

Computation of Casimir forces for dielectrics or intrinsic semiconductors based on the Boltzmann transport equation

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Abstract. The interaction between drifting carriers and traveling electromagnetic waves is considered within the context of the classical Boltzmann transport equation to compute the Casimir-Lifshitz force between media with small density of charge carriers, including dielectrics and intrinsic semiconductors. We expand upon our previous work (Phys. Rev. Lett. 2008, in press; arXiv:0805.1676) and derive in some detail the frequency-dependent reflection amplitudes in this theory and compute the corresponding Casimir free energy for a parallel plate configuration. We critically discuss the the issue of verification of the Nernst theorem of thermodynamics in Casimir physics, and explicitly show that our theory satisfies that theorem. Finally, we show how the theory of drifting carriers connects to previous computations of Casimir forces using spatial dispersion for the material boundaries.

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1. Introduction

Quantum vacuum forces acting between dielectric planar surfaces or between an atom and a dielectric semi-space were computed long ago by Lifshitz [1] in terms of the complex frequency-dependent dielectric permittivity $\bar{\epsilon}(\omega)$ of the material boundaries. In this original formulation for ideal dielectrics, $\bar{\epsilon}(\omega)$ does not include contributions from current carriers, and as such can be called the “bare” permittivity. Extensions of the Lifshitz theory to media with large free charge carrier density, such as metals or highly doped semiconductors, are typically done by adding a frequency-dependent conduction term, computed from the optical data of the material and extrapolated to low frequencies by different theoretical models (e.g., a Drude-like term $i4\pi\sigma_0/\omega$, where σ_0 is the dc Drude conductivity). The finite temperature Casimir-Lifshitz force is extremely sensitive to the optical response of the materials at low frequency, and therefore different theoretical extrapolations have resulted in conflicting conclusions about the nature of the Casimir force between metals and/or highly doped semiconductors.

Systems with small density of current carriers, such as insulators or intrinsic-semiconductors, were recently considered by Pitaevskii [2] and by us [3]. In [2] the thermal Lifshitz force between an atom and a conductor with low charge density was computed in terms of the Green function formalism, taking into account the penetration of the static component of the fluctuating EM field into the conductor. This approach is quasi-static, appropriate for the large distance regime

of the thermal Lifshitz atom-surface interaction. The relevant (longitudinal) Green function, expressed in terms of an auxiliary static potential field, can be computed assuming that the gas of carriers is nondegenerate. The static potential satisfies the equation $(\nabla^2 - \kappa^2)\varphi = 0$, where $\kappa^2 = 4\pi e^2 n_0 / \bar{\epsilon}_0 k_B T$. Here $-e$ is the electron charge, $\bar{\epsilon}_0$ is the static bare dielectric constant of the medium (which does not take into account the contribution from current carriers), and n_0 is the (uniform) carrier density. Note that $\kappa = 1/R_D$ is the inverse of the Debye-Hückel screening radius R_D . For good metals the Debye radius is very small (on the order of interatomic distances), while for insulators and intrinsic semiconductors it is much larger (on the order of microns or more).

In [3] we have extended Pitaevskii's calculation beyond the quasi-static limit and proposed a theory for the Casimir interaction taking into account Debye screening and carrier drift based on the classical Boltzmann equation. Rather than computing the force with the Green function formalism, we use the form of the Lifshitz formula written in terms of frequency-dependent reflection amplitudes $r_{\mathbf{k},j}^p(\omega)$ of the j -th material boundary. Here p denotes the polarization of incoming waves (transverse electric TE or transverse magnetic TM). For simplicity, we will assume that the material is such that there is no mixing of polarizations upon reflection (the more general case can be treated replacing the reflection amplitudes by 2×2 reflection matrices). The projection on the plane of the interface of the linear momentum of incoming waves is denoted by \mathbf{k} . The Casimir-Lifshitz pressure between two plane semi-spaces separated by a gap of length d is

$$P(d) = 2k_B T \sum_{n=0}' \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \sqrt{k^2 + \xi^2/c^2} \sum_p \frac{r_1^p r_2^p e^{-2d\sqrt{k^2 + \xi^2/c^2}}}{1 - r_1^p r_2^p e^{-2d\sqrt{k^2 + \xi^2/c^2}}}. \quad (1)$$

The prime in the sum over n means that the zero frequency $n = 0$ term has to be multiplied by a $1/2$ factor, and all reflection coefficients are evaluated at imaginary frequencies $\omega = i\xi_n$, where $\xi_n = 2\pi n k_B T / \hbar$ are the Matsubara frequencies.

2. From Boltzmann transport equation to reflection amplitudes

In order to compute the appropriate frequency-dependent reflection coefficients for materials with small density of carriers we will consider that the EM field interacts with the gas of drifting carriers, and that these can be modeled as a continuum nondegenerate system. Under these conditions, it is reasonable to model the carriers with the classical Boltzmann transport equation coupled to Maxwell's equations for the electromagnetic field [5, 6].

For a dielectric the carriers are charged particles (electrons or ions) hopping from site to site of the crystalline array. For an intrinsic semiconductor, the density of carriers and hole is equal, but their dynamics are different; however, in this work we treat them as dynamically equivalent, which doubles the charge density. Assuming that there is no external applied field on the material, and all fields have a time dependency of the form $e^{-i\omega t}$, Maxwell's equations take the form

$$\nabla \times \mathbf{E} = i\mu_0 \omega \mathbf{H}, \quad \nabla \times \mathbf{H} = -i\bar{\epsilon}(\omega)\omega \mathbf{E} + \mathbf{J}, \quad \nabla \cdot \mathbf{E} = -\frac{en}{\bar{\epsilon}(\omega)}. \quad (2)$$

Here n is the carrier density, μ_0 is the permeability of vacuum, and $\mathbf{J} = -en\mathbf{v}$ is the carrier current, where \mathbf{v} is the mean velocity of carriers. The charge transport in the system is described by the classical Boltzmann equation,

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\frac{e}{m} \mathbf{E} - \frac{v_T^2}{n} \nabla n - \frac{\mathbf{v}}{\tau}, \quad (3)$$

where m is the effective mass of charge carriers, $v_T = \sqrt{k_B T / m}$ is their mean thermal velocity, and τ is the carrier relaxation time. Now we linearize Eq.(3) with respect to the ac fields.

