Planar Graphical Models which are Easy

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Outline

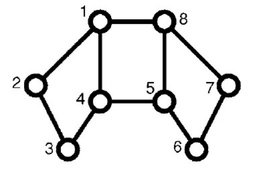
1. Introduction
   - Graphical Models
   - Easy and Difficult
   - Dimer and Ising Models on Planar Graphs
2. Planar is not necessarily easy ... but
   - Holographic Algorithms & Gauge Transformations
   - Edge-Binary models of degree ≤ 3
   - Edge-Binary Wick Models (of arbitrary degree)
3. Conclusions & Path forward
   - What did we learn?
   - Where do we go from here?

Binary Graphical Models

Forney style - variables on the edges

$$\mathcal{P}(\vec{\sigma}) = Z^{-1}\prod_a f_a(\vec{\sigma}_a)$$

$$\underbrace{Z = \sum_{\boldsymbol{\sigma}}\prod_a f_a(\vec{\sigma}_a)}_{\text{partition function}}$$



$f_a \geq 0$

$\sigma_{ab} = \sigma_{ba} = \pm 1$

$\vec{\sigma}_1 = (\sigma_{12}, \sigma_{14}, \sigma_{18})$

$\vec{\sigma}_2 = (\sigma_{12}, \sigma_{23})$

- Most Probable Configuration = Maximum Likelihood = Ground State: $\arg\max \mathcal{P}(\vec{\sigma})$
- Marginal Probability: e.g. $\mathcal{P}(\sigma_{ab}) \equiv \sum_{\vec{\sigma}\setminus\sigma_{ab}} \mathcal{P}(\vec{\sigma})$
- **Partition Function: Z** – Our main object of interest

Examples

Easy & Difficult Boolean Problems

EASY

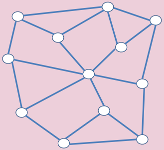
- Any graphical problems on a tree (Bethe-Peierls, Dynamical Programming, BP, TAP and other names)
- Ground State of a Rand. Field Ferrom. Ising model on any graph
- Partition function of planar Ising & Dimer models
- Finding if 2-SAT is satisfiable
- Decoding over Binary Erasure Channel = XOR-SAT
- Some network flow problems (max-flow, min-cut, shortest path, etc)
- Minimal Perfect Matching Problem
- Some special cases of Integer Programming (TUM)

Typical graphical problem, with loops and factor functions of a general position, is DIFFICULT

Glassy Ising & Dimer Models on a Planar Graph

Partition Function of $J_{ij} \gtrless 0$ Ising Model, $\sigma_i = \pm 1$

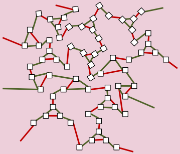
$$Z = \sum_{\vec{\sigma}} \exp\left(\frac{\sum_{(i,j)\in\Gamma} J_{ij}\sigma_i\sigma_j}{T}\right)$$



Partition Function of Dimer Model, $\pi_{ij} = 0, 1$

perfect matching

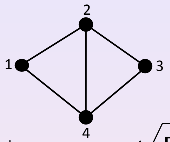
$$Z = \sum_{\vec{\pi}} \prod_{(i,j)\in\Gamma} (z_{ij})^{\pi_{ij}} \prod_{i\in\Gamma} \delta\left(\sum_{j\in i} \pi_{ij}, 1\right)$$



Ising & Dimer Classics

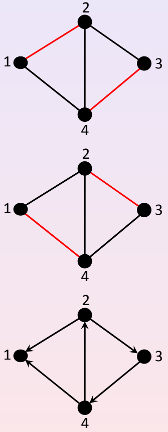
- L. Onsager, *Crystal Statistics*, Phys.Rev. **65**, 117 (1944)
- M. Kac, J.C. Ward, *A combinatorial solution of the Two-dimensional Ising Model*, Phys. Rev. **88**, 1332 (1952)
- C.A. Hurst and H.S. Green, *New Solution of the Ising Problem for a Rectangular Lattice*, J.of Chem.Phys. **33**, 1059 (1960)
- M.E. Fisher, *Statistical Mechanics on a Plane Lattice*, Phys.Rev **124**, 1664 (1961)
- P.W. Kasteleyn, *The statistics of dimers on a lattice*, Physics **27**, 1209 (1961)
- P.W. Kasteleyn, *Dimer Statistics and Phase Transitions*, J. Math. Phys. **4**, 287 (1963)
- M.E. Fisher, *On the dimer solution of planar Ising models*, J. Math. Phys. **7**, 1776 (1966)
- F. Barahona, *On the computational complexity of Ising spin glass models*, J.Phys. A **15**, 3241 (1982)

