On Dense Graphical Models and Their Potential for Coding

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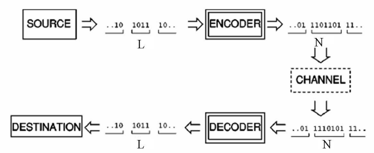
The talk is based on joint papers with V. Chernyak (Wayne State, Detroit) & S.K. Chilappagari, M. Stepanov, B. Vasic (UofA, Tucson)

Outline

Error Correction

Cell phone Hard disk Optical disk Fiber

Scheme:



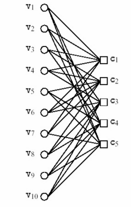
Example of Additive White Gaussian Channel:

$$P(\mathbf{x}_{out}|\mathbf{x}_{in}) = \prod_{i=bits} p(x_{out;i}|x_{in;i})$$

$$p(x|y) \sim \exp(-s^2(x-y)^2/2)$$

- Channel
  is noisy "black box" with only statistical information available
- Encoding:
  use redundancy to redistribute damaging effect of the noise
- Decoding [Algorithm]:
  reconstruct most probable codeword by noisy (polluted) channel

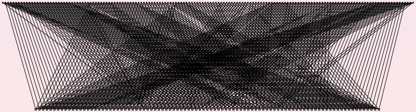
Low (and High) Density Parity Check Codes

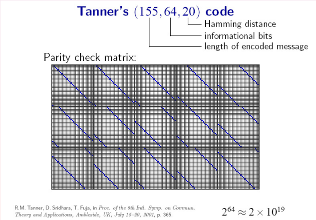


- N bits, M checks, $L = N - M$ information bits
  example: $N = 10$, $M = 5$, $L = 5$
- 2^L codewords of 2^N possible patterns
- Parity check: $\hat{H}\mathbf{v} = \mathbf{c} = \mathbf{0}$
  example:

$$\hat{H} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

- LDPC = graph (parity check matrix) is sparse
- HDPC = make it denser





Statistical Inference

$\boldsymbol{\sigma}_{\text{orig}}$	$\Rightarrow$	$\mathbf{x}$	$\Rightarrow$	$\boldsymbol{\sigma}$
original *data* $\boldsymbol{\sigma}_{\text{orig}} \in \mathcal{C}$ codeword	noisy channel $\mathcal{P}(\mathbf{x}\|\boldsymbol{\sigma})$	corrupted data: log-likelihood	statistical inference	possible preimage $\boldsymbol{\sigma} \in \mathcal{C}$

Maximum Likelihood

Marginal Probability

$$\arg\max_{\boldsymbol{\sigma}} \mathcal{P}(\boldsymbol{\sigma}|\mathbf{x}) \qquad \arg\max_{\sigma_i} \sum_{\boldsymbol{\sigma}\setminus\sigma_i} \mathcal{P}(\mathbf{x}|\boldsymbol{\sigma})$$

Exhaustive search is generally expensive:
complexity of the algorithm $\sim 2^N$

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$$\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_N), \quad N \textit{ finite}, \quad \sigma_i = \pm 1 \ (\textit{example})$$

Maximum Likelihood

Marginal Probability

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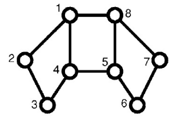
Graphical models

Factorization (Forney '01, Loeliger '01)

$$\mathcal{P}(\boldsymbol{\sigma}|\mathbf{x}) = Z^{-1}\prod_a f_a(\mathbf{x}_a|\boldsymbol{\sigma}_a)$$

$$\underbrace{Z(\mathbf{x}) = \sum_{\boldsymbol{\sigma}}\prod_a f_a(\mathbf{x}_a|\boldsymbol{\sigma}_a))}_{\text{partition function}}$$

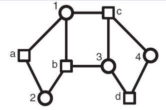


$$f_a \geq 0$$
$$\sigma_{ab} = \sigma_{ba} = \pm 1$$
$$\boldsymbol{\sigma}_1 = (\sigma_{12}, \sigma_{14}, \sigma_{18})$$
$$\boldsymbol{\sigma}_2 = (\sigma_{12}, \sigma_{13})$$

Error-Correction (linear code, bipartite Tanner graph)

$$f_i(h_i|\boldsymbol{\sigma}_i) = \exp(\sigma_i h_i)\cdot\begin{cases} 1, & \forall \alpha,\beta \ni i, \quad \sigma_{i\alpha} = \sigma_{i\beta} \\ 0, & \text{otherwise}\end{cases}$$

$$f_\alpha(\boldsymbol{\sigma}_\alpha) = \delta\left(\prod_{i\in\alpha}\sigma_i, +1\right)$$



h_i - log-likelihoods

Suboptimal but Efficient Decoding

MAP≈BP=Belief-Propagation (Bethe-Pieirls): iterative ⇒ Gallager '61; MacKay '98

- Exact on a tree [Derivation Sketch]
- Trading optimality for reduction in complexity: $\sim 2^L \to \sim L$
- BP = solving equations on the graph:

$$\eta_{\alpha j} = h_j + \sum_{\beta \neq \alpha}^{j \in \beta} \tanh^{-1}\left(\prod_{i \neq j}^{i \in \beta} \tanh \eta_{\beta i}\right) \quad \Leftarrow \text{LDPC representation}$$

- Message Passing = iterative BP
- Convergence of MP to minimum of Bethe Free energy can be enforced (Stepanov, Chertkov '06)