Writing the charge density as $n \rightarrow n_0 + n(\mathbf{r})e^{-i\omega t}$, the current is $\mathbf{J} = -en_0\mathbf{v}$ (after discarding the term $n(\mathbf{r})\mathbf{v}e^{-2i\omega t}$). Since by assumption there is no external field applied, there is no net motion of charges, $\mathbf{v}_0 = 0$, so the term $\mathbf{v} \cdot \nabla\mathbf{v}$ in Eq.(3) behaves as $e^{-2i\omega t}$, and can also be discarded. Finally, the linearized Boltzmann equation is

$$\left(-i\omega + \frac{1}{\tau}\right)\mathbf{v} = -\frac{e}{m}\mathbf{E} - \frac{v_T^2}{n_0}\nabla n. \quad (4)$$

Inserting $n = -\bar{\epsilon}\nabla \cdot \mathbf{E}/e$ into this equation, we can solve for \mathbf{v} :

$$\mathbf{v} = \frac{\tau}{1 - i\omega\tau} \left[-\frac{e}{m}\mathbf{E} + \frac{v_T^2\bar{\epsilon}}{en_0}\nabla \cdot (\nabla \cdot \mathbf{E}) \right]. \quad (5)$$

Plugging this expression into Maxwell's equations we derive the fundamental equation for the electric field inside the material,

$$\left[\nabla^2 + \mu_0\bar{\epsilon}(\omega)\omega^2 \left(1 + i\frac{\omega_c}{\omega(1 - i\omega\tau)} \right) \right] \mathbf{E} = \left[1 + i\mu_0\bar{\epsilon}(\omega)\frac{\omega D}{1 - i\omega\tau} \right] \nabla \cdot (\nabla \cdot \mathbf{E}). \quad (6)$$

Here $\omega_c = 4\pi en_0\mu/\bar{\omega}$, $\mu = e\tau/m$ is the mobility of carriers, and $D = v_T^2\tau$ is the diffusion constant. Note that the frequency-dependent ratio $\omega_c/D = 4\pi e^2 n_0/\bar{\epsilon}(\omega)k_B T$ coincides with $\kappa^2 = 1/R_D^2$ in the quasi-static limit.

As we will show below, Eq. (6) allows TM and TE solutions, so that there is no cross-polarization upon reflection on the material, and we can safely use reflection amplitudes rather than reflection matrices in the Lifshitz formula. Let us assume that the material occupies the semi-space region $z < 0$ and the region $z > 0$ is vacuum.

2.1. TM modes

For transverse magnetic modes $e_y = 0$, so that the electric field is

$$\mathbf{E}(\mathbf{r}) = [e_x(z)\hat{\mathbf{x}} + e_z(z)\hat{\mathbf{z}}]e^{ikx}, \quad (7)$$

where, from now on, we are omitting the phase factors $e^{-i\omega t}$. Substituting this into Eq. (6) we obtain two coupled second-order differential equations for e_x and e_z :

$$\begin{aligned} \left[\partial_z^2 + \mu_0\bar{\epsilon}(\omega)\omega \left(1 + i\frac{\tilde{\omega}_c}{\omega} + i\frac{\tilde{D}k^2}{\omega} \right) \right] e_x &= ik[1 + i\mu_0\bar{\epsilon}(\omega)\omega\tilde{D}]\partial_z e_z, \\ \left[-i\mu_0\bar{\epsilon}(\omega)\omega\tilde{D}\partial_z^2 - k^2 + \mu_0\bar{\epsilon}(\omega)\omega^2 \left(1 + i\frac{\tilde{\omega}_c}{\omega} \right) \right] e_z &= ik[1 + i\mu_0\bar{\epsilon}(\omega)\omega\tilde{D}]\partial_z e_x, \end{aligned}$$

where $\tilde{\omega}_c \equiv \omega_c/(1 - i\omega\tau)$ and $\tilde{D} \equiv D/(1 - i\omega\tau)$. It is possible to combine these two coupled equations into two uncoupled fourth-order differential equations. To this end one takes the ∂_z^2 derivative of the first equation above, which results in terms proportional to $\partial_z^4 e_x$, $\partial_z^2 e_x$, and $\partial_z^3 e_z$. This last term $\partial_z^3 e_z$ can be obtained from taking the ∂_z derivative of the second equation above, which results in terms proportional to $\partial_z^2 e_x$ and e_x . Putting all together, one can derive the following fourth-order differential equation for e_x :

$$(\partial_z^2 - \eta_T^2)(\partial_z^2 - \eta_L^2)e_x = 0, \quad (8)$$

where

$$\eta_T^2(\omega) = k^2 - \mu_0 \bar{\epsilon}(\omega) \omega^2 \left(1 + i \frac{\tilde{\omega}_c}{\omega}\right), \quad (9)$$

$$\eta_L^2(\omega) = k^2 - i \frac{\omega}{D} \left(1 + i \frac{\tilde{\omega}}{\omega}\right). \quad (10)$$

In a similar fashion one obtains the following equation for e_z :

$$(\partial_z^2 - \eta_T^2) (\partial_z^2 - \eta_L^2) e_z = 0. \quad (11)$$

The solutions of these equations that vanish for $z \rightarrow -\infty$ are $e_x(z) = A_T e^{\eta_T z} + A_L e^{\eta_L z}$ and $e_z(z) = A'_T e^{\eta_T z} + A'_L e^{\eta_L z}$, where we assume $\text{Re} \eta_T > 0$ and $\text{Re} \eta_L > 0$. The amplitudes are related as $A'_L = -i \eta_L A_L / k$ and $A'_T = -ik A_T / \eta_T$. Given this TM electric field, the associated TM magnetic field can be readily computed, $\mathbf{H} = i \hat{\mathbf{y}} A_T e^{\eta_T z} e^{ikx} (k^2 - \eta_T^2) / \mu_0 \omega \eta_T$.