Pfaffian solution of the Matching problem



$$Z = z_{12}z_{34} + z_{14}z_{23} = \sqrt{\mathrm{Det}\hat{A}} = \mathrm{Pf}[\hat{A}]$$

$$\hat{A} = \begin{pmatrix} 0 & -z_{12} & 0 & -z_{14} \\ +z_{12} & 0 & +z_{23} & -z_{24} \\ 0 & -z_{23} & 0 & +z_{34} \\ +z_{14} & +z_{24} & -z_{34} & 0 \end{pmatrix}$$



Odd-face [Kasteleyn] rule (for signs)

Direct edges of the graph such that for every internal face the number of edges oriented clockwise is odd

Fermion/Grassman Representation

Planar Spin Glass and Dimer Matching Problems

The Pfaffian formula with the "odd-face" orientation rule extends to any planar graph thus proving constructively that

- Counting weighted number of dimer matchings on a planar graph is easy
- Calculating partition function of the spin glass Ising model on a planar graph is easy

Planar is generally difficult [Barahona '82]

- Planar spin-glass problem with magnetic field is difficult
- Dimer-monomer matching is difficult even in the planar case

Outline

Are there other graphical models which are easy?

Holographic Algorithms [Valiant '02-'08]

- reduction to dimers via
- "classical" one-to-one gadgets
  (e.g. Ising model to dimer model)
- "holographic" gadgets (e.g. ice model to dimer model)
- resulted in discovery of variety of new easy planar models

Gauge Transformations [Chertkov, Chernyak '06-'09]

- Equivalent to the holographic gadgets Gauge Transformations
  (different gauges = different transformations)
- Belief Propagation (BP) Loop Calculus/Series
  is one special choice of the gauge freedom

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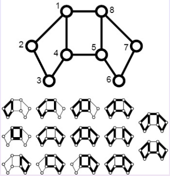
BP+ for Planar [degree $\leq$ 3]

Loop Series (general)
[MC,Chernyak '06]

$$Z = Z_0 \cdot z, \;\; z \equiv 1 + \sum_C r_C$$

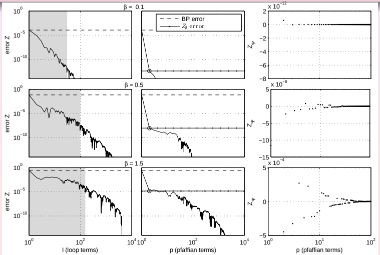


Summing 2-regular partition is easy!! [MC,Chernyak,Teodorescu '08]

$$Z_s = Z_0 \cdot z_s, \quad z_s = 1 + \sum_{C \in \mathcal{G}}^{\forall a \in C,\; |\delta(a)|_C = 2} r_C$$

Efficient Approximate Scheme
[Gomez,MC,Kappen '09]

http://arXiv.org/abs/0901.0786



Easy Models of degree ≤ 3 [MC,Chernyak,Teodorescu '08]

Generic planar problem is difficult

A planar problem is easy if

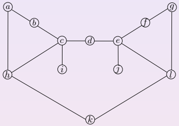
- the factor functions satisfy

$$\forall\ a \in \mathcal{G}: \quad \sum_{\vec{\sigma}_a} f_a(\vec{\sigma}_a) \times \prod_b^{(a,b)\in\mathcal{E}} \exp\left(\eta_{ab}\sigma_{ab}\right) \times \left(\sigma_{ab} - \tanh\left(\eta_{ab} + \eta_{ba}\right)\right) = 0$$