Bethe free energy: variational approach (Yedidia,Freeman,Weiss '01 - inspired by Bethe '35, Peierls '36)

$$F = \underbrace{-\sum_a \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) \ln f_a(\boldsymbol{\sigma}_a)}_{\text{self-energy}} + \underbrace{\sum_a \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) \ln b_a(\boldsymbol{\sigma}_a) - \sum_{(a,c)} b_{ac}(\sigma_{ac}) \ln b_{ac}(\sigma_{ac})}_{\text{configurational entropy}}$$

$$\forall\ a;\ c \in a: \quad \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) = 1, \quad b_{ac}(\sigma_{ac}) = \sum_{\boldsymbol{\sigma}_a \setminus \sigma_{ac}} b_a(\boldsymbol{\sigma}_a)$$

Linear Programming version of Belief Propagation

In the limit of large SNR, $\ln f_a \to \pm\infty$: BP→LP

Minimize $F \approx E = -\sum_a \sum_{\boldsymbol{\sigma}_a} b_a(\boldsymbol{\sigma}_a) \ln f_a(\boldsymbol{\sigma}_a) =$ self energy
under set of linear constraints

LP decoding of LDPC codes — Feldman, Wainwright, Karger '03

- ML can be restated as an LP over a codeword polytope
- LP decoding is a "local codewords" relaxation of LP-ML
- Codeword convergence certificate
- Discrete and Nice for Analysis
- Large polytope $\{b_\alpha, b_i\} \Rightarrow$ Small polytope $\{b_i\}$

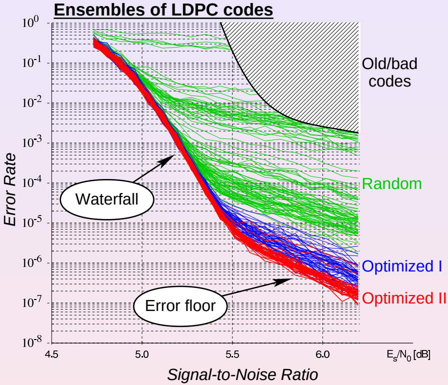
Linear Programming Decoding

Motivational Remarks

- Low- to High- ... not a "phase transition"
- How do we leave with loops?
- Try to learn from other easy but dense problems

Outline

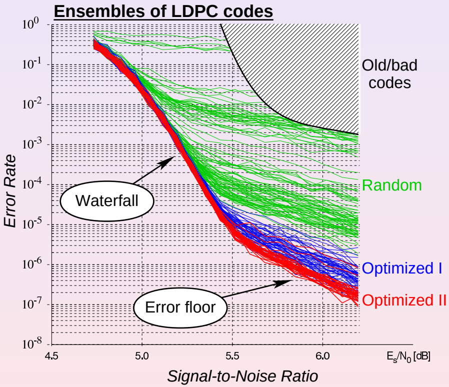
Error-Floor



- T. Richardson '03 (EF)
- Density evolution does not apply (to EF)

- BER vs SNR = measure of performance
- Finite size effects
- Waterfall $\leftrightarrow$ Error-floor
- Error-floor typically emerges due to sub-optimality of decoding, i.e. due to unaccounted loops
- Monte-Carlo is useless at FER $\lesssim 10^{-8}$

Error-floor Challenges



- Understanding the Error Floor (Inflection point, Asymptotics), Need an efficient method to analyze error-floor
- Improving Decoding
- Constructing Better Codes (Graphs)

Optimal Fluctuation Approach for Extracting Rare but Dominant Events

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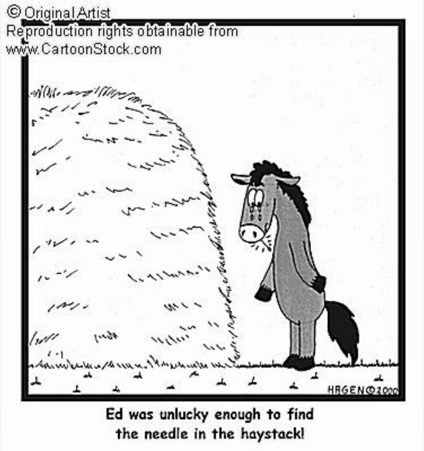
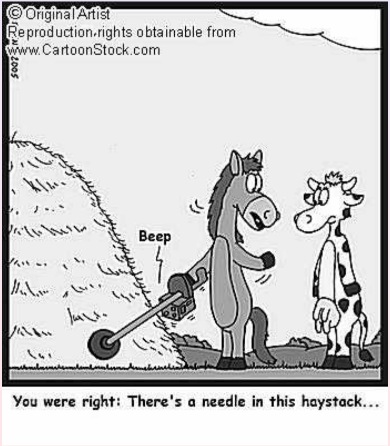
Ed was unlucky enough to find the needle in the haystack!

Optimal Fluctuation Approach for Extracting Rare but Dominant Events





Pseudo-codewords and Instantons

Error-floor is caused by Pseudo-codewords:

Wiberg '96; Forney et.al'99; Frey et.al '01; Richardson '03; Vontobel, Koetter '04-'06

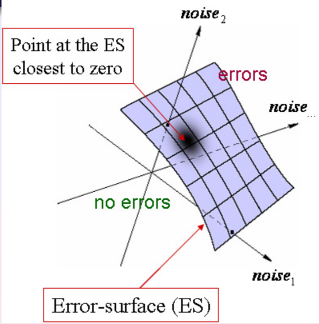
Instanton = optimal config. of the noise

$$BER = \int d(noise)\ WEIGHT(noise)$$

$$BER \sim WEIGHT\begin{pmatrix} optimal\ \ conf \\ of\ \ the\ \ noise \end{pmatrix}$$

$$\begin{matrix} optimal\ \ conf \\ of\ \ the\ \ noise \end{matrix} = \begin{matrix} Point\ \ at\ \ the\ \ ES \\ closest\ \ to\ \ "0" \end{matrix}$$