Now we compute the TM reflection amplitude, imposing the boundary conditions on the $z = 0$ interface. These conditions are E_x , H_y , and $\bar{\epsilon} E_z$ continuous (the continuity of B_z is automatically satisfied for TM modes). On the vacuum side ($z > 0$) the condition $\nabla \cdot \mathbf{E} = 0$ implies $ik e_x + \partial_z e_z = 0$, so that the fields incident on the interface from the $z > 0$ side are $\mathbf{E}^{\text{in}} = E_0 [-(k_z/k) \hat{\mathbf{x}} + \hat{\mathbf{z}}] e^{ik_z z} e^{ikx}$ and $\mathbf{H}^{\text{in}} = -E_0 \omega \hat{\mathbf{y}} e^{ik_z z} e^{ikx} / \mu_0 k c^2$, where we have used that in vacuum $k^2 + k_z^2 = \omega^2 / c^2$. The reflected fields are $\mathbf{E}^{\text{r}} = r E_0 [(k_z/k) \hat{\mathbf{x}} + \hat{\mathbf{z}}] e^{-ik_z z} e^{ikx}$ and $\mathbf{H}^{\text{r}} = -r E_0 \omega \hat{\mathbf{y}} e^{-ik_z z} e^{ikx} / \mu_0 k c^2$, where r is the reflection amplitude. The transmitted fields into the material ($z < 0$) are $\mathbf{E}^{\text{t}} = \left[(A_T e^{\eta_T z} + A_L e^{\eta_L z}) \hat{\mathbf{x}} + \left(-\frac{ik}{\eta_T} A_T e^{\eta_T z} - \frac{i \eta_L}{k} A_L e^{\eta_L z} \right) \hat{\mathbf{z}} \right] e^{ikx}$ and $\mathbf{H}^{\text{t}} = i \hat{\mathbf{y}} (k^2 - \eta_T^2) A_T e^{\eta_T z} e^{ikx} / \mu_0 \omega \eta_T$. Imposing the boundary conditions, and after some straightforward algebra, the reflection amplitude can be written as $r(\omega) = (1 - \alpha) / (1 + \alpha)$, where $\alpha = \frac{k^2}{i \eta_L k z} \left[\frac{1}{\bar{\epsilon}(\omega)} - \frac{\omega^2 / c^2}{k^2 - \eta_T^2} + \frac{\eta_L \eta_T \omega^2 / c^2}{k^2 (k^2 - \eta_T^2)} \right]$. Expressed along imaginary frequencies $\omega = i \xi$, the TM reflection amplitude is

$$r_{\mathbf{k}}^{\text{TM}}(i \xi) = \frac{\bar{\epsilon}(i \xi) \sqrt{k^2 + \xi^2 / c^2} - \chi}{\bar{\epsilon}(i \xi) \sqrt{k^2 + \xi^2 / c^2} + \chi}, \quad (12)$$

where

$$\chi = \frac{1}{\eta_L} \left[k^2 + \bar{\epsilon}(i \xi) \frac{\xi^2}{c^2} \frac{\eta_L \eta_T - k^2}{\eta_T^2 - k^2} \right]. \quad (13)$$

Along imaginary frequencies, η_L and η_T take the form:

$$\eta_L(i \xi) = \sqrt{k^2 + \frac{4 \pi e^2 n_0}{\bar{\epsilon}(i \xi) k_B T} + \frac{\xi(1 + \xi \tau)}{v_T^2 \tau}}, \quad (14)$$

$$\begin{aligned} \eta_T(i \xi) &= \sqrt{k^2 + \bar{\epsilon}(i \xi) \frac{\xi^2}{c^2} \left(1 + \frac{4 \pi e^2 n_0 \tau}{m \bar{\epsilon}(i \xi) \xi (1 + \xi \tau)} \right)} \\ &= \sqrt{k^2 + [\bar{\epsilon}(i \xi) + 4 \pi \sigma(i \xi) / \xi] \xi^2 / c^2}, \end{aligned} \quad (15)$$

where $\sigma(i \xi) = \sigma_0 / (1 + \xi \tau)$ and $\sigma_0 = e^2 n_0 \tau / m$ are the ac and dc Drude conductivities, respectively. Therefore, Eq. (12) gives a modified Fresnel TM coefficient due to the presence of Debye-Hückel screening and charge drift in the material.

2.2. TE modes

For transverse electric modes $e_z = 0$, so that the electric field is

$$\mathbf{E}(\mathbf{r}) = [e_x(z) \hat{\mathbf{x}} + e_y(z) \hat{\mathbf{y}}] e^{ikx}. \quad (16)$$

Substituting this into Eq. (6) we obtain two second-order differential equations for e_x and e_z :

$$\begin{aligned} \left[\partial_z^2 + \mu_0 \bar{\epsilon}(\omega) \omega^2 \left(1 + i \frac{\tilde{\omega}_c}{\omega} + i \frac{\tilde{D} k^2}{\omega} \right) \right] e_x &= 0, \\ \left[\partial_z^2 - k^2 + \mu_0 \bar{\epsilon}(\omega) \omega^2 \left(1 + i \frac{\tilde{\omega}_c}{\omega} \right) \right] e_y &= ik [1 + i \mu_0 \bar{\epsilon}(\omega) \omega \tilde{D}] \partial_z e_x. \end{aligned}$$

The solution to the first equation is $e_x(z) = Ae^{\beta z}$, where A is a constant and $\beta^2 = -i\mu_0 \bar{\epsilon}(\omega) \omega \tilde{D} \eta_L^2$ (we assume $\text{Re}\beta > 0$). Plugging this solution into the second equation we obtain $e_y(z) = Be^{\eta_T z} + Ce^{\beta z}$, where $C = ikA\beta(1 + i\mu_0 \bar{\epsilon}(\omega) \omega \tilde{D})/(\beta^2 - \eta_T^2)$. Given this TE electric field, the associated TE magnetic field is $\mathbf{H} = (1/i\mu_0 \omega)[- (B\eta_T e^{\eta_T z} + C\beta e^{\beta z}) \hat{\mathbf{x}} + A\beta e^{\beta z} \hat{\mathbf{y}} + ik(Be^{\eta_T z} + Ce^{\beta z}) \hat{\mathbf{z}}] e^{ikx}$.

