Statistical Inference with Gauges, Loops and Fermions

Michael Chertkov

Center for Nonlinear Studies & Theory Division, LANL

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The talk is based on joint papers with V. Chernyak (Wayne State, Detroit)

Outline

Example (1): Statistical Physics

Ising model $\sigma_i = \pm 1$

$$\mathcal{P}(\vec{\sigma}) = Z^{-1} \exp\left(\sum_{(i,j)} J_{ij}\sigma_i\sigma_j\right)$$

J_{ij} defines the graph (lattice)

- Ferromagnetic ($J_{ij} < 0$), Anti-ferromagnetic ($J_{ij} > 0$) or Frustrated/Glassy
- Magnetization (order parameter) and Ground State
- Thermodynamic Limit, $N \to \infty$
- Phase Transitions

Example (2): Information Theory, Machine Learning, etc

Probabilistic Reconstruction (Statistical Inference)

$\vec{\sigma}_{\text{orig}}$	$\Rightarrow$	$\vec{x}$	$\Rightarrow$	$\vec{\sigma}$
original *data* $\vec{\sigma}_{\text{orig}} \in \mathcal{C}$ codeword	noisy channel $\mathcal{P}(\vec{x}\|\vec{\sigma})$	corrupted data: log-likelihood magnetic field	statistical inference	possible preimage $\vec{\sigma} \in \mathcal{C}$

Maximum Likelihood [ground state]

Marginalization

$$\text{ML}(\vec{x}) = \arg\max_{\vec{\sigma}} \mathcal{P}(\vec{x}|\vec{\sigma}) \qquad \sigma_i^*(\vec{x}) = \arg\max_{\sigma_i} \sum_{\vec{\sigma}\setminus\sigma_i} \mathcal{P}(\vec{x}|\vec{\sigma})$$

e.g., decoding in forward error correction

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e.g., decoding in forward error correction

Example (3): Combinatorial Optimization, K-SAT

$$F(\vec{x}) = \begin{array}{c} (x_1 \vee x_2 \vee \bar{x}_3)\wedge \\ (x_5 \vee \bar{x}_1 \vee \bar{x}_4)\wedge \\ (x_2 \vee x_7 \vee x_3)\wedge \\ (\bar{x}_7 \vee x_5 \vee \bar{x}_5)\wedge \\ \cdots \end{array}$$

$1, 2, \cdots, N$ – *variables*

$F(\vec{x})$ is a conjunction of M clauses

$x_i = 0(\text{bad}), 1(\text{good})$

$\bar{x}_i$ is negation of x_i

$\vee = \text{OR} \quad \wedge = \text{AND}$

$\vec{x}$ is a "valid assignment" if $F(\vec{x}) = 1$

Probabilistic interpretation

$$P(\vec{x}) = Z^{-1} F(\vec{x}), \quad Z \equiv \sum_{\vec{x}} F(\vec{x})$$

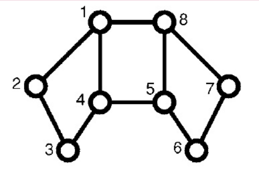
- Finding a Valid Assignment, Counting Number of Assignments
- Graphical Representation, Sparseness
- Random, non-Random formulas
- SAT/UNSAT transition wrt $\alpha = M/N$, $M, N \to \infty$
- Focus on Algorithms for given finite instance

Graphical Models = The Language

Forney style - variables on the edges

$$\mathcal{P}(\vec{\sigma}) = Z^{-1}\prod_a f_a(\vec{\sigma}_a)$$

$$\underbrace{Z = \sum_{\vec{\sigma}}\prod_a f_a(\vec{\sigma}_a)}_{\text{partition function}}$$



$$f_a \geq 0$$

$$\sigma_{ab} = \sigma_{ba} = \pm 1$$

$$\vec{\sigma}_1 = (\sigma_{12}, \sigma_{14}, \sigma_{18})$$

$$\vec{\sigma}_2 = (\sigma_{12}, \sigma_{23})$$

Objects of Interest

- Most Probable Configuration = Maximum Likelihood = Ground State: $\arg\max \mathcal{P}(\vec{\sigma})$
- Marginal Probability: e.g. $\mathcal{P}(\sigma_{ab}) \equiv \sum_{\vec{\sigma}\setminus\sigma_{ab}} \mathcal{P}(\vec{\sigma})$
- Partition Function: Z

