Non-Equilibrium Statistical Physics of Currents in Queuing Networks

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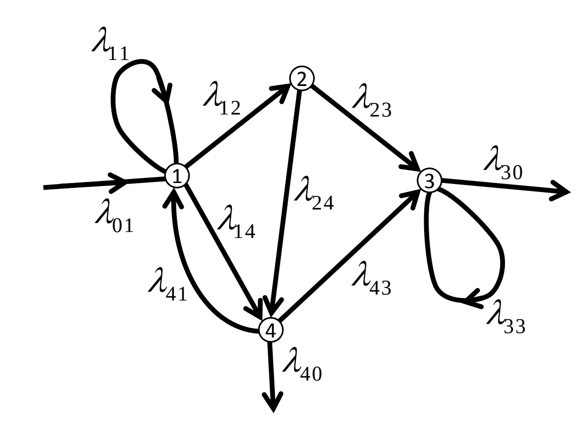
Outline

Definitions: Network of $M/M/m/\infty$ queues

Open Jackson (Queuing) Network



- Particles are identical (single class)
- Out of Detailed Balance
- $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1) =$ (set of vertexes, set of edges) - directed graph
- self-loops are allowed
- $M/M/m_i/\infty$ or shorter $M/M/m_i$
- Markovian arrival
- Markovian departure
- m_i servers at the node i
- ∞ - waiting room capacity

"Say-you don't suppose she LIVES in there?"

Agner Krarup Erlang (1878-1929)
Danish engineer, who worked for the Copenhagen Telephone Exchange, published the first paper on queueing theory in 1909.

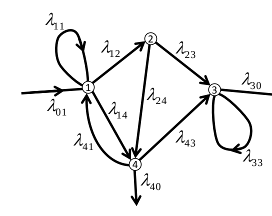
Queuing Theory Classics [Kelly '76]

Explicit Solution for Steady Queues over ANY graph!

$$P(\mathbf{n}) = Z^{-1}\prod_{i\in\mathcal{G}_0}\frac{h_i^{n_i}}{\prod_{l_i=1}^{n_i}\theta_{m_i}(l_i)},\quad \theta_m(l)=\min(m,l)$$

$$\forall i\in\mathcal{G}_0:-\quad h_i\sum_{j\neq 0}^{(i,j)\in\mathcal{G}_1}\lambda_{ij}+\sum_{j\neq 0}^{(j,i)\in\mathcal{G}_1}\lambda_{ji}h_j+\lambda_{0i}-\lambda_{i0}h_i=0$$

- Product Form .. but not a Gibbs distribution!
- Stability: $h_i < m_i$
- $\mathbf{h}$ is a "single-particle" object
- It is a version of the "zero range process" model in math/physics [Spitzer '70]
- ... not the story of the Asymmetric Exclusion Process



Master Equation

Many-particle Interacting System

$$\frac{\partial}{\partial t}P(\mathbf{n};t) = \sum_{(i,j)\in\mathcal{G}_1} \lambda_{ij}\Big(\theta_{m_i}(n_i+1)P(\cdots,n_i+1,\cdots,n_j-1,\cdots;t)$$

$$-\theta_{m_i}(n_i)P(\cdots,n_i,\cdots,n_j,\cdots;t)\Big)$$

$$+\sum_{i\in\mathcal{G}_0}\lambda_{0i}\left(P(\cdots,n_i-1,\cdots;t)-P(\cdots,n_i,\cdots;t)\right)$$

$$+\sum_{i\in\mathcal{G}_0}\lambda_{i0}\left(\theta_{m_i}(n_i+1)P(\cdots,n_i+1,\cdots;t)-\theta_{m_i}(n_i)P(\cdots,n_i,\cdots;t)\right)$$

$$\theta_m(n) = \min(n,m).$$

And still the system allows explicit solution for the stationary distribution !

Doi-Peliti Technique

Allows interpetation as an auxiliary Quantum Mechanics

$$\hat{H} = \sum_{(i,j)\in\mathcal{G}_1}^{i,j\neq 0} \lambda_{ij}(\hat{a}_j^+ - \hat{a}_i^+)\hat{b}_i^{(m_i)} + \sum_{i\in\mathcal{G}_0}\left(\lambda_{0i}(\hat{a}_i^+ - 1) + \lambda_{i0}(1-\hat{a}_i^+)\hat{b}_i^{(m_i)}\right)$$

$$\hat{b}^{(m)}|n\rangle = \theta_m(n)|n-1\rangle.$$

Peliti ['85] for death-birth + extended to the Q-networks by Zeitak ['07]