Now we compute the TE reflection amplitude imposing the boundary conditions on the interface (the continuity of $\bar{\epsilon}E_z$ is automatically satisfied for TE modes). On the vacuum side we have $e_x = 0$, so that the incident fields are $\mathbf{E}^{\text{in}} = E_0 e^{ik_z z} e^{ikx} \hat{\mathbf{y}}$ and $\mathbf{H}^{\text{in}} = E_0(-k_z \hat{\mathbf{x}} + k \hat{\mathbf{z}}) e^{ik_z z} e^{ikx} / \mu_0 \omega$, and the reflected fields are $\mathbf{E}^{\text{r}} = r E_0 e^{-ik_z z} e^{ikx} \hat{\mathbf{y}}$ and $\mathbf{H}^{\text{r}} = r E_0(+k_z \hat{\mathbf{x}} + k \hat{\mathbf{z}}) e^{-ik_z z} e^{ikx} / \mu_0 \omega$. The transmitted fields into the material are given above. A simple calculation leads to the expression of the reflection amplitude $r = (ik_z - \eta_T)/(ik_z + \eta_T)$. Upon performing the rotation $\omega \rightarrow i\xi$, we get

$$r_{\mathbf{k}}^{\text{TE}}(i\xi) = \frac{\sqrt{k^2 + \xi^2/c^2} - \eta_T}{\sqrt{k^2 + \xi^2/c^2} + \eta_T}. \quad (17)$$

Using Eq. (15) we see that r^{TE} is the usual Fresnel TE reflection coefficient with a dielectric permittivity equal to the sum of the “bare” one and the ac Drude (conduction) permittivity, $\epsilon(i\xi) = \bar{\epsilon}(i\xi) + 4\pi\sigma(i\xi)/\xi$.

As discussed in detail in [3], these modified TE and TM reflection coefficients have appropriate limiting behaviors. In the quasi-static limit ($\xi \rightarrow 0$) they coincide with the ones derived in [2] for conductors with small density of carriers in the large distance (low frequency) regime, namely $r_{\mathbf{k}}^{\text{TE}}(0) = 0$ (in the static limit the TE polarized field is a pure magnetic field, which fully penetrates the nonmagnetic material) and $r_{\mathbf{k}}^{\text{TM}}(0) = (\bar{\epsilon}_0 q - k)/(\bar{\epsilon}_0 q + k)$, with $q = \sqrt{k^2 + \kappa^2}$ (in the static limit r^{TM} interpolates between a good conductor and an ideal dielectric). On the other hand, for any frequency $\xi \geq 0$, and in the limit of ideal dielectrics (small free charge density and small effective thermal velocity), we recover the usual Fresnel equations written in terms of the bare permittivity $\bar{\epsilon}(\omega)$.

3. Influence of drifting carriers in the Casimir-Lifshitz free energy

We now study the implications of our theory in the computation of the Casimir-Lifshitz free energy

$$\frac{E}{A} = k_B T \sum_p \sum_{n=0}^{\infty'} \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \ln[1 - r_{\mathbf{k},1}^p(i\xi_n) r_{\mathbf{k},2}^p(i\xi_n) e^{-2d\sqrt{k^2 + \xi_n^2/c^2}}], \quad (18)$$

between two identical planar semi-spaces with small density of charge carriers, such as intrinsic semiconductor media. As examples, we consider the cases of pure germanium and pure silicon. The reflection coefficients Eqs.(12,17) that enter into this equation depend on temperature explicitly through the Matsubara frequencies and implicitly through the optical and conductivity parameters, which we proceed to quote.

For intrinsic Ge, the bare permittivity can be approximately fitted with a Sellmeier-type expression

$$\bar{\epsilon}(i\xi) = \bar{\epsilon}_\infty + \omega_0^2 \frac{\bar{\epsilon}_0 - \bar{\epsilon}_\infty}{\xi^2 + \omega_0^2}, \quad (19)$$

where $\bar{\epsilon}_0 \approx 16.2$, $\bar{\epsilon}_\infty \approx 1.1$, and $\omega_0 \approx 5.0 \times 10^{15}$ rad/sec at $T \approx 300$ K. The temperature dependence of the permittivity has been measured in the 20 – 300 K range at wavelengths 1.9 – 5.5 μm [7], and shown to be very weak. Therefore, in this paper we assume that the permittivity is approximately constant as a function of temperature, and given by the above Sellmeier fitting function. The intrinsic carrier density varies with temperature as

$$n_0(T) = \sqrt{n_c n_v} e^{-\frac{E_g}{2k_B T}}, \quad (20)$$

where n_c and n_v are the effective density of states in the conduction and valence band, respectively. These depend on temperature as $n_c(T) = 1.98 \times 10^{15} T^{3/2} \text{cm}^{-3}$ and $n_v(T) = 9.6 \times 10^{14} T^{3/2} \text{cm}^{-3}$ (temperature is measured in degrees K). The band gap energy also depends on temperature, $E_g(T) = 0.742 - 4.8 \times 10^{-4} T^2 / (T + 235) \text{eV}$. The effective mass of conductivity is $m = 0.12 m_e$, where m_e is the free electron mass [8]. The relaxation time depends on temperature as $\tau(T) = \tau_0 + \tau_1 e^{C_1(T/300)^2 + C_2(T/300)}$ [9], where $\tau_0 = 0.26 \text{ps}$, $\tau_1 = 1.49 \text{ps}$, $C_1 = -0.434$, and $C_2 = 1.322$. At $T = 300 \text{K}$ one has $E_g = 0.66 \text{eV}$, $n_c = 1.0 \times 10^{19} \text{cm}^{-3}$, $n_v = 5.0 \times 10^{18} \text{cm}^{-3}$, and $\tau = 3.9 \text{ps}$.

For intrinsic Si, $\bar{\epsilon}_0 \approx 11.87$, $\bar{\epsilon}_\infty \approx 1.035$, and $\omega_0 \approx 6.6 \times 10^{15}$ rad/sec at $T \approx 300$ K. The temperature dependence of the permittivity has been measured in the 20 – 300 K range at wavelengths 1.1 – 5.6 μm [7], and also shown to be very weak. The effective density of states in the conduction and valence bands are $n_c(T) = 6.2 \times 10^{15} T^{3/2} \text{cm}^{-3}$ and $n_v(T) = 3.5 \times 10^{15} T^{3/2} \text{cm}^{-3}$ respectively. The band gap energy is $E_g(T) = 1.17 - 4.73 \times 10^{-4} T^2 / (T + 636) \text{eV}$. The effective mass of conductivity is $m = 0.26 m_e$ [8]. The relaxation time parameters are $\tau_0 = 1.0 \text{ps}$, $\tau_1 = -0.538 \text{ps}$, $C_1 = 0.0015$, and $C_2 = -0.09$ [9]. At $T = 300 \text{K}$ one has $E_g = 1.12 \text{eV}$, $n_c = 3.2 \times 10^{19} \text{cm}^{-3}$, $n_v = 1.8 \times 10^{19} \text{cm}^{-3}$, and $\tau = 0.5 \text{ps}$.

