)} dx [\psi \dot{\xi}^* - \dot{\psi} \xi^*]$.

$\leq \Lambda$ and $F(t) = t/\Lambda$ for $-\infty \leq t \leq 0$. The different ranges for the functions G and F follow directly from the nonlocal structure of Moore equations.

Making the shift $t \rightarrow t - \Lambda$ in Eq. (14) and replacing the result in Eq. (15) we get an equation for the first-order correction to the function G , namely

$$\begin{aligned} G_1(t + \Lambda) - G_1(t - \Lambda) &= \theta(t - \Lambda) \delta L(t - \Lambda) [G'_0(t - \Lambda) + F'_0(t - \Lambda)] \\ &\quad - \theta(t) \delta R(t) [G'_0(t + \Lambda) + F'_0(t - \Lambda)]. \end{aligned} \quad (17)$$

Since this equation is linear, the solution is of the form $G_1 = G_1^{(1)} + G_1^{(2)}$, where

$$\begin{aligned} G_1^{(1)}(t + \Lambda) - G_1^{(1)}(t - \Lambda) &= \theta(t - \Lambda) \delta L(t - \Lambda) \\ &\quad \times [G'_0(t - \Lambda) + F'_0(t - \Lambda)], \end{aligned} \quad (18)$$

$$\begin{aligned} G_1^{(2)}(t + \Lambda) - G_1^{(2)}(t - \Lambda) &= -\theta(t) \delta R(t) \\ &\quad \times [G'_0(t + \Lambda) + F'_0(t - \Lambda)], \end{aligned} \quad (19)$$

whose general solutions read

$$\begin{aligned} G_1^{(1)}(t) &= \frac{A_L}{\Lambda} \frac{t}{\Lambda} \theta(t) \sin(q\pi t/\Lambda) \\ &\quad \times \left\{ 1 + \pi \sum_{n \geq 1} n [A_n \cos(n\pi t/\Lambda) \right. \\ &\quad \left. - B_n \sin(n\pi t/\Lambda)] \right\} + g^{(1)}(t) \end{aligned} \quad (20)$$

for Eq. (18), and for Eq. (19) we get

$$\begin{aligned} G_1^{(2)}(t) &= \frac{A_R}{\Lambda} \frac{t}{\Lambda} \theta(t + \Lambda) \\ &\quad \times \{ \sin(\phi) + (-1)^{q+1} \sin(q\pi t/\Lambda + \phi) \} \\ &\quad \times \left\{ 1 + \pi \sum_{n \geq 1} n [A_n \cos(n\pi t/\Lambda) \right. \\ &\quad \left. - B_n \sin(n\pi t/\Lambda)] \right\} + g^{(2)}(t), \end{aligned} \quad (21)$$

where $g^{(1)}$ and $g^{(2)}$ are arbitrary periodic functions of period 2Λ . The first-order correction for F can be deduced from the first-order correction for G that we have just found using Eq. (14) or (15) interchangeably. We see that the perturbative corrections contain secular terms that grow linearly in time. Therefore, the approximation will be valid only for short times, that is, $\epsilon t/\Lambda \ll 1$.

In order to determine the two unknown periodic functions we have to consider the boundary conditions for the functions G and F . We have already said that the nonlocal structure of Moore's equations implies that, although at $t=0$ the motion of the mirrors starts, the expression for G for times up to $t=\Lambda$ is given by the solution for motionless walls,

namely, $G(t) = t/\Lambda$ for $t \leq \Lambda$, while that for F reads $F(t) = t/\Lambda$ for $t \leq 0$. If we assume that these boundary conditions are already satisfied by the zeroth-order solutions $G_0(t)$ and $F_0(t)$, then the periodic functions must be chosen so that $G_1(t) = 0$ and $F_1(t) = 0$ in the respective intervals. This fact, when translated to the functions $G_1^{(1)}$ and $G_1^{(2)}$, implies the following boundary conditions:

$$\begin{aligned} G_1^{(1)}(t) &= 0 \quad \text{for } 0 \leq t \leq 2\Lambda, \\ G_1^{(2)}(t) &= 0 \quad \text{for } -\Lambda \leq t \leq \Lambda, \end{aligned} \quad (22)$$

which leads to the following expressions for the periodic functions:

$$\begin{aligned} g^{(1)}[(2k+1)\Lambda + z] &= -\frac{A_L}{\Lambda} \frac{z + \Lambda}{\Lambda} \sin(q\pi z/\Lambda) \\ &\quad \times \left\{ 1 + \pi \sum_{n \geq 1} n \right. \\ &\quad \left. \times [A_n \cos(n\pi z/\Lambda) - B_n \sin(n\pi z/\Lambda)] \right\} \end{aligned} \quad (23)$$

and

$$\begin{aligned} g^{(2)}(2p\Lambda + \omega) &= -\frac{A_R}{\Lambda} \frac{\omega}{\Lambda} \\ &\quad \times \{ \sin(\phi) + (-1)^{q+1} \sin(q\pi \omega/\Lambda + \phi) \} \\ &\quad \times \left\{ 1 + \pi \sum_{n \geq 1} n \right. \\ &\quad \left. \times [A_n \cos(n\pi \omega/\Lambda) - B_n \sin(n\pi \omega/\Lambda)] \right\}, \end{aligned} \quad (24)$$