$$|s\rangle = \sum_{\mathbf{n}} P(\mathbf{n})|\mathbf{n}\rangle, \quad \sum_{\mathbf{n}} P(\mathbf{n}) = 1,$$

$$\hat{a}_j^+|\cdots, n_j, \cdots\rangle = |\cdots, n_j+1, \cdots\rangle$$

$$\hat{a}_j|\cdots, n_j, \cdots\rangle = n_j|\cdots, n_j-1, \cdots\rangle.$$

Coherent State Solutions

Ground State of the Hamiltonian is constructed explicitly

$$\hat{H}|\mathrm{coh}_m(h)\rangle = 0$$

$$\hat{b}^{(m)}|\mathrm{coh}_m(h)\rangle = h|\mathrm{coh}_m(h)\rangle, \quad |\mathrm{coh}_m(h)\rangle \equiv g_m(h\hat{a}^+)|0\rangle$$

$$g_m(x) \equiv \sum_{k=0}^{\infty} \frac{x^k}{\prod_{l=1}^{k} \theta_m(l)} = \frac{m^m}{m!}\frac{1}{1-x/m} + \sum_{k=0}^{m-1} x^k \left(\frac{1}{k!} - \frac{m^{m-k}}{m!}\right)$$

$$\forall i \in \mathcal{G}_0 : - \quad h_i \sum_{j\neq 0}^{(i,j)\in\mathcal{G}_1} \lambda_{ij} + \sum_{j\neq 0}^{(j,i)\in\mathcal{G}_1} \lambda_{ji} h_j + \lambda_{0i} - \lambda_{i0} h_i = 0$$

- Related Operator Technique (and results !!) by Massey ['84]

Notice: the entire spectrum does not allow explicit expression!

Outline

Joint Distribution function of queues and currents

- Assume that the system is stable: $\forall i: \quad h_i < m_i$
- Currents are defined on edges: J_{ij} is the number of particles which traversed the edge in time t

Our study:

- Joint probability distribution function of queues and currents accumulated in time t: $P(\mathbf{n}, \mathbf{J}|t)$
- When t is sufficiently large (larger than correlation time) we are in the Large Deviation regime:
  $P(\mathbf{n}, \mathbf{J}|t) \sim \exp(-tS(\mathbf{J}/t))\Phi(\mathbf{n}; \mathbf{J}/t)$
- How universal is the Crámer function, $S(\mathbf{J}/t)$?
- How does $\Phi(\mathbf{n}; \mathbf{J}/t)$ change with increase in $\mathbf{J}$? Anomaly?

Non-Congested to Congested (I)

$$P(\mathbf{n},\mathbf{J}|t)\sim\exp(-tS(\mathbf{J}/t))\Phi(\mathbf{n};\mathbf{J}/t)$$

Our statements (I)

- For typical and moderate $\mathbf{J}$ the network is uncongested
- In this regime $S(\mathbf{J}/t)$ is a "single-particle" object - does not feel interaction, and in particular does not depend on the number of servers
- Uncongested= all moments of $\mathbf{n}$, calculated according to $\Phi(\mathbf{n};\mathbf{J}/t)$, are finite
- Pick direction in $\mathbf{J}/t$-space, start from sufficiently small value and go towards the length of the vector increase. At sufficiently large $\mathbf{J}/t$ you will observe a non-equilibrium second order transition from uncongested regime to congested regime.

▸ Technical Details

Non-Congested to Congested (II)

$$P(\mathbf{n}, \mathbf{J}|t) \sim \exp(-tS(\mathbf{J}/t))\Phi(\mathbf{n}; \mathbf{J}/t)$$

Our statements (II)

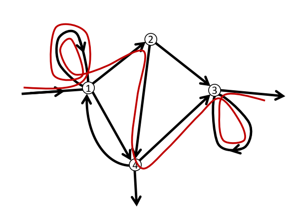
- In the congested regime one or more nodes are congested = all respective moments of $\mathbf{n}$ grow (to become infinite)
- The transition is second order = $1/\langle n_i \rangle$ is the order parameter, $\to 0$ at the transition point

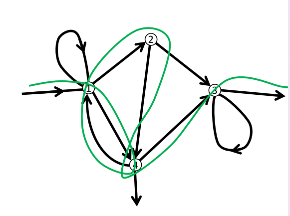
Technical Details

- Extension of the Doi-Peliti technique for generation function (over currents)
- Explicit construction of the effective Hamiltonian ground state in the coherent state form

Universality of the uncongested regime

Consider a trajectory (red or green)





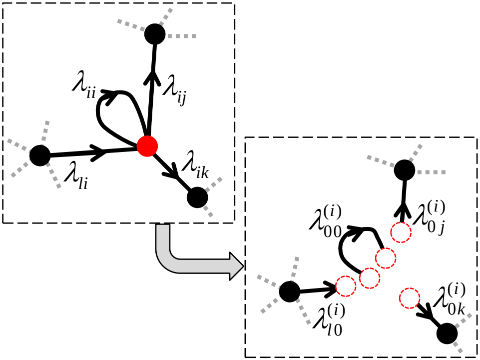
Statistics of currents is interaction independent !