In Fig 1 we plot the Casimir-Lifshitz free energy between two identical planar intrinsic semiconducting (Ge and Si) semi-spaces as a function of the distance between them. We use our theory of Casimir forces with account of Debye-Hückel screening and charge drift to compute the reflection coefficient Eqs. (12,17) and compare these predictions with the simple model in which the reflection coefficients are given by the usual Fresnel formulas in which the permittivity of the materials $\epsilon(i\xi)$ is computed by adding to the bare permittivity $\bar{\epsilon}(i\xi)$ a dc conductivity term $4\pi\sigma_0/\xi$. In both models we normalize the free energies to the free energy computed using the standard Lifshitz theory using the bare permittivity only. For intrinsic Ge and intrinsic Si, $\tilde{\omega}_c$ and \tilde{D}/ξ are both very small in the relevant range of frequencies for the Lifshitz formula. Therefore only the $n = 0$ TM mode is significantly modified by the screening and charge drift effects, $r_{\mathbf{k}}^{\text{TE}}(0) = 0$ and $r_{\mathbf{k}}^{\text{TM}}(0) = (\bar{\epsilon}(0)q - k) / (\bar{\epsilon}(0)q + k)$, with $q = \sqrt{k^2 + \kappa^2}$, as in [2]. In all other $n \geq 1$ terms in Eq. (18) the reflection coefficients can be replaced by the standard Fresnel expressions in terms of the bare permittivity $\bar{\epsilon}(i\xi)$. Since for intrinsic carrier density for Ge ($\approx 10^{13} \text{cm}^{-3}$) is much larger than that for Si ($\approx 10^{10} \text{cm}^{-3}$), the Debye radius is of Ge, $R_D = 1/\kappa = 0.68 \mu\text{m}$ is much smaller than that of Si, $R_D = 24 \mu\text{m}$. As follows from Fig. 1, the effect of Debye screening and drifting carriers (denoted as “drift” in Fig. 1) becomes important for distances much larger than the Debye radius, so that this effect is more likely to be detected in Ge than in Si. In the latter case, for distances $d > R_D$, the Casimir force is too weak, at such a large distance, to be measured by any current or proposed experimental technique. From Fig. 1 we also note that when $d \gg R_D$ the plates appear as perfect conductors for the TM $n = 0$ mode, while in the case of the additive term $(\bar{\epsilon}(i\xi) + 4\pi\sigma_0/\xi)$, denoted as “cond” in the figure), the plates appear as perfect conductors for the TM $n = 0$ mode at distances of the order of $\lambda_T = \hbar c/k_B T$ ($\approx 7 \mu\text{m}$ at $T = 300 \text{K}$), independent of the material properties.

4. On the satisfaction of Nernst theorem as a prerequisite for a Casimir theory

Calculations of finite temperature Casimir-Lifshitz forces between media with large charge density, such as metals and highly doped semiconductors, have resulted in a heated debate on the

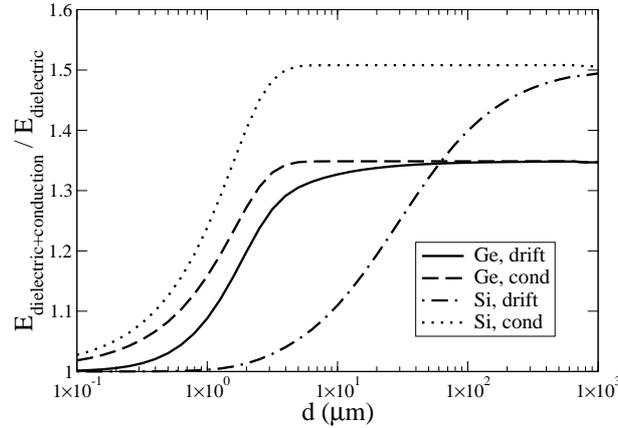


Figure 1. Casimir-Lifshitz free energy at $T = 300\text{K}$ for intrinsic Ge and Si taking into account charge drift Eqs. (12,17) (curves denoted by “drift”) or an additive dc conductivity term $4\pi\sigma_0/\xi$ to the bare permittivity (curves denoted by “cond”). For Ge, $\sigma_0 = 1/(43\Omega\text{ cm})$, and for Si, $\sigma_0 = 1/(2.3 \times 10^5\Omega\text{ cm})$. The free energies are normalized to those computed using only the bare permittivity in the usual Lifshitz theory.

adequate way to describe the optical properties of such systems within the Lifshitz formalism. Different phenomenological ways to extrapolate optical data to low frequencies, either with a Drude model including dissipation or with a plasma model setting dissipation to zero from the start, result in completely different predictions for the force at finite temperature [10].

It has been suggested that the Nernst theorem of thermodynamics serves as a way to accept or discard conductivity models when applied to the computation of the Casimir-Lifshitz entropy $S(T) = -\partial E(T)/\partial T$. The Nernst theorem states that the entropy of a physical system of N particles in thermal equilibrium at zero temperature is a well-defined constant, determined only by the degeneracy Ω_N of the ground state of the system, that is, $S(T = 0) = k_B \ln \Omega_N$. For systems with non-degenerate ground states $\Omega_N = 1$ (e.g., a perfect crystal lattice), the entropy should vanish at zero temperature. This is not the case for a large class of systems, including spin networks and glasses, that can have a large collection of degenerate ground states (with degeneracy Ω_N depending on the total number of particles), so that $S(T = 0) > 0$ in such systems. In some textbooks [11] it is further required as part of Nernst theorem that the degeneracy Ω_N be independent of any varying parameters of the system (such as pressure, volume, field intensities, etc). This has been used by some authors to discard Casimir theories which lead to a zero-temperature entropy that depends on the separation d between the Casimir plates. The requirement of independence of $S(T = 0)$ on volume is at odds with the fact that entropy is an extensive quantity, and should grow with system size [12]. It is not clear to us that one can simply discard a model of conductivity for the Casimir plates based on the fact that the zero temperature entropy depends on the distance between plates. We believe that this issue requires further study. Of course, if for a given theoretical model for the Casimir plates $S(T = 0) < 0$, then such model violates Nernst theorem.