where $t = (2k+1)\Lambda + z$, $k = 0, 1, 2, \dots$, and $-\Lambda \leq z \leq \Lambda$ for the function $g^{(1)}$, while for the function $g^{(2)}$ we have $t = 2p\Lambda + \omega$, $p = 0, 1, 2, \dots$, and $-\Lambda \leq \omega \leq \Lambda$. Given t , the values of the integers k and p are obtained as $k = p = \frac{1}{2} \text{int}(t/\Lambda)$ for $\text{int}(t/\Lambda)$ even and $k = \frac{1}{2} [\text{int}(t/\Lambda) - 1]$, $p = \frac{1}{2} [\text{int}(t/\Lambda) + 1]$ for $\text{int}(t/\Lambda)$ odd. Note that during the first period $k=0$ ($p=0$), the function $g^{(1)}$ ($g^{(2)}$) makes $G_1^{(1)}$ ($G_1^{(2)}$) vanish identically. As we have already seen, since the mirrors were at rest for $t < 0$, we must impose $G(t) = t/\Lambda$ for $t \leq \Lambda$ and $F(t) = t/\Lambda$ for $t \leq 0$. Therefore, $c = A_n = B_n = 0$, and the perturbative solution for $t > 0$, to order $O(\epsilon^2)$, is

$$\begin{aligned} G(t) &= \frac{t}{\Lambda} + \epsilon \frac{A_L}{\Lambda} \frac{t - z - \Lambda}{\Lambda} \sin(q\pi t/\Lambda) + \epsilon \frac{A_R}{\Lambda} \frac{t - \omega}{\Lambda} \\ &\quad \times [\sin(\phi) + (-1)^{q+1} \sin(q\pi t/\Lambda + \phi)], \end{aligned} \quad (25)$$

$$F(t) = G(t) + 2\epsilon \frac{A_L}{\Lambda} \sin(q\pi t/\Lambda). \quad (26)$$

These perturbative solutions suffer from the aforementioned secularity problems, being valid for times $t/\Lambda \ll \epsilon^{-1}$. In order to deal with this drawback and get improved solutions valid for longer times, in the rest of this section we describe the RG method we mentioned before for this problem of an oscillating cavity.

What the renormalization-group method does is to improve the perturbative expansion by resumming an infinite number of secular terms. In general, if one performs the perturbative expansion to higher orders, there appear different time scales, ϵt to first order, $\epsilon^2 t$ and $\epsilon^2 t^2$ to second order, $\epsilon^3 t$, $\epsilon^3 t^2$, and $\epsilon^3 t^3$ to third order, and so on. The RG technique sums the most secular terms of each order ($\epsilon^n t^n$), and it is therefore valid for times $t/\Lambda \ll \epsilon^{-2}$. The way to carry out the resummation is nicely described in [19] and it basically consists in introducing an arbitrary time τ , splitting the time in the secular terms of the first-order perturbative corrections as $t = (t - \tau) + \tau$, and absorbing the terms proportional to τ into the ‘‘bare’’ parameters of the zeroth-order perturbative solution, thereby becoming ‘‘renormalized.’’ Introducing the arbitrary time τ and splitting t as stated, the perturbative solution can be written as

$$\begin{aligned}
G(t, \tau) = & c(\tau) + \sum_{n \geq 1} [A_n(\tau) \sin(n\pi t/\Lambda) \\
& + B_n(\tau) \cos(n\pi t/\Lambda)] + \frac{t - \tau}{\Lambda} \\
& + \epsilon \frac{t - \tau}{\Lambda} \frac{A_L}{\Lambda} \sin(q\pi t/\Lambda) \left\{ 1 + \pi \sum_{n \geq 1} n \right. \\
& \left. \times [A_n(\tau) \cos(n\pi t/\Lambda) - B_n(\tau) \sin(n\pi t/\Lambda)] \right\} \\
& + \epsilon \frac{t - \tau}{\Lambda} \frac{A_R}{\Lambda} \{ \sin(\phi) + (-1)^{q+1} \sin(q\pi t/\Lambda + \phi) \} \\
& \times \left\{ 1 + \pi \sum_{n \geq 1} n [A_n(\tau) \cos(n\pi t/\Lambda) \right. \\
& \left. - B_n(\tau) \sin(n\pi t/\Lambda)] \right\} + \epsilon g^{(1)}(t, \tau) + \epsilon g^{(2)}(t, \tau),
\end{aligned} \tag{27}$$

where the bare parameters c , A_n , and B_n have been replaced by their renormalized counterparts $c(\tau)$, $A_n(\tau)$, and $B_n(\tau)$. Here $g^{(1)}(t, \tau)$ and $g^{(2)}(t, \tau)$, respectively, denote the functions $g^{(1)}(t)$ and $g^{(2)}(t)$ with the same replacement. Note that these functions are no longer periodic due to the RG improvement.

Since the time τ is arbitrary, the solution for G should not depend on it, which implies the following RG equation $(\partial G / \partial \tau)_t = 0$. In our case it consists of three independent equations

$$\frac{\partial c(\tau)}{\partial \tau} = \frac{1}{\Lambda} + \frac{2}{\pi} (-1)^{q+1} b + O(\epsilon^2), \tag{28}$$

$$\begin{aligned}
\frac{\partial A_n(\tau)}{\partial \tau} = & \frac{2}{\pi} a \delta_{nq} - 2(-1)^{q+1} b n B_n + |n - q| \\
& \times [a A_{|n-q|} - b \operatorname{sgn}(n - q) B_{|n-q|}] - (n + q) \\
& \times [a A_{n+q} + b B_{n+q}] + O(\epsilon^2),
\end{aligned} \tag{29}$$