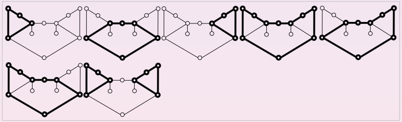
where η are messages from a BP solution for the model

- i.e. when all (!!) "three-colorings" are zero after a BP-transformation [BP gauge= all (!!) "one-colorings" are zero]

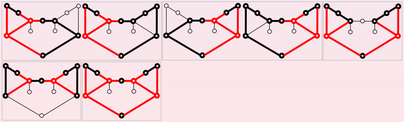
(a)



(b)



(c)

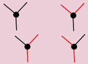


"three-colorings" are shown in red

Easy Models of degree ≤ 3 (II)

To describe the family of easy edge-binary models of degree not larger than three (partition function is reducible to Pfaffian of a $|\mathcal{G}_1| \times |\mathcal{G}_1|$-dimensional skew-symmetric matrix) one needs to:

Item #1: Generate an arbitrary factor-function set which satisfies: $\forall a: \quad W^{(a)}(\vec{\sigma}_a) = 0 \quad \text{if} \quad \sum_{b\sim a} \sigma_{ab} \neq 0 (\text{mod } 2)$



Item #2: Apply an arbitrary skew-orthogonal Gauge-transformation:

$$W^{(a)}(\boldsymbol{\pi}_a) \to f_a(\boldsymbol{\pi}_a) = \sum_{\boldsymbol{\pi}'_a} \left(\prod_{b\sim a} G_{ab}(\pi_{ab}, \pi'_{ab}) \right) W^{(a)}(\boldsymbol{\pi}'_a)$$

$$\forall \{a,b\} \in \mathcal{G}_1 : \quad \sum_{\pi} G_{ab}(\pi, \pi') G_{ba}(\pi, \pi'') = \delta(\pi', \pi'')$$

$$Z = \sum_{\boldsymbol{\pi}} \prod_{a\in\mathcal{G}_0} f_a(\boldsymbol{\pi}_a) = \sum_{\boldsymbol{\pi}} \prod_{a\in\mathcal{G}_0} \left(\sum_{\boldsymbol{\pi}'_a} \left(\prod_{b\sim a} G_{ab}(\pi_{ab}, \pi'_{ab}) \right) W^{(a)}(\boldsymbol{\pi}_a) \right)$$

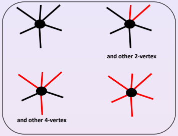
Next Step:

Generalize construction (Item #1) to an arbitrary planar graph

Edge Binary Wick (EBW) Models

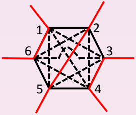
[Chernyak, MC '09]

$$Z_{EBW}(W) = \sum_{\gamma=\{\gamma_{ab}\}\in\mathcal{Z}_1(\mathcal{G};\mathbb{Z}_2)} \prod_{b\in\mathcal{G}_0}^{\sum_{a\sim b}\gamma_{ab}\neq 0} W^{(b)}_{\{a_1,\cdots,a_{2k}\}\equiv\{a|a\sim b;\gamma_{ab}=1\}}$$

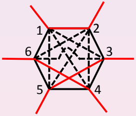


$$W^{(b)}_{\{a_1,\cdots,a_{2k}\}} \equiv \sum_{\xi\in P([2k-1])} W^{(b)}_{\xi,a_1\cdots a_{2k}}, \quad W^{(b)}_{\xi,a_1\cdots a_{2k}} \equiv (-1)^{\overbrace{\sum_{p,p'\in\xi}^{p<p'} C_{\alpha(p)}\cdot C_{\alpha(p)'}}^{\text{number of crossings (mod 2)}}} \cdot \prod_{p\in\xi} W^{(b)}_{\alpha(p)}$$

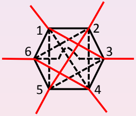
Examples of 6-colorings of a EBW-model 6 vertex



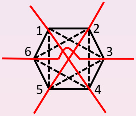
$W_{16}W_{25}W_{34}$ [zero crossing]



$-W_{12}W_{35}W_{46}$ [one crossing]



$W_{13}W_{25}W_{46}$ [two crossings]