In the remainder of this section we explicitly prove that our theory for Casimir forces with intrinsic semiconductor media is compatible with Nernst theorem of thermodynamics, resulting in $S(T = 0) = 0$, as for systems with a non-degenerate ground state ($\Omega_N = 1$). We will closely follow the approach in [13]. We start by expressing the Casimir free energy as

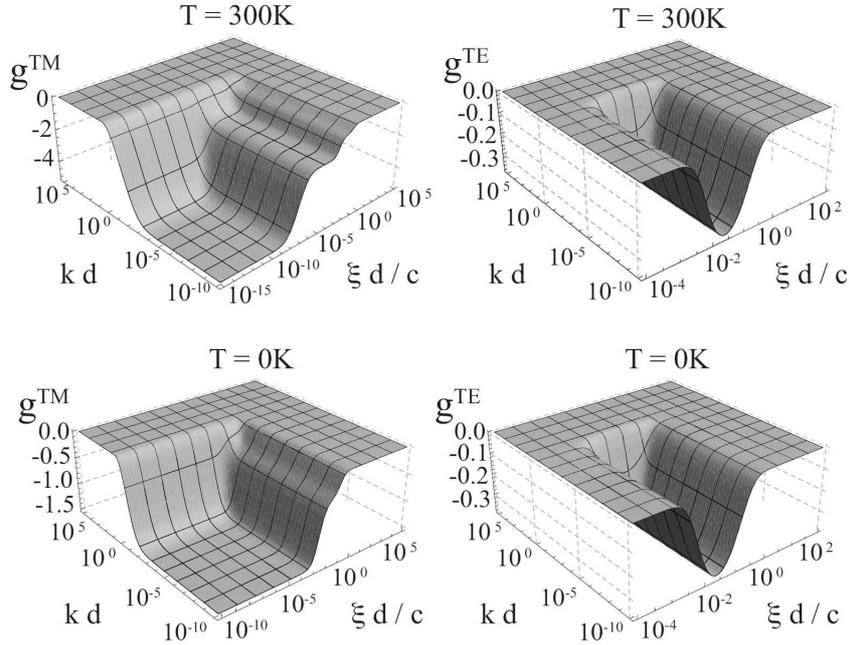


Figure 2. Behavior of the functions $g^p(i\xi, \mathbf{k})$ used to compute the Casimir-Lifshitz free energy and entropy for semiconductor materials with account of drifting carriers. The reflection coefficients are given by (12) and (17), parameters are for intrinsic Ge (see text), and the distance is set to $d = 1\mu\text{m}$. The variation with temperature (in the range $T = 0 - 300\text{K}$) of the TE function is not perceptible on the scale of the figure. The corresponding functions without account of Debye screening and carrier drift correspond to the $T = 0\text{K}$ plots in this figure.

$$E = \frac{\hbar}{2\pi} \sum_{p,\mathbf{k}} \sum_{n=0}^{\infty'} \theta g^p(in\theta, \mathbf{k}; \theta), \text{ where}$$

$$g^p(\omega, \mathbf{k}; \theta) = \ln[1 - r_{\mathbf{k},1}^p(\omega, \theta) r_{\mathbf{k},2}^p(\omega, \theta) e^{-2d\sqrt{k^2 - \omega^2/c^2}}], \quad (21)$$

and $\theta = 2\pi k_B T/\hbar$. Note that we have allowed for an implicit dependence of the reflection coefficients on temperature. The Casimir-Lifshitz entropy is $S = -(2\pi/\hbar)\partial E/\partial\theta$. In Fig. 2 we plot the behaviour of $g^p(i\xi, \mathbf{k}; \theta)$ as a function of the imaginary frequency $\omega = i\xi$ and as a function of $k = |\mathbf{k}|$ for TM and TE polarizations for different temperatures (the corresponding reflection amplitudes are obtained from Eqs. (12,17)). Let us consider the TE and TM contributions to the entropy separately.

For TE modes, since the reflection coefficient (17) depends implicitly on temperature only through $\tilde{\omega}_c$, which becomes exponentially small as low temperatures because the carrier density vanishes as $T \rightarrow 0$, it is possible to show that the $\theta \rightarrow 0$ and $\omega \rightarrow 0$ limits of $r_{\mathbf{k}}^{\text{TE}}(\omega, \theta)$ commute, that $g^p(\omega, \mathbf{k}; \theta)$ is analytic in the upper-half complex ω plane, and that the sum over n and the derivative with respect to θ in the expression for the entropy can be interchanged. Therefore, the contribution of TE modes to the entropy is [13]

$$S^{\text{TE}}(T) = - \sum_{\mathbf{k}} \sum_{n=0}^{\infty'} [g_{\omega}^{\text{TE}}(in\theta, \mathbf{k}; \theta) + in\theta g_{\omega}^{\text{TE}}(in\theta, \mathbf{k}; \theta) + \theta g_{\theta}^{\text{TE}}(in\theta, \mathbf{k}; \theta)]. \quad (22)$$

Here we have defined $g_{\omega}^{\text{TE}} \equiv \partial_{\omega} g^{\text{TE}}(\omega, \mathbf{k}; \theta)$ and $g_{\theta}^{\text{TM}} \equiv \partial_{\theta} g^{\text{TM}}(\omega, \mathbf{k}; \theta)$. Using the analytical properties of the function g^{TE} it is possible to write an expansion of the first two terms in

(22) in powers of temperature, resulting in $(g_{\xi}^{\text{TE}}(0, \mathbf{k})/6)\tau + (5g_{\xi^2}^{\text{TE}}(0, \mathbf{k})/12)\tau^2 + \dots$, where $g_{\xi}^{\text{TE}}(0, \mathbf{k}) = \lim_{\xi \rightarrow 0} \partial g^{\text{TE}}(i\xi, \mathbf{k}; 0)/\partial \xi$ and $g_{\xi^2}^{\text{TE}}(0, \mathbf{k}) = \lim_{\xi \rightarrow 0} \partial^2 g(i\xi, \mathbf{k}; 0)/\partial \xi^2$ [13]. Thus, the first two terms in (22) give a vanishing entropy at $T = 0$, and imply a low-temperature behaviour of the entropy proportional to T^2 , since $g_{\xi}^{\text{TE}}(0, \mathbf{k}) = 0$ and $g_{\xi^2}^{\text{TE}}(0, \mathbf{k}) = -e^{-2kd}\bar{\epsilon}_0^2\omega_c^2/8k^4 < 0$ (see TE plots in Fig. 2). The last term in (22) is proportional to $\partial_{\theta}\tilde{\omega}_c$, which is exponentially small at low temperatures. Therefore, the full TE contribution to the entropy vanishes at zero temperature, namely $S^{\text{TE}}(0) = 0$.