$$\begin{aligned}
\frac{\partial B_n(\tau)}{\partial \tau} = & \frac{2}{\pi} b \delta_{nq} + 2(-1)^{q+1} b n A_n + |n - q| \\
& \times [a \operatorname{sgn}(n - q) B_{|n-q|} + b A_{|n-q|}] + (n + q) \\
& \times [-a B_{n+q} + b A_{n+q}] + O(\epsilon^2),
\end{aligned} \tag{30}$$

where

$$a \equiv \frac{\epsilon}{\Lambda} \frac{\pi}{2} \left[\frac{A_L}{\Lambda} + \frac{A_R}{\Lambda} (-1)^{q+1} \cos(\phi) \right], \tag{31}$$

$$b \equiv \frac{\epsilon}{\Lambda} \frac{\pi}{2} \frac{A_R}{\Lambda} (-1)^{q+1} \sin(\phi). \tag{32}$$

These parameters a and b play a crucial role because they determine the behavior of the solutions to the set of generalized Moore’s equations. There are four distinct cases. The simplest one is for $a = b = 0$, which happens, for example, for equal amplitudes $A_L = A_R$, zero dephasing, and even frequencies. In this case there is no secular behavior at the level of the perturbative solutions Eqs. (25) and (26), which are then valid also for long times. The energy inside the cavity oscillates around the static Casimir value and there is no motion-induced radiation. A second case is $a \neq 0$ and $b = 0$, which occurs, for example, for equal amplitudes, zero dephasing, and odd frequencies. In this case secular terms do appear in the perturbative solutions and the RG method is useful for finding the long time behavior, which shows an exponential increase of the energy in the cavity and motion-induced photons. This case will be the subject matter of the next section. A third case is $a = 0$ and $b \neq 0$, which takes place, for example, for a static left mirror $A_L = 0$ and dephasing $\phi = \pi/2$. Here there are also secular terms at the perturbative level, and for long times the energy does not grow exponentially but quadratically, photons also being generated. We shall deal with this case in the Sec. V. Finally, the case $a \neq 0$ and $b \neq 0$ is similar to the second case in that there is motion-induced radiation and an exponential increase of the energy. We shall not cover this case in detail, since the expressions for the solutions to Moore’s equations are cumbersome.

Now we solve the RG equations (28)–(30). The solution for c is trivial, $c(\tau) = [1/\Lambda + (\epsilon/\Lambda)(A_R/\Lambda)\sin(\phi)]\tau + \kappa$, with κ a constant to be determined. Writing $A_n = \tilde{A}_n - \tilde{A}_{-n}$ and $B_n = \tilde{B}_n + \tilde{B}_{-n}$, the new variables satisfy

$$\begin{aligned}
\frac{\partial \tilde{A}_m(\tau)}{\partial \tau} = & \frac{2}{\pi} a \delta_{mq} - 2(-1)^{q+1} b m \tilde{B}_m + (m - q) \\
& \times [a \tilde{A}_{m-q} - b \tilde{B}_{m-q}] - (m + q) \\
& \times [a \tilde{A}_{m+q} + b \tilde{B}_{m+q}] + O(\epsilon^2),
\end{aligned} \tag{33}$$

$$\begin{aligned} \frac{\partial \tilde{B}_m(\tau)}{\partial \tau} &= \frac{2}{\pi} b \delta_{mq} + 2(-1)^{q+1} b m \tilde{A}_m + (m-q) \\ &\times [a \tilde{B}_{m-q} + b \tilde{A}_{m-q}] + (m+q) \\ &\times [-a \tilde{B}_{m+q} + b \tilde{A}_{m+q}] + O(\epsilon^2). \end{aligned} \quad (34)$$

$$\tilde{B}_{m=qj} = \frac{i \tanh^j(qa\tau)}{\pi q j} \left\{ \frac{(1-ib/a)^j}{[1+i(-1)^{q+1}(b/a)\tanh(qa\tau)]^j} - \frac{(1+ib/a)^j}{[1-i(-1)^{q+1}(b/a)\tanh(qa\tau)]^j} \right\}, \quad (40)$$

The initial conditions for these differential equations are dictated by the perturbative solution $c(0) = \tilde{A}_m(0) = \tilde{B}_m(0) = 0$. This implies that the constant κ is zero. In order to solve for \tilde{A}_m and \tilde{B}_m , we first decouple the equations through the transformation $\tilde{C}_m = \tilde{A}_m - i\tilde{B}_m$ and $\tilde{D}_m = \tilde{A}_m + i\tilde{B}_m$, and introduce a generating functional $M(s, \tau) = \sum_m s^m \tilde{C}_m(\tau)$. It is easy to see that this functional verifies the following differential equation:

$$\begin{aligned} \frac{\partial M}{\partial \tau} &= \frac{2}{\pi} (a-ib)s^{q+1} + [-2i(-1)^{q+1}bs + (a-ib)s^{q+1} \\ &- (a+ib)s^{1-q}] \frac{\partial M}{\partial s}, \end{aligned} \quad (35)$$

with boundary condition $M(s, \tau=0) = 0$. The solution can be obtained by proposing an ansatz $M(s, \tau) = \Phi[e^{-\tau}\alpha(s)] + \beta(s)$, where $\Phi[\dots]$, $\alpha(s)$, and $\beta(s)$ are functions to be determined. We shall not dwell on the details of finding these functions, but suffice it to say that the last two are straightforwardly derived after introducing the ansatz in Eq. (35), while the first one is obtained once the initial condition on M is imposed. The solution reads

$$\begin{aligned} M(s, \tau) &= \frac{2}{\pi} (i(-1)^{q+1}qb\tau - \ln \cosh(qa\tau) \\ &- \ln\{[1+i(-1)^{q+1}(b/a)\tanh(qa\tau)] \\ &- (1-ib/a)\tanh(qa\tau)s^q\}). \end{aligned} \quad (36)$$