- Statistics of trajectories (and of the passing time!) is not sensitive (in the leading order) to interaction
- Statistics of passing time is interaction sensitive
- # of particles which went through the system in a large time is much larger than # of these which did not leave the system

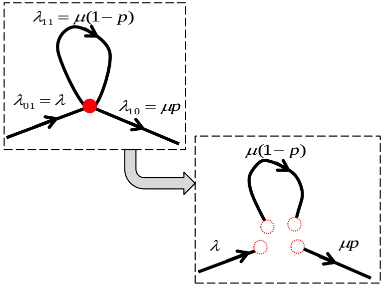
Congested regime is as simple to describe

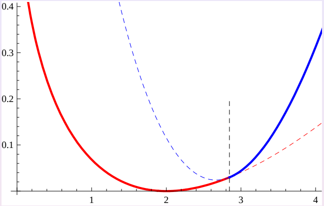
Graph Reduction Procedure

in the congested regime reduces the problem to the previous one on a less connected network



Station with a Feedback: Uncongested to Congested



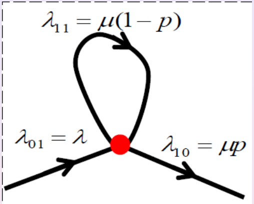


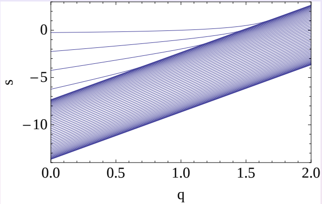
$$j < j_c: \quad \mathcal{S}(j) = \lambda(1-p/2) - \frac{\sqrt{p\lambda(4j+p\lambda)}}{2} + j\ln\left(\frac{2j+p\lambda-\sqrt{p\lambda(4j+p\lambda)}}{2j(1-p)}\right).$$

$$j > j_c: \quad \mathcal{S}(j) = \frac{\lambda^2 + m^2 p\mu^2 - 2\lambda m p\mu - m^2 p(1-p)\mu^2\ln(\mu p m/\lambda)}{\lambda} + j\ln\left(\frac{j}{em\mu(1-p)}\right)$$

- The transition is of the second order = jump in the Crámer function derivative

Numerical Exploration of the Spectrum





$m = 5, \lambda = 0.5, \mu = 2, p = 0.5, N = 50$

- At $q = 1$ $\Delta = 0$. Three discrete states and continuous band
- Gap is closing at q_c (variable conjugated to j_c)
- Supported by analytical analysis
- Consistent with the general (ground state) picture

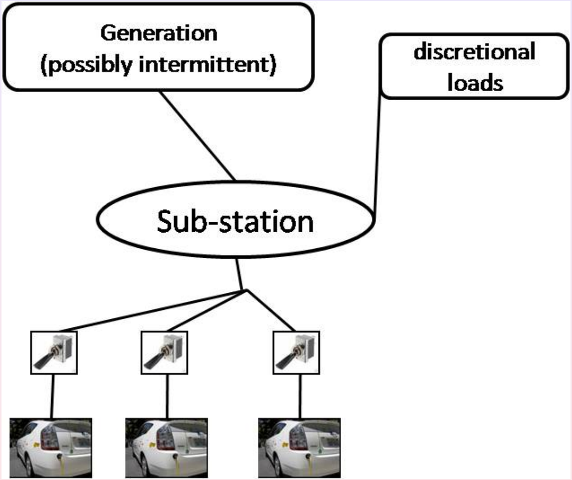
Conclusions

- Studied Generating function of currents over Jackson (queuing) network
- Adopted Doi-Peliti (Massey) technique
- Explicit coherent state (product form) solution for the ground state
- Observed second order dynamical transition from uncongested to congested network
- Illustrated and validated the story on example of a single station with feedback

Path Forward

- Sequence of phase transitions. Algorithmically and Analytically.
- Random Queuing Network - thermodynamic limit.
- Generalization to rates dependent on the outgoing (and much more difficult incoming or all) queues
- Generalization to multi-class (priorities)
- Finite waiting room - makes the problem really difficult (truly multi-particle, quantum)
- Extension to semi-open and open networks
- Changing with time rates - fluctuation theorem and Jarzynski relations?
- Some rigorous proofs are required (would not hurt)
- Applications to optimization and control, e.g. in energy and more specifically power grid networks

An application of interest



PHEV charging, power and traffic control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OOPS! SORRY!
TRAFFIC CONTROL
TOM
search ID: twl0088

Thank you !