For TM modes, the reflection coefficient (12) depends implicitly on temperature both through $\tilde{\omega}_c$ and \tilde{D} in a complicated fashion. Contrary to the TE case, the $\theta \rightarrow 0$ and $\omega \rightarrow 0$ limits of $r^{\text{TM}}(\omega, \mathbf{k}; \theta)$ do not commute, and therefore it is not possible to write the contribution of TM modes to the entropy in the simple form (22). The $n \geq 1$ and $n = 0$ terms have to be treated separately. This can be done by defining a new function $\tilde{g}^{\text{TM}}(in\theta, \mathbf{k})$ which is identical to $g^{\text{TM}}(in\theta, \mathbf{k}; \theta)$ for $n \geq 1$, and for $n = 0$ it is defined as $\tilde{g}^{\text{TM}}(0, \mathbf{k}) \equiv \lim_{\theta \rightarrow 0} g^{\text{TM}}(\alpha\theta, \mathbf{k}; \theta)$, where we approach zero along the path $\omega = \alpha\theta$ [13]. The TM contribution to the entropy is

$$S^{\text{TM}}(T) = - \sum_{\mathbf{k}} \left\{ \frac{g^{\text{TM}}(0, \mathbf{k}; \theta) - \tilde{g}^{\text{TM}}(0, \mathbf{k})}{2} + \sum_{n=0}^{\infty'} [\tilde{g}^{\text{TM}}(in\theta, \mathbf{k}; \theta) + in\theta\tilde{g}_{\omega}^{\text{TM}}(in\theta, \mathbf{k}; \theta) + \theta\tilde{g}_{\theta}^{\text{TM}}(in\theta, \mathbf{k}; \theta)] \right\}. \quad (23)$$

As in the TE case, the first two terms in the sum $\sum_{n=0}^{\infty'}$ vanish in the zero temperature limit $\theta \rightarrow 0$. Their first non-vanishing contribution to the entropy is linear in T , since $g_{\xi}^{\text{TM}}(0, \mathbf{k}) > 0$ (see plots TM in Fig. 2). On the other hand, in the low-temperature limit the third term in the sum over n in (23) is,

$$\sum_{n=0}^{\infty'} \theta\tilde{g}_{\theta}^{\text{TM}}(in\theta, \mathbf{k}; \theta) \xrightarrow{\theta \rightarrow 0} \partial_{\theta}\omega_c(\theta) \int_0^{\infty} d\xi \frac{\partial \tilde{g}^{\text{TM}}(i\xi, \mathbf{k}; \theta)}{\partial \omega_c} + \partial_{\theta}D(\theta) \int_0^{\infty} d\xi \frac{\partial \tilde{g}^{\text{TM}}(i\xi, \mathbf{k}; \theta)}{\partial D}. \quad (24)$$

The first term in (24) is zero at $T = 0$ due to the exponential decay of ω_c at low temperatures. Although $\partial_{\theta}D$ does not vanish at $T = 0$ (since $\tau(T)$ goes to a non-zero constant at $T = 0$ and then $D(T) \propto T$ at low temperatures), the second term in (24) is also zero at $T = 0$ because the integrand is exponentially small. Finally, in the limit $\theta \rightarrow 0$, the first line in (23) vanishes since $g^{\text{TM}}(0, \mathbf{k}; 0) = \tilde{g}^{\text{TM}}(0, \mathbf{k})$ by definition. Therefore, the TM contribution to the zero temperature entropy is $S^{\text{TM}}(T = 0) = 0$. We conclude that our theory for Casimir-Lifshitz forces in systems with low density of carriers (intrinsic semiconductors, dielectrics, etc) with account of Debye screening and charge drift is in agreement with Nernst theorem.

5. Connection between this theory of drifting carriers and spatial dispersion

As we have already mentioned, the static limit of the reflection coefficients (12,17) obtained by us using the Boltzmann transport approach coincide with those previously derived in [2], where it was noted that the same static reflection amplitudes can be interpreted in terms of spatial dispersion. Indeed, using as the static permittivity tensor $\epsilon = \text{diag}(\epsilon^{\perp}, \epsilon^{\perp}, \epsilon^{\parallel})$ with the transverse permittivity $\epsilon^{\perp} = \bar{\epsilon}_0$, and the longitudinal permittivity depending on wavevector as $\epsilon^{\parallel}(k) = \bar{\epsilon}_0[1 + 1/(kR_D)^2]$, and computing the reflection coefficients for an anisotropic (uniaxial) material [14], it is straightforward to recover the static versions of (12,17).

Although the original Lifshitz paper [1] for dispersion forces between bodies separated by vacuum did not allow for spatial dispersion, it can certainly be generalized to include nonlocal dielectric response in those setups, for example by averaging the vacuum Maxwell stress tensor

and calculating field strengths via the retarded Green tensor of the field, that should include spatial dispersion when it is important [2, 15]. As noted in [2], problems arise when the bodies are separated by a liquid instead of vacuum.