Complexity & Algorithms

- How many operations are required to calculate partition function, find a ground state, etc ... for a graphical model of size N?
- What is the exact algorithm with the least number of operations?
- If one is ready to trade optimality for efficiency, what is the best (or just good) approximate algorithm he/she can find for a given (small) number of operations?
- Given an approximate algorithm, how to decide if the algorithm is good or bad? What is the measure of success?
- How one can systematically improve an approximate algorithm?

- Linear (or Algebraic) in N is EASY, Exponential is DIFFICULT

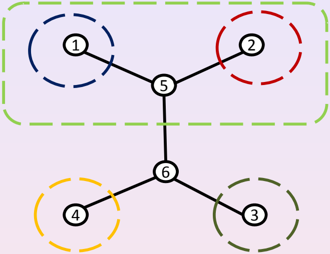
Easy & Difficult Problems

EASY

- Any graphical problems on a tree (Bethe-Pieirls, dynamical programming, belief propagation, and other names)
- Ground State of a Rand. Field Ferrom. Ising model on any graph
- Partition function of a planar Ising model
- Finding if 2-SAT is satisfiable
- Decoding over Binary Erasure Channel = XOR-SAT
- Some network flow problems (max-flow, min-cut, shortest path, etc)
- Minimal Perfect Matching Problem
- Some special cases of Integer Programming (TUM)

Typical graphical problem, with loops and factor functions of a general position, is DIFFICULT

BP is Exact on a Tree   Bethe '35, Pieirls '36



$$Z_{15}(\sigma_{15}) = f_1(\sigma_{15}), \quad Z_{25}(\sigma_{25}) = f_2(\sigma_{25}),$$
$$Z_{36}(\sigma_{36}) = f_3(\sigma_{36}), \quad Z_{46}(\sigma_{46}) = f_4(\sigma_{46})$$
$$Z_{56}(\sigma_{56}) = \sum_{\vec{\sigma}_5 \setminus \sigma_{56}} f_5(\vec{\sigma}_5) Z_{15}(\sigma_{15}) Z_{25}(\sigma_{25})$$
$$Z = \sum_{\vec{\sigma}_6} f_6(\vec{\sigma}_6) Z_{36}(\sigma_{36}) Z_{46}(\sigma_{46}) Z_{56}(\sigma_{56})$$

$$Z_{ba}(\sigma_{ab}) = \sum_{\vec{\sigma}_a \setminus \sigma_{ab}} f_a(\vec{\sigma}_a) Z_{ac}(\sigma_{ac}) Z_{ad}(\sigma_{ad}) \Rightarrow Z_{ab}(\sigma_{ab}) = A_{ab} \exp(\eta_{ab}\sigma_{ab})$$

Belief Propagation Equations

$$\sum_{\vec{\sigma}_a} f_a(\vec{\sigma}_a) \exp(\sum_{c \in a} \eta_{ac}\sigma_{ac}) \left(\sigma_{ab} - \tanh\left(\eta_{ab} + \eta_{ba}\right)\right) = 0$$

Variational Method in Statistical Mechanics

$$P(\vec{\sigma}) = \frac{\prod_a f_a(\vec{\sigma}_a)}{Z}, \quad Z \equiv \sum_{\vec{\sigma}} \prod_a f_a(\vec{\sigma}_a)$$