Expanding this solution in powers of s (and doing the same for its complex conjugate), we get to our final objective, i.e., the coefficients \tilde{A}_m and \tilde{B}_m . The only nonvanishing coefficients are

$$\begin{aligned} \tilde{A}_{m=0} &= \frac{1}{\pi q} \{-2 \ln \cosh(qa\tau) - \ln[1+i(b/a)\tanh(qa\tau)] \\ &- \ln[1-i(b/a)\tanh(qa\tau)]\}, \end{aligned} \quad (37)$$

$$\begin{aligned} \tilde{B}_{m=0} &= \frac{i}{\pi q} (-1)^{q+1} \{2iqb\tau - \ln[1+i(b/a)\tanh(qa\tau)] \\ &+ \ln[1-i(b/a)\tanh(qa\tau)]\}, \end{aligned} \quad (38)$$

$$\begin{aligned} \tilde{A}_{m=qj} &= \frac{\tanh^j(qa\tau)}{\pi q j} \left\{ \frac{(1-ib/a)^j}{[1+i(-1)^{q+1}(b/a)\tanh(qa\tau)]^j} \right. \\ &\left. + \frac{(1+ib/a)^j}{[1-i(-1)^{q+1}(b/a)\tanh(qa\tau)]^j} \right\}, \end{aligned} \quad (39)$$

where $j \in N$. Note that since $\tilde{A}_{m<0} = \tilde{B}_{m<0} = 0$, the original coefficients A_m and B_m are equal to the \tilde{A}_m 's and \tilde{B}_m 's, respectively.

The expressions for the RG-improved coefficients ensure that the solution for G and F does not depend on τ . We still have the freedom to choose the arbitrary time τ at will, and the obvious choice is $t = \tau$, since in this way the secular terms proportional to $t - \tau$ disappear. Given the RG-improved coefficients, we still have to plug them into Eq. (27) and perform the necessary summations to finally get the RG-improved solutions $G(t, t)$ and $F(t, t)$.

For a general dephasing, the resulting expressions are rather lengthy, so in the next two sections we will concentrate on particular cases. First, we study the case of dephasing $\phi = 0$, which corresponds to translational modes, and dephasing $\phi = \pi$, which corresponds to breathing modes. Second, we analyze a case with only one mirror oscillating, similar to the one we studied in [15], but with a dephasing $\phi = \pi/2$, which gives qualitatively different results.

IV. TRANSLATIONAL AND BREATHING MODES

In the present section we consider that the cavity has translational modes ($\phi = 0$), or that it has breathing modes ($\phi = \pi$). For the particular case of equal amplitudes $A_L = A_R$, the former type of motion corresponds to the cavity oscillating as a whole, with its mechanical length kept constant (pictorially called an ‘‘electromagnetic shaker’’ [20]), while in the latter type of motion the mirrors oscillate symmetrically with respect to the center of the cavity, the mechanical length changing periodically (an ‘‘antishaker’’). Both for translational and breathing modes the expressions for the coefficients in Eqs. (37)–(40) simplify considerably because $b = 0$, and the summations to get the functions G and F are straightforward. Setting $t = \tau$ in Eq. (27), we get the RG-improved solutions

$$\begin{aligned} G(t, t) &= \frac{t}{\Lambda} - \frac{2}{\pi q} \text{Im} \ln[1 + \zeta + (1 - \zeta)e^{iq\pi t/\Lambda}] + \epsilon g^{(1)}(t, t) \\ &+ \epsilon g^{(2)}(t, t), \end{aligned} \quad (41)$$

$$\begin{aligned} F(t, t) &= G(t) + 2\epsilon \frac{A_L}{\Lambda} \\ &\times \sin(q\pi t/\Lambda) \frac{2\zeta}{1 + \zeta^2 + (1 - \zeta^2)\cos(q\pi t/\Lambda)}, \end{aligned} \quad (42)$$

where we have defined $\zeta \equiv \exp[2qat]$. The (now nonperiodic) RG-improved functions $g^{(1)}(t, t)$ and $g^{(2)}(t, t)$ are

$$g^{(1)}(t,t) = -\frac{A_L}{\Lambda} \frac{z+\Lambda}{\Lambda} \times \sin(q\pi t/\Lambda) \frac{2\zeta}{1+\zeta^2+(1-\zeta^2)\cos(q\pi t/\Lambda)}, \quad (43)$$

$$g^{(2)}(t,t) = \mp (-1)^{q+1} \frac{A_R}{\Lambda} \frac{\omega}{\Lambda} \times \sin(q\pi t/\Lambda) \frac{2\zeta}{1+\zeta^2+(1-\zeta^2)\cos(q\pi t/\Lambda)}, \quad (44)$$

where, in the last formula, the upper sign corresponds to $\phi = 0$ and the lower sign to $\phi = \pi$. These solutions for G and F are qualitatively similar to the one we obtained in the case for one oscillating wall with zero dephasing [15]. For the same reasons described in that reference, both RG-improved nonperiodic functions give negligible corrections to G and F in the long time limit ($\epsilon^{-1} \ll t/\Lambda \ll \epsilon^{-2}$). However, they are crucial for the solution to satisfy the correct boundary conditions at short times ($t/\Lambda \ll \epsilon^{-1}$).

