Null-length springs: Some curious properties

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Null-length springs, in spite of being apparently simple, reveal several unexpected properties. All of these stem from the fact that these springs, lacking a characteristic length, show no differences between longitudinal and transverse vibrations. Some of these qualities are demonstrated in an elementary way so as to show the underlying physics. The properties proved are used to obtain certain well-known results of introductory physics. Finally, an experimental setup is described briefly to simulate null-length springs.

Elementary demonstrations of physical properties of chosen systems, being unencumbered by excessive mathematics, tend to build up intuition and lead to surer and more physical reasoning. Hence their pedagogical value.[1-3] Some problems dealing with null-length springs are particularly interesting for this kind of discussion.

In Secs. I, II, and III we show some examples dealing with null-length springs. In Sec. IV we apply the obtained results to two well-known problems of elementary mechanics. Finally, in Sec. V we describe briefly a way of making null-length springs and we make reference to a commercial device that uses them.

I. A SINGLE SPRING

We shall start by considering a spring that behaves like an ideal Hooke's law spring, no matter how much it is stretched. One of its ends is joined to a mass that can move only along a straight line without damping, and the other one is fixed at a distance d from this line (Fig. 1). If the unstretched length of the spring L_0— i.e., its length when no forces are applied— is different from zero, the motion of the mass can be considered harmonic only when its amplitude is small enough (unless $d = 0$). Besides, the frequency of the motion depends to a great extent on the distance d from the fixed point of the spring to the line of movement. When the unstretched length L_0 is zero (a "null-length" spring) the result changes appreciably: The movement is exactly harmonic for any amplitude and its frequency does not depend on the position of the spring's fixed end.

To understand these results, consider (Fig. 1) a spring with a fixed end at B. (Springs will be null length hereafter, unless we expressly state the contrary.) A mass is attached to the spring and is constrained to move along the line AC; the mass has its equilibrium position at A. At that point, the spring exerts on the mass a force F_A that balances the constraint force F_V. Due to the fact that the force exerted by the spring is strictly proportional to its length (and not merely a linear function as would be the case if we had a nonzero unstretched length), the forces at points A and C are in the geometric relation shown in Fig. 1 (this is just Thales' theorem of elementary geometry). It follows that the restoring force along the line of movement is equal to the displacement from the equilibrium position times K, the spring constant; i.e., the motion is harmonic for any amplitude.

As far as the angular frequency is concerned, we can find it in the following way: It can only depend on the three relevant parameters of the problem— the spring constant K, the mass M, and the distance d from the fixed point to the line of movement. On dimensional grounds, the only combination of these magnitudes having frequency units is $\sqrt{K/M}$;[4] hence the frequency must necessarily be proportional to such a value. Since the frequency is obviously $\sqrt{K/M}$ when $d = 0$, it is $\sqrt{K/M}$ in all cases. In particular, it cannot depend on the distance d.

The previous results are not altered if we suppose that there exists a uniform gravitational field, regardless of its direction: Since the force exerted on the mass by such a field is constant at every point of space, its only effect is to modify the equilibrium position of the mass. The oscillation frequency is unaffected.

In fact, a mass hanging from a spring in a uniform gravitational field has a vertical oscillation period equal to that corresponding to horizontal oscillations.

To see this we refer to Fig. 2. In the first place, it is clear from what we have said that the frequencies of the motions (a) and (b) are equal. Second, the constraint force F_V [case (b)] is independent of the position of the mass on the line of movement, as can be seen in Fig. 1. This last property enables us to replace the constraint AC by a uniform gravitational field $g = F_V/M = Kd/M$ without altering the motion. It is interesting to notice that the mass now performs a pendular motion with the peculiarity that it remains on a straight line. This characteristic may be immediately generalized to a uniform gravitational field of arbitrary value g: We just have to take the distance d to satisfy the relation $Mg = Kd$. From the two points we have mentioned it follows that vertical oscillations and pendular oscillations (without constraint) have the same frequency.

The previous result is useful to explain the curious fact that a mass hanging from a spring—of nonzero unstretched length— in a uniform gravitational field has a

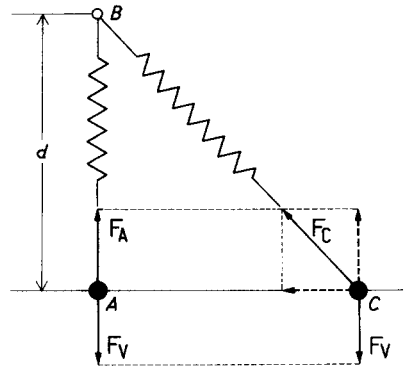

Fig. 1. A mass joined to a null-length spring fixed at B moves along the line AC. The forces exerted on the mass at two different positions are shown.