Instead of proceeding via the Green function method, here we use the scattering formalism generalized to spatial dispersion to compute the force between plates separated by vacuum. The Casimir pressure between the plates is given by the same Eq. (1), and the effects of spatial dispersion are incorporated by appropriately writing the reflection amplitudes $r_{\mathbf{k}}^p(\omega)$ in terms of the permittivity tensor $\epsilon(\omega, k)$ [16, 17]. In [16] the permittivity tensor is computed in the random phase approximation (Lindhard dielectric function) including dissipation [18]. The reflection amplitudes are written as

$$r_{\mathbf{k}}^p(\omega) = \frac{H^p(k, \omega) - 1}{H^p(k, \omega) + 1}, \quad (25)$$

where the TM and TE H -functions are (our H, h functions correspond to the G, g functions in [16])

$$H^{\text{TM}}(k, \omega) = \frac{k}{\gamma^0} \tilde{h}_a(k, \omega) - \frac{(\omega/c)^2}{(\gamma^0)^2} \tilde{h}_b(k, \omega) + \frac{k(k - \gamma^0)}{(\gamma^0)^2} \tilde{h}_c(k, \omega) + 1, \quad (26)$$

$$H^{\text{TE}}(k, \omega) = \tilde{h}(k, \omega) + 1, \quad (27)$$

where

$$\begin{aligned} h_a(k, \omega) &= 2k \int_{-\infty}^{\infty} \frac{dq_z}{2\pi} \frac{1}{q^2 \epsilon^{\parallel}(q, \omega)}, \\ h_b(k, \omega) &= 2\gamma^0(k, \omega) \int_{-\infty}^{\infty} \frac{dq_z}{2\pi} \frac{1}{q^2 - \epsilon^{\perp}(q, \omega)(\omega/c)^2}, \\ h_c(k, \omega) &= \frac{2(\omega/c)^2 k \gamma^0(k, \omega)}{k - \gamma^0(k, \omega)} \int_{-\infty}^{\infty} \frac{dq_z}{2\pi} \frac{1}{q^2 [q^2 - \epsilon^{\perp}(q, \omega)(\omega/c)^2]}. \end{aligned} \quad (28)$$

Here $\gamma^0(k, \omega) = \sqrt{k^2 - (\omega/c)^2}$, $\mathbf{q} = (\mathbf{k}, q_z)$, and the tilde above an h -function means the h -function minus the same function except that the dielectric function is set to unity [16]. In the limit of negligible spatial dispersion (ϵ^{\parallel} and ϵ^{\perp} independent of q) one gets the usual Fresnel expressions for the reflection coefficients.

We now show that our reflection coefficients can be linked to spatial dispersion even beyond the static limit, connecting in this way our approach based on Boltzmann transport equation with spatial nonlocality. We closely follow the approach of [16]. As in the static case, we assume that the nonlocal permittivity depends solely on k , being independent of q_z . This allows us to straightforwardly compute the integrals in (28). From the H^{TE} function we obtain

$$\frac{H^{\text{TE}}(k, i\xi) - 1}{H^{\text{TE}}(k, i\xi) + 1} = \frac{\sqrt{k^2 + \xi^2/c^2} - \sqrt{k^2 + \epsilon^{\perp}(k, i\xi)\xi^2/c^2}}{\sqrt{k^2 + \xi^2/c^2} + \sqrt{k^2 + \epsilon^{\perp}(k, i\xi)\xi^2/c^2}}. \quad (29)$$

We recover our TE reflection coefficient (17) for a transverse dielectric function independent of k ,

$$\epsilon^{\perp}(k, i\xi) = \bar{\epsilon}(i\xi) \left[1 + \frac{\omega_c}{\xi(1 + \xi\tau)} \right]. \quad (30)$$

For H^{TM} we obtain

$$\begin{aligned} H^{\text{TM}}(k, i\xi) &= 1 + \frac{k}{\gamma^0} \left[\frac{1}{\epsilon^{\parallel}(k, \omega)} - 1 \right] + \frac{\xi^2/c^2}{\gamma^0} \left[\frac{1}{\eta_T} - \frac{1}{\gamma^0} \right] \\ &\quad - \frac{k\xi^2/c^2}{\gamma^0} \left[\frac{1}{k\eta_T - \eta_T^2} - \frac{1}{k\gamma^0 - (\gamma^0)^2} \right], \end{aligned} \quad (31)$$

where, after rotation $\omega \rightarrow i\xi$, $\gamma^0 = \sqrt{k^2 + \xi^2/c^2}$ and $\eta_T = \sqrt{k^2 + \epsilon^\perp(k, i\xi)\xi^2/c^2}$ from (17) and (30). Equating $(H^{\text{TM}} - 1)/(H^{\text{TM}} + 1)$ to the expression (12) for the TM reflection coefficient, one obtains $H^{\text{TM}}(k, i\xi) = \bar{\epsilon}(i\xi)\gamma^0/\chi$. Using this expression in (31) one can derive the longitudinal permittivity

$$\epsilon^\parallel(k, i\xi) = \frac{k}{\gamma^0} \quad (32)$$

$$\times \left[\frac{\bar{\epsilon}(i\xi)\gamma^0}{\chi} - 1 + \frac{k}{\gamma^0} + \frac{\xi^2/c^2}{\gamma^0} \left(\frac{1}{\eta_T} - \frac{1}{\gamma^0} \right) + \frac{k\xi^2/c^2}{\gamma^0} \left(\frac{1}{k\eta_T - \eta_T^2} - \frac{1}{k\gamma^0 - (\gamma^0)^2} \right) \right]^{-1}.$$

As follows from the above considerations, our theory for Casimir forces with media with low density of charge carriers can be directly connected to spatial dispersion, as mentioned by us in [3].

6. Conclusions

In this paper we have expanded on our previous work [3] to compute Casimir-Lifshitz forces between bodies with low density of charge carriers (intrinsic semiconductors, dielectrics, disordered systems, etc) taking into account Debye-Hückel screening and charge drift. Our approach is based on the classical Boltzmann transport equation, and is applicable to non-degenerate systems with an energy gap. We have shown how the finite conductivity of such systems modifies the Casimir-Lifshitz force between such materials, and made numerical predictions for the force using germanium and silicon plates. Our theory can be seen as a special case of spatial dispersion, and provides a simple way to take into account nonlocal effects in terms of readily available material properties. We have explicitly shown that our theory is compatible with Nernst theorem of thermodynamics. This is in agreement with previous work [16] that demonstrated that spatial dispersion resolves the issues with Nernst theorem. Work related to our approach, totally based on nonlocal dielectric responses, recently appeared [19], extending our analysis to degenerate systems.

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number of ground states for a many-body system (see, for example, Kardar M 2007 *Statistical Physics of Particles* (Cambridge University Press, Cambridge)). Note that as long as the ground-state degeneracy does not grow exponentially with the system size, then the entropy per particle does vanish at zero temperature in the thermodynamic limit.

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