The energy density inside the cavity is given by Eq. (9) in terms of derivatives of G and F . Since these expressions involve second and third derivatives of these functions, and since there is an initial discontinuity of the velocities of the mirrors, the energy density will develop δ -function singularities that will be infinitely reflected back and forth between the mirrors. In what follows we will ignore these singularities.⁴

The structure of the electromagnetic field within the cavity at long times strongly depends on the relation among amplitudes, frequencies, and dephasings. If these are such that the coefficient a is equal to zero (remember that for the motions considered in this section the other coefficient b is always null), then there is destructive interference. For equal amplitudes $A_L = A_R$, this takes place for even q and dephasing $\phi = 0$, or for odd q and dephasing $\phi = \pi$: all RG-improved coefficients A_n and B_n are null, and there is no motion-induced radiation enhancement whatsoever. If, on the other hand, $a \neq 0$, then we have constructive interference, which, for $A_L = A_R$, is maximal for odd q and $\phi = 0$, or for even q and $\phi = \pi$. Radiation enhancement takes place: the electromagnetic shaker and antishaker have “explosive cocktails” at long times. In particular, for $q \geq 2$, the RG solutions $G(t,t)$ and $F(t,t)$ develop a staircase form. Within regions of t between odd multiples of Λ , there are a total of q jumps located at values of t for which the argument of the logarithm in Eq. (41) vanishes, i.e., $\cos(q\pi t/\Lambda) = \pm 1$, where the upper sign corresponds to $a > 0$ and the lower one to $a < 0$. In Fig. 1 we show the form of the functions G and F for short times and in Fig. 2 for long times. Note that in the long time limit they are practically the same.

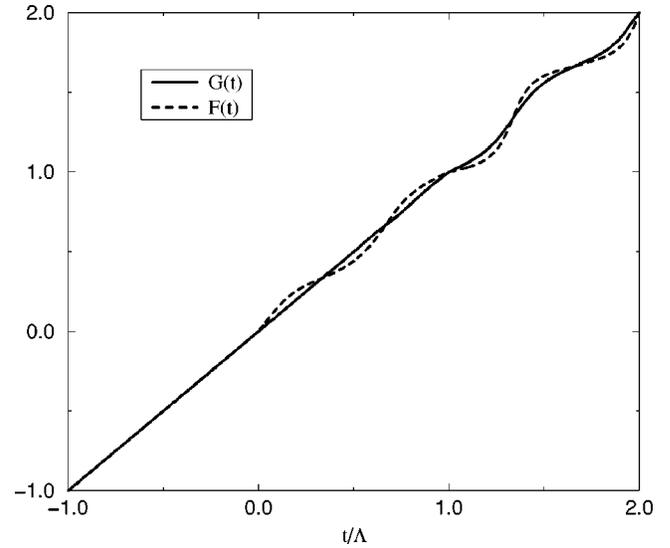


FIG. 1. $G(t)$ and $F(t)$ vs t/Λ as given by Eqs. (41) and (42) for small times $q|a|t/\Lambda \ll 1$. Note that the function $F(t)$ departs from the straight line at $t=0$, while the function $G(t)$ departs from it at $t=\Lambda$, as dictated by the initial boundary conditions. The parameters are those for a shaker with $A_L/\Lambda = A_R/\Lambda = 1$, $q=3$, $\phi=0$, and $\epsilon = 0.03$.

The energy density builds up a number of q traveling wave packets which become narrower as $\exp(-2q|a|t)$ and higher as $\exp(4q|a|t)$, so that the total energy inside the cavity grows like $\exp(2q|a|t)$ at the expense of the energy pumped into the system to keep the mirrors moving as predetermined. In Fig. 3 the profile of the energy density inside the cavity at a fixed time is depicted. We compare the case of the shaker with that of a single oscillating mirror. The difference in height and width of the peaks between these two situations is due to the fact that the parameter a for the shaker is twice that of the single mirror. This reflects how the cavity can enhance vacuum radiation.

A rather different picture appears when one considers the $q=1$ case, which corresponds to an oscillation frequency

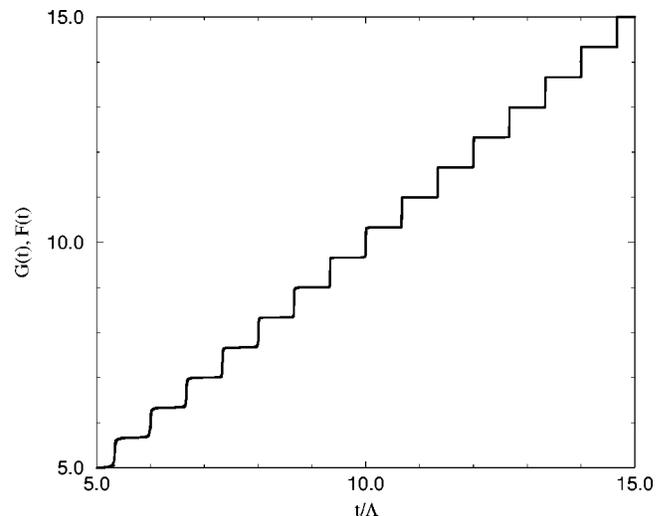


FIG. 2. $G(t)$ and $F(t)$ vs t/Λ as given by Eqs. (41) and (42) for long times $q|a|t/\Lambda \gg 1$. At these times both functions coincide and take a staircase profile. The parameters are the same as in Fig. 1.