Instantons are decoded to Pseudo-Codewords



Instanton-amoeba

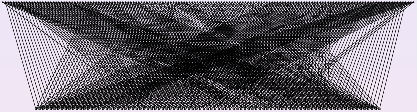
= optimization heuristics

M.Stepanov, MC, Chernyak, B.Vasic '04,'05; MS,MC '06

LP-specific: MC,MS '08

+Chillapagari,MS,MC,BV '09-

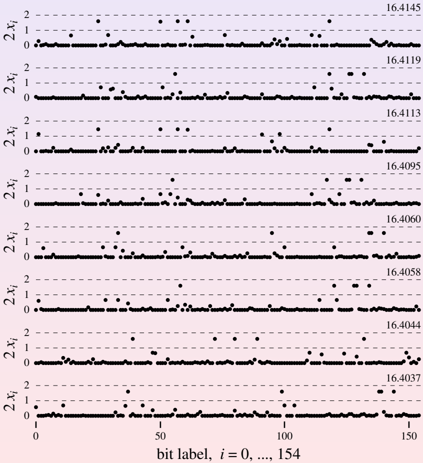
Most Probable Pseudo-Codewords Test $(155, 64, 20)$, LP, AWGN



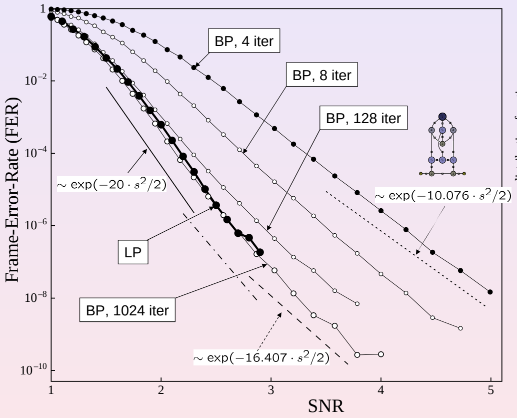
- Fast Convergence ($5-10$ iterations)
- ~ 200 pseudo-codewords within $16.4037 < d < 20$

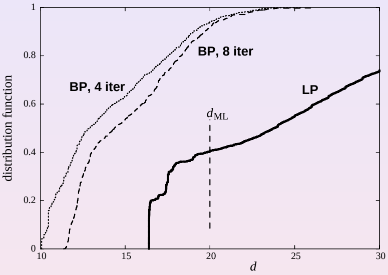
Reducing Complexity of LP

- Degree Three: MC, Stepanov '07 also Yang, Wang, Feldman '07
- Adaptive LP: Taghavi, Siegel '06; Taghavi, Siegel, Shokrollahi '08
- More recent developments: A. Tanatmis, et al '09



FER vs SNR AWGN





Instanton-amoeba:
Stepanov, et.al '04,'05,'06
LP-based PCS-search:
Chertkov, Stepanov '06,'07

▸ Instanton-Search Algorithm for BSC/LP

▸ Instantons for BSC/LP

Explaining the Error-Floor: Results and Future Tasks

Results:

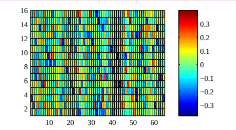
- Instantons are responsible for the Error-Floor
- Finding an instanton is an optimization problem
- Instanton-Search Toolbox is Efficient for getting correct Error-Floor asymptotic

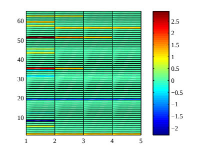
Future Tasks:

- Need an efficient algorithm constructing the entire FER vs SNR curve at once, at least finding the inflection point
- Accelerate the LP- and iterative alg.- Instanton Search
- Extend the test-base e.g. to HDPC

Other Applications for the Instanton-Search

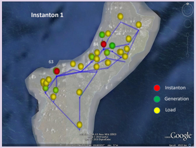
Compressed Sensing [Chillapagari,MC,Vasic '10]

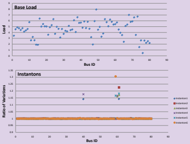




Given a measurement matrix and a probabilistic measure for error-configuration/noise: find the most probable error-configuration not-recoverable in l_1-optimization

Distance to Failure in Power Grids [MC,Pen,Stepanov '10]





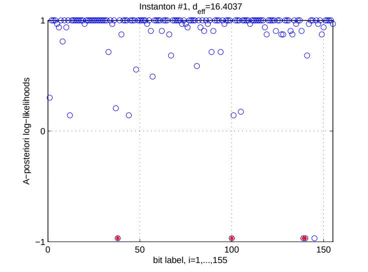
Given a DC-power flow with graph-local constraints, the problem of minimizing the load-shedding (LP-DC), and a probabilistic measure of load-distribution (centered about a good operational point): find the most probable configuration of loads which requires shedding

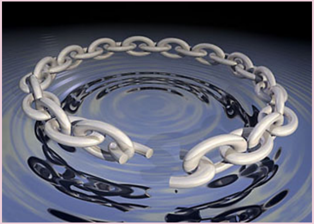
How to improve decoding?

Correcting along the critical loop

- Decode with BP/LP
- Find the critical loop in the output
  with the help of Loop Calculus [wait a bit for description]

Improve decoding by breaking the loop





Breaking the critical loop locally

Chertkov '07

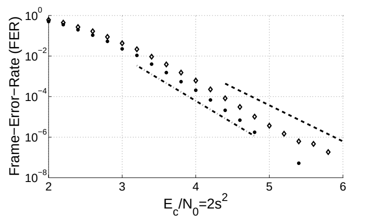
Loop Guided Guessing (LGG)

- **1.** Run the LP algorithm. Terminate if LP succeeds.
- **2.** If LP fails, find the critical loop, Γ.
- **3.** Pick any bit along the critical loop and "fix the bit" running two corrected LP schemes. Terminate if any of LPs succeeds.
- **4.** If not return to **Step 3** selecting another bit along the critical loop or to **Step 2** for an improved selection principle for Γ.