$-W_{14}W_{25}W_{36}$ [three crossings]

Any EBW model on a planar graph is EASY

- Equivalent to Gaussian Grassman Models on the same graph
- Partition function is Pfaffian of a $|\mathcal{G}_1|\times|\mathcal{G}_1|$ matrix

Related Grassmann/Fermion Models

Vertex Gaussian Grassmann Graphical (VG^3) Models

$$Z_{VG^3}(\varsigma,\boldsymbol{\sigma};\mathbf{W}) = \frac{\int \exp\left(\frac{1}{2}\sum\limits_{(b\to a\to c)\in\mathcal{G}_1}\varphi_{ab}\varsigma_{bc}^{(a)}W_{bc}^{(a)}\varphi_{ac}\right)\exp\left(\frac{1}{2}\sum_{(a,b)\in\mathcal{G}_1}\varphi_{ab}\sigma_{ab}\varphi_{ba}\right)\prod\limits_{(a,b)} d\varphi_{ab}}{\int \exp\left(\frac{1}{2}\sum_{(a,b)\in\mathcal{G}_1}\varphi_{ab}\sigma_{ab}\varphi_{ba}\right)\prod\limits_{(a,b)} d\varphi_{ab}}$$

$$= \frac{\mathrm{Pf}(H(\varsigma,\boldsymbol{\sigma};\mathbf{W}))}{\mathrm{Pf}(H(\varsigma,\boldsymbol{\sigma};\mathbf{0}))},\qquad H_{ij} = \begin{cases}\varsigma_{bc}^{(a)}W_{bc}^{(a)}, & i=(a,b)\ \&\ j=(a,c),\ \text{where } b\neq c\sim a,\\ \sigma_{ab}, & i=(a,b),\ \&\ j=(b,a).\end{cases}$$

Grassmann (anti-commuting) variables: $\forall (a,b),(c,d)\in\mathcal{G}_1\quad \varphi_{ab}\varphi_{cd} = -\varphi_{cd}\varphi_{ab}$
Berezin (formal) integration rules: $\forall (a,b)\in\mathcal{G}_1:\quad \int d\varphi_{ab} = 0,\quad \int \varphi_{ab}d\varphi_{ab} = 1$

Main Theorem of [Chernyak,MC '09]

- $\exists \sigma,\varsigma = \pm 1:$ s.t. $Z_{VG^3}(\varsigma,\boldsymbol{\sigma};\mathbf{W}) = Z_{EBW}(\mathbf{W})$
- The special configuration of σ,ς corresponds to Kastelyan (spinor) orientation on the planar graph

Q:

To describe the family of easy edge-binary models on an arbitrary planar graph $\mathcal{G}$ (partition function is reducible to Pfaffian of a $|\mathcal{G}_1| \times |\mathcal{G}_1|$-dimensional skew-symmetric matrix)

A: [constructive]

- Generate an arbitrary Vertex Gaussian Grassmann binary-Gauge (VG^3) Model on the graph
- Fix the binary-gauge according to the Kasteleyn (spinor) rule on the extended graph
- Construct respective Edge-Binary Wick model on the original graph
- Apply an arbitrary skew-orthogonal (holographic) gauge/transformation

The partition function of the resulting model is an explicitly known Pfaffian

Future work

- Use the described hierarchy of easy planar models as a basis for efficient variational approximation of generic (difficult) planar problems. (The approach may also be useful for building efficient variational matrix-product state wave functions for quantum planar models.)
- Extend the construction to Wick Gaussian models on surface graphs of nonzero genus, in the spirit of Kastelyan, Gallucio-Loebl, Cimasoni-Reshetikhin.
- Study Wick Gaussian models on non-planar but Pfaffian orientable or k-Pfaffian orientable graphs (where any dimer model on surface graph of genus g is 2^{2g}-Pfaffian orientable).