⁴These singularities are, of course, artifacts of the sudden approximation we are using to describe the motion of the mirrors at $t=0$.

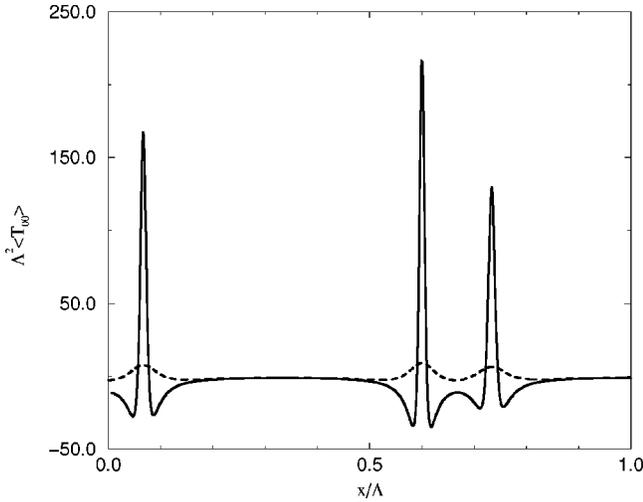


FIG. 3. Energy density profile between plates for fixed time $t/\Lambda = 15.4$. The solid line corresponds to a shaker with $A_L/\Lambda = A_R/\Lambda = 1$, $q=3$, $\phi=0$, and $\epsilon=0.01$. The dashed line corresponds to a cavity with a single oscillating mirror ($A_L=0$) with the same parameters. Note that the height of the peaks for the shaker grows as $\exp(4\pi q\epsilon t/\Lambda)$, while that for the single mirror grows as $\exp(2\pi q\epsilon t/\Lambda)$.

equal to the lowest eigenfrequency of the cavity. In this case the energy does not grow exponentially, but oscillates around the static Casimir value.

Now we calculate the number of motion-induced photons inside the cavity. We assume that before the mirrors started to move the state of the field was vacuum, and that at time $t=T$, when both walls come once again to rest,⁵ we define a new vacuum, in which there is an amount of real photons given by the Bogoliubov coefficients Eq. (10). To calculate these coefficients at a time $t>T$ we need to know the form for the functions G and F in the corresponding time intervals as they appear in the integral expression Eq. (10).

To this end let us discuss briefly how the RG-improved solutions $G(t,t)$ and $F(t,t)$ match the solutions to the problem of motionless walls for $t>T$. The nonlocal structure of Moore's equation implies that the solution for $F(t)$ is the RG one $F(t,t)$ up to $t\leq T$, and that for $G(t)$ is the RG one $G(t,t)$ up to $t\leq T+\Lambda$. Also, evaluating the Moore equation for times $t\geq T$, it follows that $F(t)=G(t)$ for $t\geq T$. Finally, for $t\geq T+\Lambda$ both Moore equations can be combined to obtain the usual equation for a static cavity, so $G(t)=F(t)=t/\Lambda+\Delta(t)$, where $\Delta(t)$ is a 2Λ -periodic function that we must determine. If due care is taken of the boundary conditions at the moment when the walls stop, it is easy to see that this function can be written by periodizing the RG-improved functions $G(t,t)$ and $F(t,t)$ as follows:

$$\Delta(t=T+2\Lambda+z) = \begin{cases} F(T+z, T+z) & \text{for } -\Lambda \leq z \leq 0, \\ G(T+z, T+z) & \text{for } 0 \leq z \leq \Lambda, \end{cases} \quad (45)$$

and $\Delta(t) = \Delta(t+2\Lambda)$.

⁵For the motions $L(t)$ and $R(t)$ we are considering in this section, this happens for times T such that $T/\Lambda = 2k/q$, ($k \in \mathbb{N}$).

Having now the form of the solutions for times after the stopping of the walls, we can calculate the Bogoliubov coefficients for late times $t/\Lambda > T/\Lambda \gg \epsilon^{-1}$ in a manner similar to that of [12]. Let us split the solution $G(t)$ in Eq. (41) in the form $G(t) = G_s(t) + G_{np}(t)$, where the first part is $G_s(t) = t/\Lambda - 2/(q\pi) \text{Im} \ln[1 + \zeta + (1 - \zeta)e^{iq\pi t/\Lambda}]$ and the last part involves the RG-improved functions $g^{(1)}$ and $g^{(2)}$. From a graph of these nonperiodic functions and of the difference between $G(t)$ and $F(t)$ one can see that, for long times, they are all bounded, much smaller than G_s , and that they are effectively zero except in small time intervals which tend to zero as time increases. Therefore, we can drop their contribution in the imaginary exponents of the integral representation of the Bogoliubov coefficient, and get

$$\beta_{nm}(t, T) = \frac{1}{2} \sqrt{\frac{m}{n}} \int_{t/\Lambda-1}^{t/\Lambda+1} dx \exp\{-i\pi[nG_s(\Lambda x) + mx]\}. \quad (46)$$