Loop-Corrected BP   LP-erasure

- Dimakis, Gohari, Wainwright '06-'08- Guessing facets
- Johnson '07 - Convex Relaxation for GM
- Sontag, Jaakkola '07 - tightening LP relaxation

$[155, 64, 20]$ test of LGG



- Complexity of LGG is the same as of LP
- LGG corrects 9 out of 10 errors at $E_b/N_0 = 4.8$!!
- Error Floor is Reduced !!

What to do with the remaining 1/10 ?

Seek for synthesis with other relevant ideas

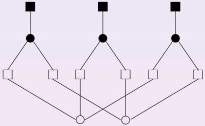
- Draper, Yedidia, Wang ISIT'07: Fixing $1, 2, ..., k$ bits $= 2^k$ LPs till decode to a codeword (ML certificate enforced).
- Weiss, Yanover, Meltzer '07: Sufficient condition for bits decoded by the bare LP to integers to show the right values.

Future Challenges:

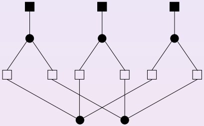
- Use Loop Calculus in sequential selection of the fixed bits
- Longer codes & HDPC
- Back to iterative BP

Some of the ideas has been already tried on HDPCC [Yufit, Lifshitz, Be'ery '09]

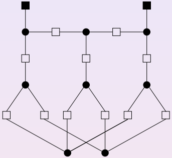
Relation Between Instantons for Different Channels and Decoders $[155, 64, 20]$



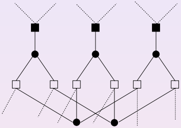
$(5,3)$ trapping set of Gallager A.



Support structure of the BSC/LP instanton.



Trapping set, $(8,2)$, over BEC.

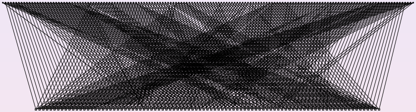


Essential part of the AWGN-instanton.

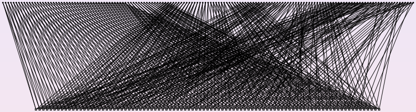
All have a common backbone!!

Chilappagari, Chertkov, Stepanov, Vasic '08

Excluding the “bad” structure



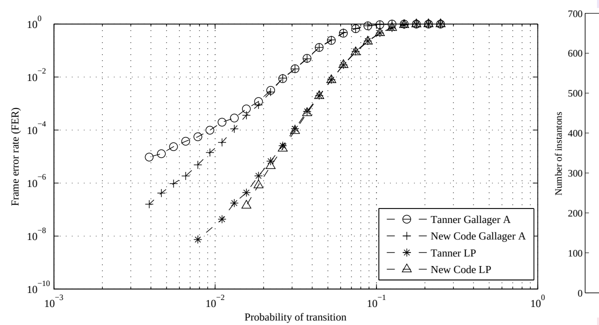
Tanner $[155, 64, 20]$ code

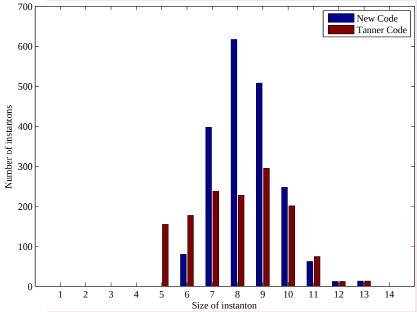


Similar code with the $(5, 3)$ backbone excluded

Random construction in the spirit of [Chilappagari, Krishnan, Vasic ’08]

Old and New Codes – Comparison BSC





Chilappagari, Chertkov, Stepanov, Vasic '08

Optimizing the Parity Check

The same code, new parity check!

- Pseudocodewords can be avoided by adding redundant constraints
- Cutting plane method [Miwa, Wadayama and Takumi '08] ... see also previously mentioned refs
- The working plan: Get a pseudocodeword. Find the parity constraint it violates. Add it to LP.

Challenges

- LP decoding complexity grows
- Too many instantons/pseudo-codewords

Preliminary Results

- Tanner code. Utilized cutting plane method.
- Found: a small set of redundant constraints (20) that cut all 155 instantons of size 5.

What did we learn from the error-floor story?

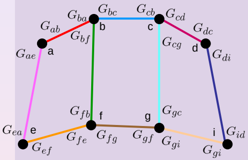
- Error floor is due to low-weight (dangerous) pseudo-codewords
- Instanton-search algorithms are efficient and allows to reconstruct FER/BER vs SNR curve efficiently.
- LP decoding into a Pseudo-codeword can be efficiently improved ... all the way to ML. Need more work on extending it to iterative algorithms.
- The pseudo-codeword-related techniques can (and should) be used more for code/graph design

Outline

Gauge Transformations

Chertkov, Chernyak '06

Local Gauge, G, Transformations



$$Z=\sum_{\vec{\sigma}}\prod_a f_a(\vec{\sigma}_a),\ \ \vec{\sigma}_a=(\sigma_{ab},\sigma_{ac},\cdots)$$

$$\sigma_{ab}=\sigma_{ba}=\pm 1$$

$$f_a(\vec{\sigma}_a=(\sigma_{ab},\cdots))\to \sum_{\sigma'_{ab}} G_{ab}\left(\sigma_{ab},\sigma'_{ab}\right) f_a(\sigma'_{ab},\cdots)$$

$$\sum_{\sigma_{ab}} G_{ab}(\sigma_{ab},\sigma')G_{ba}(\sigma_{ab},\sigma'')=\delta(\sigma',\sigma'')$$