Example (1): Statistical Physics

Ising model $\sigma_i = \pm 1$

$$\mathcal{P}(\vec{\sigma}) = Z^{-1} \exp\left(\sum_{(i,j)} J_{ij}\sigma_i\sigma_j\right)$$

J_{ij} defines the graph (lattice)

Graphical Representation

Variables are usually associated with vertexes ... but transformation to the Forney graph (variables on the edges) is straightforward

- Ferromagnetic ($J_{ij} < 0$), Anti-ferromagnetic ($J_{ij} > 0$) and Frustrated/Glassy
- Magnetization (order parameter) and Ground State
- Thermodynamic Limit, $N \to \infty$
- Phase Transitions

◂ Binary Graphical Models

Example (2): Information Theory, Machine Learning, etc

Probabilistic Reconstruction (Statistical Inference)

$\vec{\sigma}_{\text{orig}}$	$\Rightarrow$	$\vec{x}$	$\Rightarrow$	$\vec{\sigma}$
original *data* $\vec{\sigma}_{\text{orig}} \in \mathcal{C}$ codeword	noisy channel $\mathcal{P}(\vec{x}\vert\vec{\sigma})$	corrupted data: log-likelihood magnetic field	statistical inference	possible preimage $\vec{\sigma} \in \mathcal{C}$

Maximum Likelihood [ground state]

Marginalization

$$\text{ML}(\vec{x}) = \arg\max_{\vec{\sigma}} \mathcal{P}(\vec{x}|\vec{\sigma}) \qquad \sigma_i^*(\vec{x}) = \arg\max_{\sigma_i} \sum_{\vec{\sigma}\setminus\sigma_i} \mathcal{P}(\vec{x}|\vec{\sigma})$$

Counting (Partition Function): $Z(\vec{x}) = \sum_{\vec{\sigma}} \mathcal{P}(\vec{x}|\vec{\sigma})$

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Binary Graphical Models

Grassmann (fermion) Calculus for Pfaffians

Grassman Variables on Vertexes

$$\forall(a,b)\in\mathcal{G}_e:\quad \theta_a\theta_b+\theta_b\theta_a=0\quad \int d\theta=0,\quad \int \theta d\theta=1$$

Pfaffian as a Gaussian Berezin Integral over the Fermions

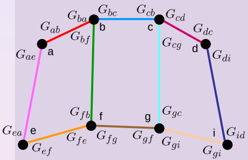
$$\int \exp\left(-\frac{1}{2}\vec{\theta}^t\hat{A}\vec{\theta}\right)d\vec{\theta}=\mathrm{Pf}(\hat{A})=\sqrt{\det(\hat{A})}$$

Pfaffian Formula

Gauge Transformations

Chertkov, Chernyak '06

Local Gauge, G, Transformations



$$Z=\sum_{\vec\sigma}\prod_a f_a(\vec\sigma_a),\ \ \vec\sigma_a=(\sigma_{ab},\sigma_{ac},\cdots)$$

$$\sigma_{ab}=\sigma_{ba}=\pm 1$$

$$f_a(\vec\sigma_a=(\sigma_{ab},\cdots))\to \sum_{\sigma'_{ab}} G_{ab}(\sigma_{ab},\sigma'_{ab})\,f_a(\sigma'_{ab},\cdots)$$

$$\sum_{\sigma_{ab}} G_{ab}(\sigma_{ab},\sigma')G_{ba}(\sigma_{ab},\sigma'')=\delta(\sigma',\sigma'')$$

The partition function is invariant under any G-gauge!

$$Z=\sum_{\vec\sigma}\prod_a f_a(\vec\sigma_a)=\sum_{\vec\sigma}\prod_a\left(\sum_{\vec\sigma'_a} f_a(\vec\sigma'_a)\prod_{b\in a} G_{ab}(\sigma_{ab},\sigma'_{ab})\right)$$

Belief Propagation as a Gauge Fixing

Chertkov, Chernyak '06

$$Z = \sum_{\vec\sigma} \prod_a f_a(\vec\sigma_a) = \sum_{\boldsymbol{\sigma}} \prod_a \left(\sum_{\vec\sigma'_a} f_a(\vec\sigma'_a) \prod_{b\in a} G_{ab}(\sigma_{ab}, \sigma'_{ab}) \right)$$

$$Z = \underbrace{Z_0(G)}_{\substack{\text{ground state}\\ \vec\sigma = +\vec 1}} + \underbrace{\sum_{\text{all possible colorings of the graph}} Z_c(G)}_{\vec\sigma \neq +\vec 1, \quad \text{excited states}}$$