The function G_s has a first term, linear in time, and a second one, which for late times becomes an oscillating function, its period being $2\Lambda/q$ and the amplitude of its oscillations being independent of ϵ . Then the Bogoliubov coefficient can be rewritten as follows:⁶

$$\beta_{nm}(t, T) = \frac{1}{2} \sqrt{\frac{m}{n}} e^{-i\pi(n+m)(t/\Lambda-1)} \sum_{k=0}^{q-1} e^{-i\pi(n+m)(2/q)k} \times \int_0^{2/q} dx e^{-i\pi[(n+m)x + nf(x)]}, \quad (47)$$

with $f(x) = -2/(q\pi) \text{Im} \ln[1 + \zeta + (1 - \zeta)\exp(iq\pi x)]$, and $\zeta = \exp(2qaT)$. To go further and to be able to perform the integral, we make a piecewise linear approximation for the function f , valid for late times. We concentrate on the case $a < 0$, for which $\zeta \rightarrow 0$ at late times.⁷ From the graph of $f(x)$ one can see that it can be approximated by

$$\tilde{f}(x) = \begin{cases} -(1-q\delta)x & \text{for } 0 \leq x \leq \frac{1}{q} - \delta, \\ -\frac{1}{q\delta}(1-q\delta)^2 \left(x - \frac{1}{q}\right) & \text{for } \frac{1}{q} - \delta \leq x \leq \frac{1}{q} + \delta, \\ -(1-q\delta) \left(x - \frac{2}{q}\right) & \text{for } \frac{1}{q} + \delta \leq x \leq \frac{2}{q}, \end{cases} \quad (48)$$

where $\delta = 2\sqrt{\zeta}/(q\pi)$. With this approximate form the integrals become trivial, and after neglecting the integral over the middle interval which is proportional to δ , one can get a closed expression for the Bogoliubov coefficient, valid for $m\delta \ll 1$. For the particular case $q=2$ we get

⁶As we have anticipated, the dependence of the Bogoliubov coefficient on the time $t>T$ is just a phase.

⁷The case $a>0$ gives similar results for the amount of created particles. The technical difference is that since $\zeta \rightarrow \infty$ for late times, the approximate function is different.

$$|\beta_{nm}(T)|^2 = \frac{m}{n\pi^2} [1 + (-1)^{m+n}] \frac{1 - (-1)^m \cos(2\pi n \delta)}{(m + 2\pi n \delta)^2}. \quad (49)$$

Next we need to calculate the sum over n in order to find the amount of motion-induced photons in the n th mode after the stopping time T . Using the summation formulas of [12] we get

$$N_m(T) = \frac{1}{m^2} \left[\ln\left(\frac{m}{2\delta}\right) - (-1)^m \ln\left(\frac{1}{2\pi\delta}\right) \right]. \quad (50)$$

Recalling that δ is a function of T and taking the T derivative we find the rate of photoproduction

$$\frac{dN_m(T)}{dT} = -\frac{2a}{m\pi^2} [1 - (-1)^m]. \quad (51)$$

These results are valid for $\epsilon T/\Lambda \gg 1$ and not for very large wave numbers $m\delta \ll 1$.⁸ The number of photons per mode grows linearly in the stopping time and the rate, for late times, approaches an asymptotic value that depends on the value of a , i.e., on the relation among amplitudes, frequency, and dephasing. Both for the shaker and the antishaker, motion-induced radiation is enhanced in comparison to the case of a single oscillating mirror in a cavity. Indeed, in the former cases the rate of photoproduction is twice that of the

latter case. Photons are created in the odd modes only, whereas their amount in even modes is zero (it may be different from zero in the next-to-leading-order approximation). This situation is typical of processes involving parametric excitations [12].

V. A DEPHASED OSCILLATING BOUNDARY

In this section we discuss another particular motion of the walls, namely one for which the left mirror is static and the right one oscillates resonantly, with a dephasing $\phi = \pi/2$. The motion we consider is then $R(t) = \Lambda - 2\epsilon A_R \sin^2(q\pi t/2\Lambda)$, which for $q=2$ corresponds to the small ϵ expansion of an exact solution to Moore's equation studied in [8]. Our motivation for studying this peculiar case is that for this motion we have $a=0$ and $b \neq 0$, which, as we have anticipated, gives qualitatively different physical results.

The expressions for the functions $G(t,t)$ and $F(t,t)$ are obtained taking the limit $a \rightarrow 0$ of Eqs. (37)–(40). The result is

$$G(t,t) = F(t,t) = \frac{t}{\Lambda_{\text{eff}}} - \frac{2}{\pi q} \text{Im} \ln \left[1 - \frac{iqbte^{iq\pi t/\Lambda}}{1 - i(-1)^{q+1}qbt} \right] + \epsilon g^{(2)}(t,t), \quad (52)$$

$$g^{(1)}(t,t) = 0, \quad (53)$$

$$g^{(2)}(t,t) = -\frac{A_R}{\Lambda} \frac{1 + (-1)^{q+1} \cos(q\pi t/\Lambda)}{1 + 2qbt \sin(q\pi t/\Lambda) + 2(qbt)^2 [1 + (-1)^{q+1} \cos(q\pi t/\Lambda)]}, \quad (54)$$

where $\Lambda_{\text{eff}} = \Lambda(1 - \epsilon A_R/\Lambda)$ is the time-averaged length of the cavity for $t > 0$.

The solution $G(t,t)$ develops a staircase profile, the jumps being located at values of t for which the argument of the logarithm in Eq. (52) vanishes, i.e., for $\cos(q\pi t/\Lambda) = \pm 1$, where the plus sign corresponds to even q and the minus sign to odd q . The energy density for this type of motion also consists of a series of q peaks that travel between the mirrors. The qualitative difference is that in this case the height of the peaks grows as $(qbt)^4$, their width decreases as $(qbt)^{-2}$, and the total energy contained in the cavity grows quadratically rather than exponentially. This follows from the fact that time enters into the logarithm of Eq. (52) as a power law instead of an exponential, as in Eqs. (41) and (42).