The partition function is invariant under any G-gauge!

$$Z=\sum_{\vec{\sigma}}\prod_a f_a\left(\vec{\sigma}_a\right)=\sum_{\vec{\sigma}}\prod_a\left(\sum_{\vec{\sigma}'_a} f_a(\vec{\sigma}'_a)\prod_{b\in a} G_{ab}(\sigma_{ab},\sigma'_{ab})\right)$$

Belief Propagation as a Gauge Fixing

Chertkov, Chernyak '06

$$Z=\sum_{\vec\sigma}\prod_a f_a\left(\vec\sigma_a\right)=\sum_{\boldsymbol{\sigma}}\prod_a\left(\sum_{\vec\sigma'_a} f_a(\vec\sigma'_a)\prod_{b\in a}G_{ab}(\sigma_{ab},\sigma'_{ab})\right)$$

$$Z=\underbrace{Z_{+\vec 1}(G)}_{\substack{\text{ground state}\\ \vec\sigma=+\vec 1}}+\underbrace{\sum_{\text{all possible colorings of the graph}}}_{\vec\sigma\neq+\vec 1,\quad \text{excited states}} Z_c(G)$$

Belief Propagation Gauge $\quad$ $\forall a\ \&\ \forall b\in a:$

$$\sum_{\vec\sigma'_a} f_a(\vec\sigma')G^{(bp)}_{ab}(\sigma_{ab}=-1,\sigma'_{ab})\prod_{c\in a}^{c\neq b}G^{(bp)}_{ac}(+1,\sigma'_{ac})=0$$

No loose BLUE=colored edges at any vertex of the graph!

Loop Series:

Chertkov,Chernyak '06

Exact (!!) expression in terms of BP

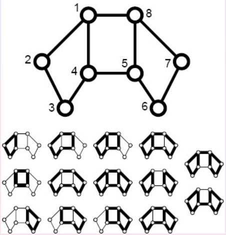
$$Z = \sum_{\boldsymbol{\sigma}} \prod_a f_a(\boldsymbol{\sigma}_a) = Z_{BP} \left(1 + \sum_C r(C)\right)$$

$$r(C) = \frac{\prod_{a \in C} \mu_a}{\prod_{(ab) \in C} (1 - m_{ab}^2)} = \prod_{a \in C} \tilde{\mu}_a$$

$C \in$ Generalized Loops $=$ Loops without loose ends

$$m_{ab} = \int d\boldsymbol{\sigma}_a b_a^{(bp)}(\boldsymbol{\sigma}_a)\sigma_{ab}$$

$$\mu_a = \int d\boldsymbol{\sigma}_a b_a^{(bp)}(\boldsymbol{\sigma}_a) \prod_{b \in a, C} (\sigma_{ab} - m_{ab})$$



- The Loop Series is finite
- All terms in the series are calculated within BP
- BP is exact on a tree

BP (Loop Calculus) + results ('06-...)

- Exact Algorithm & Efficient Truncation of Loops [V. Gómez, J.M. Mooij, H.J. Kappen '06]
- Improving LP/BP decoding with loops [MC '07]
- Loop Tower (general finite alphabet) [VC,MC '07]
- Low bound on partition function for some special (attractive) graphical models [Sudderth, Wainwright, Willsky '07]
- Fermions & Loops, e.g. monomer-dimer =series over dets [VC,MC '08]
- Counting Independent Sets Using the Bethe Approximation [V. Chandrasekaran, MC, D. Gamarnik, D. Shah, J. Shin '09]
- Beyond Gaussian BP (det=BP*det & orbit product) [J. Johnson, VC, MC '09-'10]
- ... also ... Particle Tracking (Learning with BP), Phase Transitions in Power Grids, Interdiction and OTHER APPLICATIONS

- BP+ and gauges on planar and surface graphs [VC, MC '09-'10]
- BP+ for Permanents (of non-negative matrices) [Y. Watanabe, MC '09]

Can BELIEF PROPAGATION be exact
for some graphical models with LOOPS?

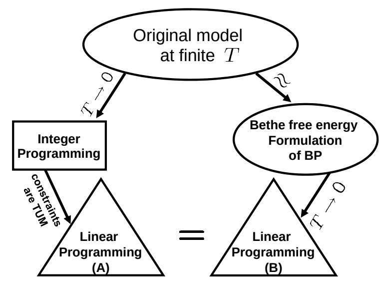
Tree reweighted BP of Kolmogorov & Wainwright '05

At $T \to 0$ BP solves the Ferromagnetic Random Field Ising model exactly on any graph!

Another Easy Example with Loops: Bayati, Shah and Sharma '06

Maximum Weight Matching of a Bi-partite graph

Proof of the BP-exactness via the Bethe Free energy approach



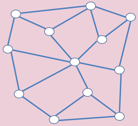
ML only!!