Belief Propagation Gauge

$\forall a$ & $\forall b \in a$:

$$\sum_{\vec\sigma'_a} f_a(\vec\sigma') G^{(bp)}_{ab}(\sigma_{ab} = -1, \sigma'_{ab}) \prod_{c\in a}^{c\neq b} G^{(bp)}_{ac}(+1, \sigma'_{ac}) = 0$$

No loose BLUE=colored edges at any vertex of the graph!

Holographic Gadgets & Gauges

Belief Propagation as a Gauge Fixing (II)

$\underline{\forall a \ \& \ \forall b \in a:}$

$$\left\{\begin{array}{l}\sum_{\vec{\sigma}'_a} f_a(\vec{\sigma}') G^{(bp)}_{ab}(-1,\sigma'_{ab}) \prod_{c\in a}^{c\neq b} G^{(bp)}_{ac}(+1,\sigma'_{ac}) = 0 \\ \sum_{\sigma_{ab}} G_{ab}(\sigma_{ab},\sigma') G_{ba}(\sigma_{ab},\sigma'') = \delta(\sigma',\sigma'')\end{array}\right. \Rightarrow \left\{\begin{array}{l} G^{(bp)}_{ba}(+1,\sigma'_{ab}) = \rho_a^{-1} \overbrace{\sum_{\vec{\sigma}'_a\setminus\sigma'_{ab}} f_a(\vec{\sigma}') \prod_{c\in a}^{c\neq b} G^{(bp)}_{ac}(+1,\sigma'_{ac})}^{sum-product} \\ \rho_a = \sum_{\vec{\sigma}'_a} f_a(\vec{\sigma}') \prod_{c\in a} G^{(bp)}_{ac}(+1,\sigma'_{ac})\end{array}\right.$$

Belief Propagation in terms of Messages

$$G^{(bp)}_{ab}(+1,\sigma) = \frac{\exp(\sigma\eta_{ab})}{2\sqrt{\cosh(\eta_{ab}+\eta_{ba})}}, \quad G^{(bp)}_{ab}(-1,\sigma) = \sigma\frac{\exp(-\sigma\eta_{ba})}{2\sqrt{\cosh(\eta_{ab}+\eta_{ba})}} \Longrightarrow$$

$$\sum_{\vec{\sigma}_a\setminus\sigma_{ab}} f_a(\vec{\sigma}_a) \exp\left(\sum_{c\in a}\sigma_{ac}\eta_{ac}\right)(\sigma_{ab} - \tanh(\eta_{ab}+\eta_{ba})) = 0$$

$$b_a(\vec{\sigma}_a) = \frac{f_a(\vec{\sigma}_a)\exp\left(\sum_{b\in a}\sigma_{ab}\eta_{ab}\right)}{\sum_{\vec{\sigma}_a} f_a(\vec{\sigma}_a)\exp\left(\sum_{b\in a}\sigma_{ab}\eta_{ab}\right)}, \quad b_{ab}(\sigma) = \frac{\exp(\sigma(\eta_{ab}+\eta_{ba}))}{\sum_\sigma \exp(\sigma(\eta_{ab}+\eta_{ba}))}$$

▸ Holographic Gadgets & Gauges

Loop Series:

Chertkov,Chernyak '06

Exact (!!) expression in terms of BP

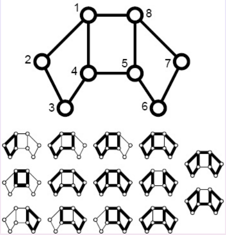
$$Z=\sum_{\vec{\sigma}_\sigma}\prod_a f_a(\vec{\sigma}_a)=Z_0\left(1+\sum_C r(C)\right)$$

$$r(C)=\frac{\prod_{a\in C}\mu_a}{\prod_{(ab)\in C}(1-m_{ab}^2)}=\prod_{a\in C}\tilde{\mu}_a$$