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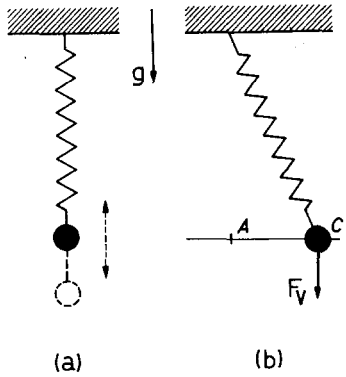

Fig. 2. Vertical and constrained pendular oscillations of a mass immersed in a uniform gravitational field g.

vertical oscillation period that cannot be greater than that corresponding to pendulumlike oscillations. To see this we can think of the nonzero unstretched length spring as a null-length spring with a rope joined to it. This rope does not alter the vertical oscillation period but it clearly raises the pendular period (as in a standard pendulum).

II. A MASS JOINED TO MANY SPRINGS

We shall now tackle the following problem: Let us suppose a mass is joined to the ends of several null-length springs fixed to arbitrary positions by their other ends. How will the mass move if it is removed from its equilibrium position and it is left free?[5]

The first thing we can assert is that it will move along the straight line joining the initial position and the equilibrium one. This is so because, as can be seen in Fig. 1, the force that each spring exerts along the transverse direction to a displacement is not altered as a result of the latter and, just as it is zero at the equilibrium position, so is it zero at every point. Consequently, the force on the mass is always directed toward the equilibrium position. Since the forces exerted by the springs vary linearly with the displacement, their sum is proportional to such displacement, the constant of proportionality being the sum of the elastic constants. Hence, the movement is harmonic regardless of the amplitude.

In order to find the oscillation frequency we can think that the mass is removed a great distance compared with the distances between the fixed ends of the springs. Therefore, we can consider, as far as the force they exert is concerned, that all of them are fixed to the same point. As a result, they behave as though they were a single spring whose spring constant is the sum of the individual ones. It is straightforward to see that the square of the oscillation frequency is the sum of the squares of all individual frequencies. In particular, it is absolutely independent of the positions to which the springs are fixed.

If some of the spring constants are smaller than zero, what has already been said is valid as long as there exists an equilibrium point for the mass. There will be such a point if and only if the sum of all spring constants is greater than zero. If this sum equals zero, it implies that a displacement along a line does not modify the force exerted on the mass because the change is proportional to the sum of all the spring constants; i.e., the system of springs is equivalent to a uniform force field in all space.

III. A MASSIVE SPRING

So far the springs we have dealt with have had no mass. We wish to show in an example another curious property of null-length springs, now having mass. We shall restrict ourselves to the case in which the mass is uniformly distributed when the spring is stretched, no other types of forces being present.

When an inextensible string hangs from its ends in the presence of a gravitational field it takes a particular shape known as a catenary. The mathematical expression for this curve is that of a hyperbolic cosine. We are interested in knowing the shape a spring takes under similar conditions. One who faces this problem for the first time may believe that the solution is more complicated than in the case of the string. Nevertheless, this is not so.

Let us first suppose that we stretch a spring between two points, S_1 and S_2, without gravity. Imagine that equally spaced marks are drawn on it. When the spring is immersed in a gravitational field, it sags (Fig. 3). To ensure the system's mechanical equilibrium, each little section of the spring exerts the same horizontal force on its neighbors. But that force is proportional to the horizontal distance between marks, each section being by itself a null-length spring. Therefore, horizontal spacing between marks is equal for all of them. Since the distance between the first (S_1) and the last (S_2) mark is constant, it is clear that each mark moves vertically as gravity is switched on. Let us take a point A on the spring. Since the portion MA is at equilibrium, the torque of all forces acting on it must be equal to zero, no matter which point the torque is referred to (these forces are shown in Fig. 3). Let A be this point. Then $F_1 y$ is equal to $F_P x'$, where F_P is the weight of the portion MA. The fact that the marks do not move horizontally implies that x' is equal to $x/2$ and F_P is proportional to x. As F_1 does not vary with A, y turns out to be proportional to x^2, i.e., the curve is a parabola.