Extension of the operator (Doi-Peliti) technique

$$\partial_t|\mathbf{s}(\mathbf{n};\mathbf{J})\rangle=\left(\hat{H}+\sum_{(i,j)\in\mathcal{G}_1}\hat{J}_{ij}\right)|\mathbf{s}(\mathbf{n};\mathbf{J})\rangle,$$

$$\forall i,j\neq 0:\quad \hat{J}_{ij}=\lambda_{ij}(\hat{a}^+_{ij}-1)\hat{a}^+_j\hat{b}_i^{(m_i)},$$

$$\hat{J}_{0i}=\lambda_{0i}(1-\hat{a}^+_{0i})\hat{a}^+_i,\quad \hat{J}_{i0}=\lambda_{i0}(\hat{a}^+_{i0}-1)\hat{b}_i^{(m_i)}$$

$$|\mathbf{s}(\mathbf{n};\mathbf{J})\rangle=P(\mathbf{n};\mathbf{J})|\mathbf{n};\mathbf{J}\rangle.\quad |\mathbf{s}_\mathbf{q}(\mathbf{n})\rangle=\sum_{\mathbf{J}}\prod_{(i,j)\in\mathcal{G}_1}q_{ij}^{J_{ij}}|\mathbf{s}(\mathbf{n};\mathbf{J})\rangle$$

$$\partial_t|\mathbf{s}_\mathbf{q}(\mathbf{n})\rangle=\hat{H}_\mathbf{q}|\mathbf{s}_\mathbf{q}(\mathbf{n})\rangle,$$

$$\hat{H}_\mathbf{q}=\sum_{(i,j)\in\mathcal{G}_1}\lambda_{ij}(\hat{a}^+_j-\hat{a}^+_i)\hat{b}_i^{(m_i)}+\sum_{(i,j)\in\mathcal{G}_1}^{i\neq 0,j\neq 0}\lambda_{ij}(q_{ij}-1)\hat{a}^+_j\hat{b}_i^{(m_i)}$$

$$+\sum_{(0,i)\in\mathcal{G}_1}\lambda_{0i}(q_{0i}a^+_i-1)+\sum_{(i,0)\in\mathcal{G}_1}\lambda_{i0}(q_{i0}-a^+_i)\hat{b}_i^{(m_i)}.$$

Key trick = Extension of the coherent state formalism

Assume that the network is not congested

$$|\mathbf{s_q}(\mathbf{n})\rangle \sim \exp(-\Delta(\mathbf{q})t)|\mathrm{coh}_{\mathbf{m}}(\mathbf{h}(\mathbf{q}))$$

$$\sum_{i\in\mathcal{G}_0}(q_{0i}\lambda_{0i}-\lambda_{i0}q_{i0}h_i(\mathbf{q}))=\Delta(\mathbf{q}),$$

$$\forall i\in\mathcal{G}_0:\quad -h_i(\mathbf{q})\sum_{j\in\mathcal{G}_0}^{(i,j)\in\mathcal{G}_1}\lambda_{ij}+\sum_{j\in\mathcal{G}_0}^{(j,i)\in\mathcal{G}_1}q_{ji}\lambda_{ji}h_j(\mathbf{q})+\lambda_{0i}-\lambda_{i0}h_i(\mathbf{q})=0$$

$$P_{\mathbf{q}}(t)=\Psi(\mathbf{q})\exp(-t\Delta(\mathbf{q}))\sim\exp\left(t\sum_{(i,j)\in\mathcal{G}_1}^{i,j\in\mathcal{G}_0}h_i(\mathbf{q})\lambda_{ij}(q_{ij}-1)\right)$$

$$\Psi(\mathbf{q})\equiv\langle\mathbf{0}|\exp\left(\sum_{i\in\mathcal{G}_0}\bar{h}_i(\mathbf{q})\hat{a}_i\right)|\mathrm{coh}_{\mathbf{m}}(\mathbf{h}(\mathbf{q}))\rangle$$

◂ Congested vs Uncongested