$C\in$ Generalized Loops $=$ Loops without loose ends

$$m_{ab}=\sum_{\vec{\sigma}_a} b_a^{(bp)}(\vec{\sigma}_a)\sigma_{ab}$$

$$\mu_a=\sum_{\vec{\sigma}_a} b_a^{(bp)}(\vec{\sigma}_a)\prod_{b\in a,C}(\sigma_{ab}-m_{ab})$$



- The Loop Series is finite
- All terms in the series are calculated within BP
- BP is exact on a tree
- BP is a Gauge fixing condition. Other choices of Gauges would lead to different representation.

▸ Holographic Gadgets & Gauges

Ice Model [vertexes of max degree 3]

#PL-3-NAE-ICE [Valiant '02]

- Input: A planar graph G = (V;E) of maximum degree 3.
- Output: The number of orientations (arrows) such that no node has all the edges directed towards it or away from it.

From arrows to binary variables

- Edge $\{a, b\}$ is broken in two by insertion of $a - b$ vertex
- Introduce binary variables s.t. if
  $a \rightarrow b \Rightarrow \pi_{a,a-b} = 0, \pi_{b,a-b} = 1$
  $b \rightarrow a \Rightarrow \pi_{a,a-b} = 1, \pi_{b,a-b} = 0$

$$Z_{ice} = \sum_{\boldsymbol{\pi}'} \left(\prod_{a \in \mathcal{G}_0} f_a(\tilde{\boldsymbol{\pi}}_a) \right) \left(\prod_{\{a,b\} \in \mathcal{G}_1} g_{a-b}(\pi_{a,a-b}, \pi_{b,a-b}) \right)$$

$$f_a(\boldsymbol{\pi}'_a) = \begin{cases} 1, & \exists\ b, c \in \delta_{\mathcal{G}}(a), \quad \text{s.t.} \quad \pi_{a,a-b} \neq \pi_{a,a-c} \\ 0, & \text{otherwise} \end{cases}$$

$$g_{a-b}(\boldsymbol{\pi}'_a) = \begin{cases} 1 & \pi_{a,a-b} \neq \pi_{b,a-b} \\ 0, & \text{otherwise} \end{cases}$$

Ice Model [vertexes of max degree 3] II

General Gauge Transformation

$$f_a(\boldsymbol{\pi}_a) \to \tilde{f}_a(\boldsymbol{\pi}_a) = \sum_{\boldsymbol{\pi}'_a} \left(\prod_{b\sim a} G_{ab}(\pi_{ab}, \pi'_{ab}) \right) f_a(\boldsymbol{\pi}'_a)$$

$$\forall \{a, b\} \in \mathcal{G}_1 : \quad \sum_{\pi} G_{ab}(\pi, \pi') G_{ba}(\pi, \pi'') = \delta(\pi', \pi'')$$

$$Z = \sum_{\boldsymbol{\pi}} \prod_{a\in\mathcal{G}_0} \tilde{f}_a(\boldsymbol{\pi}_a) = \sum_{\boldsymbol{\pi}} \prod_{a\in\mathcal{G}_0} \left(\sum_{\boldsymbol{\pi}'_a} \left(\prod_{b\sim a} G_{ab}(\pi_{ab}, \pi'_{ab}) \right) f_a(\boldsymbol{\pi}_a) \right)$$

Gauge Transformation for the Ice model

$$G^{(ice)}_{a,a-b} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad \tilde{g}_{a-b}(\boldsymbol{\pi}'_a) = \begin{cases} 1, & \pi_{a,a-b} = \pi_{b,a-b} = 0 \\ -1, & \pi_{a,a-b} = \pi_{b,a-b} = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\tilde{f}_a(\pi_{a,a-1}, \pi_{a,a-2}, \pi_{a,a-3}) = \frac{3}{\sqrt{2}} * \begin{cases} 1, & \pi_{a,a-1} = \pi_{a,a-2} = \pi_{a,a-3} = 0 \\ -1/3, & \sum_i \pi_{a,a-i} = 2 \\ 0, & \text{otherwise} \end{cases}$$