Chertkov '08

Ising & Dimer Models on a Planar Graph = det-easy

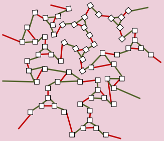
Partition Function of $J_{ij} \gtrless 0$ Ising Model, $\sigma_i = \pm 1$

$$Z = \sum_{\vec{\sigma}} \exp\left(\frac{\sum_{(i,j)\in\Gamma} J_{ij}\sigma_i\sigma_j}{T}\right) = \det(\cdots)$$



Partition Function of Dimer Model, $\pi_{ij} = 0, 1$

perfect matching



$$Z = \sum_{\vec{\pi}} \prod_{(i,j)\in\Gamma} (z_{ij})^{\pi_{ij}} \prod_{i\in\Gamma} \delta\left(\sum_{j\in i} \pi_{ij}, 1\right) = \det(\cdots)$$

Ising & Dimer Classics

- L. Onsager, *Crystal Statistics*, Phys.Rev. **65**, 117 (1944)
- M. Kac, J.C. Ward, *A combinatorial solution of the Two-dimensional Ising Model*, Phys. Rev. **88**, 1332 (1952)
- C.A. Hurst and H.S. Green, *New Solution of the Ising Problem for a Rectangular Lattice*, J.of Chem.Phys. **33**, 1059 (1960)
- M.E. Fisher, *Statistical Mechanics on a Plane Lattice*, Phys.Rev **124**, 1664 (1961)
- P.W. Kasteleyn, *The statistics of dimers on a lattice*, Physics **27**, 1209 (1961)
- P.W. Kasteleyn, *Dimer Statistics and Phase Transitions*, J. Math. Phys. **4**, 287 (1963)
- M.E. Fisher, *On the dimer solution of planar Ising models*, J. Math. Phys. **7**, 1776 (1966)
- F. Barahona, *On the computational complexity of Ising spin glass models*, J.Phys. A **15**, 3241 (1982)

Are there other (than Ising and dimer) planar graphical models which are det-easy?

Holographic Algorithms [Valiant '02-'08]

- reduction to dimers via
- "classical" one-to-one gadgets
  (e.g. Ising model to dimer model)
- "holographic" gadgets (e.g. Ice model to Dimer model)
- resulted in discovery of variety of new easy planar models

Gauge Transformations [Chertkov, Chernyak '06-'09]

- Equivalent to the holographic gadgets
  (different gauges = different transformations)
- Belief Propagation (BP) is a special choice of the gauge freedom ...
  other gauges may also be useful

Easy Planar and Surface Models of arbitrary degree [MC,VC '09-]

- Constructed family of graphical models for a given planar graph which are det-easy [arXiv:0902.0320] + degree 3 story [MC,VC, R. Teodorescu '08] + efficient computational scheme [V.Gomez, H.J.Kappen,MC '10 -JML]
- Generalized this construction to g-surface graphs: family of graphical models for a given g-surface graph which are surface-easy = partition function is a sum of 2^{2g} dets

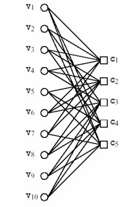
Anticipate applications in

- Decoding & Reconstruction for Planar (or approximate for close to planar) codes/graphs ... wrt approximate notice the approach of [Globerson, Jaakkola '07]
- Capacity estimation (channel-coding but also constraint-coding) for Planar codes/graphs ... wrt constrained coding notice the approach of [Schwartz, Bruck '07]

- Dense is not necessarily a handicap (LDPC $\Rightarrow$ HDPC: is it a crossover, or a phase transition still?)
- BP,LP, gauges, loops, constraints can be dealt with.
- Random Graph analysis is not the end of the story. Think finite graphs!
- Expect more synergy between the Graph-Model related disciplines.

Thank You !!

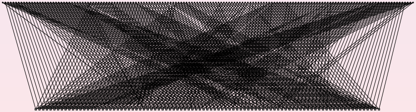
Low Density Parity Check Codes

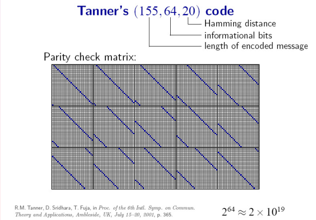


- N bits, M checks, $L = N - M$ information bits
  example: $N = 10$, $M = 5$, $L = 5$
- 2^L codewords of 2^N possible patterns
- Parity check: $\hat{H}\mathbf{v} = \mathbf{c} = \mathbf{0}$
  example:

$$\hat{H} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

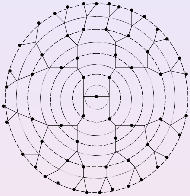
- LDPC = graph (parity check matrix) is sparse





Coding-Decoding

BP is Exact on a Tree (LDPC)



$$Z(\mathbf{h}) = \sum_{\boldsymbol{\sigma}} \prod_{\alpha=1}^{M} \delta\left(\prod_{i\in\alpha} \sigma_i, 1\right) \exp\left(\sum_{i=1}^{N} h_i \sigma_i\right)$$

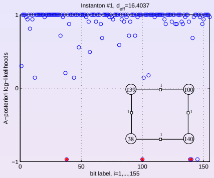
h_i is a log-likelihood at a bit (outcome of the channel)

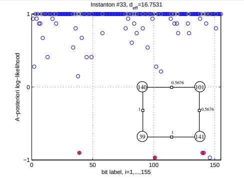
$$Z^{\pm}_{j\alpha}(\mathbf{h}^{>}) \equiv \sum_{\boldsymbol{\sigma}^{>}}^{\sigma_j=\pm1} \prod_{\beta>} \delta\left(\prod_{i\in\beta} \sigma_i, 1\right) \exp\left(\sum_{i>} h_i \sigma_i\right)$$

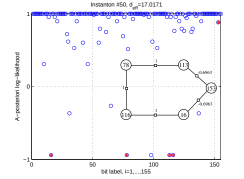
$$Z^{\pm}_{j\alpha} = \exp(\pm h_j) \prod_{\beta\neq\alpha}^{j\in\beta} \frac{1}{2}\left(\prod_{i\neq j}^{i\in\beta}(Z^{+}_{i\beta} + Z^{-}_{i\beta}) \pm \prod_{i\neq j}^{i\in\beta}(Z^{+}_{i\beta} - Z^{-}_{i\beta})\right)$$