Next we calculate the amount of motion-induced radiation for this case. To this end we assume that at time $t=T$ the wall comes to rest, $R(t) = \Lambda_{\text{eff}}$ for $t \geq T$, where T is of the

form $T/\Lambda = (2k+1)/(2q)$ ($k \in \mathbb{N}$). This choice for the stopping time simplifies the computation of the Bogoliubov coefficients β_{nm} . In such a case Eq. (46) is slightly modified,

$$\beta_{nm}(t,T) = \frac{1}{2} \sqrt{\frac{m}{n}} \int_{t/\Lambda_{\text{eff}}-1}^{t/\Lambda_{\text{eff}}+1} dx \times \exp\{-i\pi[nG_s(\Lambda_{\text{eff}}x) + mx]\}. \quad (55)$$

Now G_s consists of the first two terms of Eq. (52), the first being linear in time and the second one being an oscillating function for late times, whose period is $2\Lambda/q$. The Bogoliubov coefficient can also be rewritten in a way similar to Eq. (47),

$$\beta_{nm}(t,T) = \frac{1}{2} \sqrt{\frac{m}{n}} e^{-i\pi(n+m)(t/\Lambda_{\text{eff}}-1)} \sum_{k=0}^{q-1} e^{-i\pi(n+m)(2/q)k} \times \int_0^{2/q} dx e^{-i\pi[(n+m)x + nf(x)]}, \quad (56)$$

since in the interval $[t/\Lambda_{\text{eff}}-1, t/\Lambda_{\text{eff}}+1]$ there are a total of $q(\Lambda_{\text{eff}}/\Lambda) \approx q$ periods. Here

⁸There is a further restriction that comes from the fact that we are using a sudden approximation for the motion of the mirrors at $t=0$ and $t=T$. Indeed, if we assume that τ_s is the characteristic time for the mirror to come to rest, the sudden approximation will be valid for modes such that $m \ll \Lambda/\tau_s$.

$$f(x) = -\frac{2}{\pi q} \text{Im} \ln \left[1 - \frac{iqb\Lambda_{\text{eff}} x e^{iq\pi x \Lambda_{\text{eff}}/\Lambda}}{1 - i(-1)^{q+1}qb\Lambda_{\text{eff}} x} \right]. \quad (57)$$

The piecewise linear approximation for the function f is in this case⁹

$$\tilde{f}(x) = \begin{cases} \left(\frac{2}{q\delta} - \frac{5}{2} \right) x - \frac{1}{q} & \text{for } 0 \leq x \leq \delta, \\ -\left(1 - \frac{3}{4}q\delta \right) \left(x - \frac{2}{q} \right) - \frac{1}{q} & \text{for } \delta \leq x \leq \frac{2}{q}, \end{cases} \quad (58)$$

where $\delta = [q\pi(qbT)^2]^{-1} \ll 1$ for late times. Now the integral in Eq. (56) is straightforward, and after dropping the integral over the first interval $[0, \delta]$, which is proportional to δ , one can get a closed expression for the Bogoliubov coefficients, valid as long as $m\delta \ll 1$. For comparison with [8] we concentrate on the case $q=2$. In this case we have

$$|\beta_{nm}(T)|^2 = \frac{2m}{n\pi^2} [1 + (-1)^{m+n}] \frac{1 - (-1)^m \cos(3\pi n \delta/2)}{(m + 3\pi n \delta/2)^2}. \quad (59)$$

Finally, we perform the summation over n to get the number of created photons in the n th mode. Using the same summation formulas as in [12], we get

$$N_m(T) = \frac{2}{m\pi^2} \left[\ln \left(\frac{2m}{3\delta} \right) - (-1)^m \ln \left(\frac{2}{3\pi\delta} \right) \right]. \quad (60)$$

Replacing the value for δ and taking the T derivative, we get the following formula for the rate of photon production, valid in the limits $m\delta \ll 1$ and $\epsilon T/\Lambda \gg 1$:

$$\frac{dN_m(T)}{dT} = \frac{4}{m\pi^2} [1 - (-1)^m] \frac{1}{T}. \quad (61)$$

⁹We concentrate on even frequencies, for which $b < 0$. For odd frequencies, the results are similar.

We see that the number of photons per mode grows logarithmically in the stopping time, and as a consequence the rate of photon creation decreases towards zero. Similarly to the case of the vibrating cavity, photons are produced only in odd modes.

VI. CONCLUSIONS

In this paper we have presented a unified and analytic treatment of the dynamical Casimir effect in a one-dimensional resonantly oscillating cavity for arbitrary amplitudes and dephasings. We have derived a generalization of Moore's equation to describe the state of the electromagnetic field inside the cavity with two moving mirrors. Using a technique inspired by the renormalization-group method, we have found a solution to the set of generalized Moore's equations which is valid both for short and long times. The physical behavior of the moving cavity depends crucially on the relation among amplitudes, frequency, and dephasing. We have shown that for certain cases there is destructive interference and no radiation is generated. For others, there is constructive interference and motion-induced photons appear. When this takes place, the way the energy within the cavity and the number of created photons grow in time depends on the relation among the above variables. For certain motions the growth of the energy density is exponential and for some others it is a power law.

We hope in the future to apply the RG method to more realistic situations, such as three-dimensional oscillating cavities with rectangular or spherical shapes.

Note added. Recently, we received a paper [21] in which the problem of photon creation in a cavity with two moving mirrors is analyzed using a different method. It is shown that if the frequency of the vibrations is not exactly a resonant one, photoproduction is highly suppressed for strong detuning.

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