A curious property of this problem is that the vertical position of the minimum M stays invariant when the points S_1 and S_2 move horizontally. To justify our statement, we notice that the vertical forces at every mark do not vary because they are equal to the weight of the portion from M to each mark. The vertical distance between marks, being proportional to these forces, do not change either. Finally, it is straightforward to see that this implies the mentioned result.

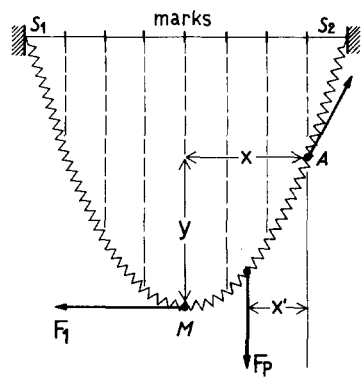

Fig. 3. A massive null-length spring hangs from S_1 and S_2 under the influence of gravity. The forces exerted on the portion MA are shown.

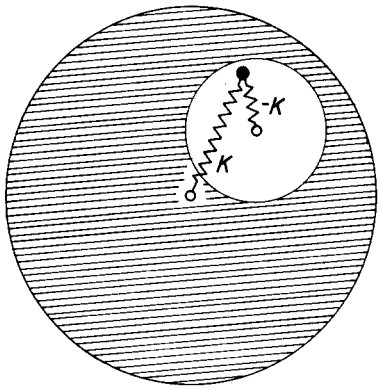

Fig. 4. The motion of a mass inside a spherical hole in a sphere of uniform mass density is equivalent to that of the mass joined to two null-length springs of elastic constants K and $-K$; the arrangement is shown in the figure.

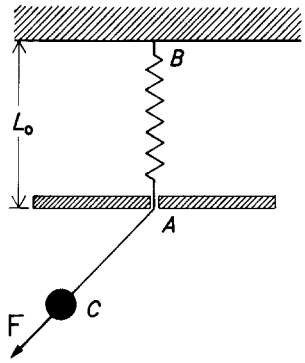

Fig. 5. An experimental setup for simulating a null-length spring: a spring of unstretched length L_0 fixed to B slips through A when a force F is exerted. The portion CA behaves like a null-length spring.

IV. APPLICATIONS TO TWO SIMPLE PROBLEMS

The properties dealt with in this article may be useful to give elementary demonstrations of some classical problems of introductory physics. In the following lines, we shall analyze two simple examples.

The problem of the frequency and harmonicity (or not) of the oscillation of a mass along a tunnel, not necessarily radial, across a homogeneous sphere, can be solved knowing that the force on the mass is proportional to the distance to the sphere's center. Since the mass can be thought of as joined to the sphere's center by means of a null-length spring, by previous reasoning we know that the motion is harmonic and of the same frequency for any direction of the tunnel.

The other example consists in determining the movement of a mass inside a spherical hole in a sphere with uniform mass density. We can consider the hollow sphere as two superimposed spherical masses of appropriate values (one positive and the other one negative). It is then clear that the movement inside the hole is equivalent to that of a mass joined to two null-length springs fixed to the centers of the spheres (Fig. 4). The spring constants of these springs have the same absolute value and opposite sign because the two spheres have the same mass density (in absolute value). Therefore, their sum is zero. It follows from what has been previously mentioned that there exists a uniform force field inside the hole having the direction of the line joining the two centers. In particular, if the hole is concentric, the field must be zero, by symmetry. This is the well-known result that a shell with uniform mass density creates no gravitational field inside it.

V. PRACTICAL REALIZATION

Finally, we shall make a comment on the possibility of making springs of this kind in the laboratory. Figure 5 shows a suitable setup: a spring with unstretched length L_0 is fixed to B and can slip through A with the least possible damping. Therefore, the force F exerted on C is proportional to the distance CA. It follows that the portion CA of the spring acts as a null-length spring. This arrangement could be used to show experimentally the properties proved, except in the case of the parabola. For the latter, a spring that can be stretched a distance much greater than its unstretched length (as in the slinky's case) could prove to be useful.

Null-length springs are not a theoretical lucubration but they are actually used, for example, in the commercial Lacoste–Romberg gravity meter, whose function is described in Refs. 6 and 7.

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