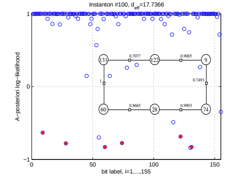
$$\eta_{j\alpha} \equiv \frac{1}{2}\ln\left(\frac{Z^{+}_{j\alpha}}{Z^{-}_{j\alpha}}\right), \quad \eta_{j\alpha} = h_j + \sum_{\beta\neq\alpha}^{j\in\beta} \tanh^{-1}\left(\prod_{i\neq j}^{i\in\beta} \tanh \eta_{i\beta}\right)$$

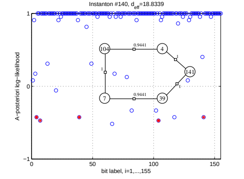
Bethe Free Energy

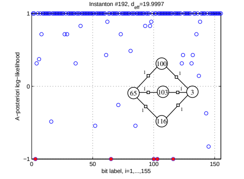

Instanton #1, d_eff=16.4037
A−posteriori log−likelihoods
bit label, i=1,...,155


Instanton #33, d_eff=16.7531
A−posteriori log−likelihood
bit label, i=1,...,155


Instanton #50, d_eff=17.0171
A−posteriori log−likelihood
bit label, i=1,...,155

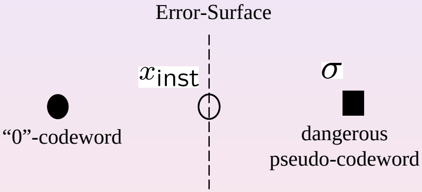

Instanton #100, d_eff=17.7366
A−posteriori log−likelihood
bit label, i=1,...,155


Instanton #140, d_eff=18.8339
A−posteriori log−likelihood
bit label, i=1,...,155


Instanton #192, d_eff=19.9997
A−posteriori log−likelihood
bit label, i=1,...,155

Back

Idea of a Smarter Search Strategy



Weighted Median: [for AWGN]

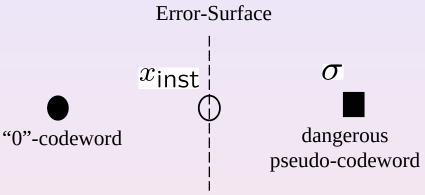
$$\mathbf{x}_{\text{inst}} = \frac{\boldsymbol{\sigma}}{2}\frac{\sum_i \sigma_i}{\sum_i \sigma_i^2}, \quad d = \frac{\left(\sum_i \sigma_i\right)^2}{\sum_i \sigma_i^2}$$

$$\text{FER} \sim \exp(-d \cdot s^2/2)$$

Wiberg '96; Forney et.al '01

Vontobel, Koetter '03,'05

Idea of a Smarter Search Strategy



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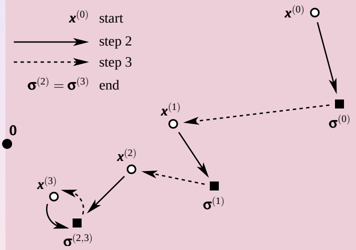
$$\text{FER} \sim \exp(-d \cdot s^2/2)$$

Wiberg '96; Forney et.al '01
Vontobel, Koetter '03,'05

Getting to the instanton in few shots:

Pseudo-Codeword Search Algorithm [Continuous Channel]

Chertkov, Stepanov '06



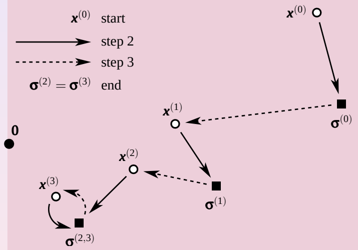
- **Start:** Initiate $\mathbf{x}^{(0)}$.
- **Step 1:** $\mathbf{x}^{(k)}$ is decoded to $\boldsymbol{\sigma}^{(k)}$.
- **Step 2:** Find $\mathbf{y}^{(k)}$ - weighted median between $\boldsymbol{\sigma}^{(k)}$, and "0"
- **Step 3:** If $\mathbf{y}^{(k)} = \mathbf{y}^{(k-1)}$, $k_* = k$ End. Otherwise go to **Step 2** with $\mathbf{x}^{(k+1)} = \mathbf{y}^{(k)} + 0$.

- Monotonicity of Iterations (e.g. observed empirically) is not proved for the AWGN version of the algorithm

Instanton-Amoeba

Pseudo-Codeword Search Algorithm [Continuous Channel]

Chertkov, Stepanov '06



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- Monotonicity of Iterations (e.g. observed empirically) is not proved for the AWGN version of the algorithm

Instanton-Amoeba

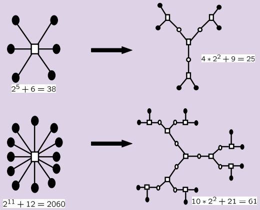
Reducing complexity of LP

Complexity of the bare LP grows exponentially with check degree

Current solutions:

- Adaptive LP (Taghavi, Siegel '06)
- BP-style relaxation of LP (Vontobel, Koetter '06)

Dendro-trick = Graph Modification (our solution) Chertkov,Stepanov'07



- MAP solutions are identical
- Set of Pseudo-codewords are identical
- Instanton spectra are very alike, $\approx$

Pseudo-Codeword Search

LP decoding [Large Polytope]

Minimize, $E=\sum_\alpha \sum_{\sigma_\alpha} b_\alpha(\sigma_\alpha)\sum_{i\in\alpha}\sigma_i(1-2x_i)/q_i$,

under the conditions:

$\forall i,\alpha \quad 0\le b_i(\sigma_i), b_\alpha(\sigma_\alpha)\le 1$

$\forall\ i: \quad \sum_{\sigma_i} b_i(\sigma_i)=1,$

$\forall\ i\ \forall\ \alpha\ni i: \quad b_i(\sigma_i)=\sum_{\sigma_\alpha\setminus\sigma_i} b_\alpha(\sigma_\alpha)$

◂ Linear Programming

Extended Variational Principe & Loop-Corrected BP

Bare BP Variational Principe: $$\left.\frac{\partial Z_0}{\partial \eta_{ab}}\right|_{\eta^{(bp)}} = 0$$

New choice of Gauges guided by the knowledge of the critical loop Γ

$$\left.\frac{\partial \exp(-\mathcal{F})}{\partial \eta_{ab}}\right|_{\eta_{\rm eff}} = 0, \quad \mathcal{F} \equiv -\ln(Z_0 + Z_\Gamma)$$

BP-equations are modified along the critical loop Γ

$$\left.\frac{\sum_{\boldsymbol{\sigma}_a}(\tanh(\eta_{ab}+\eta_{ba})-\sigma_{ab})P_a(\boldsymbol{\sigma}_a)}{\sum_{\boldsymbol{\sigma}_a} P_a(\boldsymbol{\sigma}_a)}\right|_{\eta_{\rm eff}} = \text{explicitly known contribution}\Big|_{\eta_{\rm eff}} \neq 0 \quad [\text{along } \Gamma]$$

Loop-Corrected BP Algorithm

- **1.** Run bare BP algorithm. Terminate if BP succeeds (i.e. a valid code word is found).
- **2.** If BP fails find the most relevant loop Γ that corresponds to the maximal $|r_\Gamma|$. Triad search is helping.
- **3.** Solve the modified-BP equations for the given Γ. Terminate if the improved-BP succeeds.
- **4.** Return to **Step 2** with an improved Γ-loop selection.

Breaking the Loop

LP-erasure = simple heuristics

- **1.** Run LP algorithm. Terminate if LP succeeds (i.e. a valid code word is found).
- **2.** If LP fails, find the most relevant loop Γ that corresponds to the maximal amplitude $r(\Gamma)$.
- **3.** Modify the log-likelihoods along the loop Γ introducing a shift towards zero, i.e. introduce a complete or partial erasure of the log-likelihoods at the bits. Run LP with modified log-likelihoods. Terminate if the modified LP succeeds.
- **4.** Return to **Step 2** with an improved selection principle for the critical loop.

$(155, 64, 20)$ Test

- **IT WORKS!**
  All troublemakers ($\sim$ 200 of them) previously found by LP-based Pseudo-Codeword-Search Algorithm method were successfully corrected by the LP-erasure algorithm.
- Method is invariant with respect the choice of the codeword (used to generate pseudo-codewords).

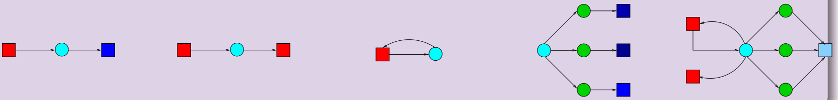
General Conjecture:

- Loop-erasure algorithm is capable of reducing the error-floor
- Local adjustment of the algorithm, anywhere along the critical loop, in the spirit of the Facet Guessing (Dimakis, Wainwright '06), may be sufficient $\Rightarrow$

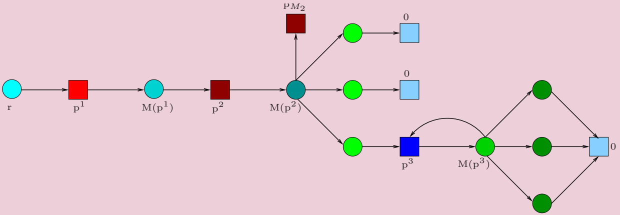
Breaking the Loop

Instanton-Search Algorithm for BSC

Required discrete channel adjustment



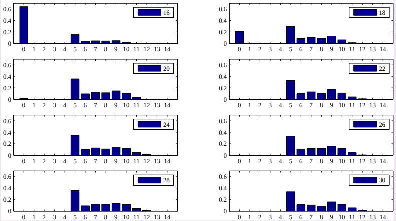
Typical Sequence for [155, 64, 20]



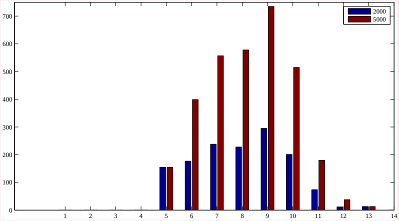
Chilappagari, Chertkov, Vasic '08

Instantons for [155, 64, 20], BSC, LP

Frequency of instanton sizes



Instanton-Bar-Graph



Chilappagari, Chertkov, Vasic '08

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- M. Chertkov and V. Chernyak, "Loop Calculus Helps to Improve Belief Propagation and Linear Programming Decodings of Low-Density-Parity-Check Codes," invited talk at 44^{th} *Allerton Conference*, September 27-29, 2006, Allerton, IL, `arXiv:cs.IT/0609154`.

All papers are available at http://cnls.lanl.gov/∼chertkov/pub.htm

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- V. Chernyak, M. Chertkov, M. Stepanov and B. Vasic, "Error correction on a tree: An instanton approach" , Phys. Rev. Lett. **93**, 198702-1 (2004).

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- M. Stepanov and M. Chertkov, "Improving convergence of belief propagation decoding," *Proceedings of 44th Allerton Conference*, September 27-29, 2006, Allerton, IL, `arXiv:cs.IT/0607112`.

All papers are available at http://cnls.lanl.gov/∼chertkov/pub.htm