Exact Variational Principle

J.W. Gibbs 1903 (or earlier)

also known as Kullback-Leibler (1951) in CS and IT

$$F\{b(\vec{\sigma})\} = -\sum_{\vec{\sigma}} b(\vec{\sigma}) \sum_a \ln f_a(\vec{\sigma}_a) + \sum_{\vec{\sigma}} b(\vec{\sigma}) \ln b(\vec{\sigma})$$

$$\left.\frac{\delta F}{\delta b(\vec{\sigma})}\right|_{b(\vec{\sigma})=p(\vec{\sigma})} = 0 \quad \text{under} \quad \sum_{\vec{\sigma}} b(\vec{\sigma}) = 1$$

Variational Ansatz

- Mean-Field: $p(\vec{\sigma}) \approx b(\vec{\sigma}) = \prod_{(a,b)} b_{ab}(\sigma_{ab})$
- Belief Propagation:

$$p(\vec{\sigma}) \approx b(\vec{\sigma}) = \frac{\prod_a b_a(\vec{\sigma}_a)}{\prod_{(a,b)} b_{ab}(\sigma_{ab})} \quad \text{(exact on a tree)}$$

$$\forall\ a;\ c \in a: \quad \sum_{\vec{\sigma}_a} b_a(\vec{\sigma}_a) = 1, \quad b_{ac}(\sigma_{ac}) = \sum_{\vec{\sigma}_a \setminus \sigma_{ac}} b_a(\vec{\sigma}_a)$$

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Bethe Free Energy: variational approach
(Yedidia,Freeman,Weiss '01 - inspired by Bethe '35, Peierls '36)

$$F = \underbrace{-\sum_a \sum_{\vec\sigma_a} b_a(\vec\sigma_a)\ln f_a(\vec\sigma_a)}_{\text{self-energy}} + \underbrace{\sum_a \sum_{\vec\sigma_a} b_a(\vec\sigma_a)\ln b_a(\vec\sigma_a) - \sum_{(a,c)} b_{ac}(\sigma_{ac})\ln b_{ac}(\sigma_{ac})}_{\text{configurational entropy}}$$

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$\Rightarrow$ Belief-Propagation Equations: $\left.\frac{\delta F}{\delta b}\right|_{\text{constr.}} = 0$

Belief-Propagation as heuristics: iterative $\Rightarrow$ Gallager '61; MacKay '98

- Exact on a tree
- Trading optimality for reduction in complexity: $\sim 2^L \to \sim L$
- (BP = solving equations on the graph) $\neq$ (Message Passing = iterative BP)
- Convergence of MP to minimum of Bethe Free energy can be enforced
- $Z_{BP} \gtrless Z_{exact}$
- Relation to Linear Programming

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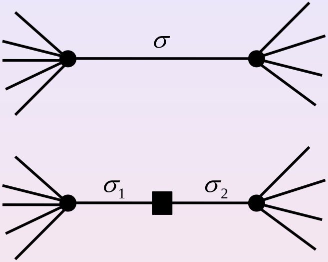
Outline

BP does not account for Loops

Questions:

- Is BP just a heuristic in a loopy case?
- Why does it (often) work so well?
- Does exact inference allow an expression in terms of BP?
- Can one correct BP systematically?

Local Gauge Freedom [preamble]



$\forall \sigma_1, \sigma_2 = \pm 1, \quad \forall \eta_1, \eta_2 \in \mathcal{C}$

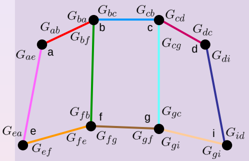
$$\delta(\sigma_1,\sigma_2) = \frac{1+\sigma_1\sigma_2}{2}$$

$$= \frac{\exp\left(\eta_1\sigma_1+\eta_2\sigma_2\right)}{2\cosh(\eta_1+\eta_2)}\left(1+\sigma_1\sigma_2\exp\left(-(\eta_1+\eta_2)(\sigma_1+\sigma_2)\right)\right)$$

Gauge Transformations

Chertkov, Chernyak '06

Local Gauge, G, Transformations



$$Z = \sum_{\vec\sigma} \prod_a f_a(\vec\sigma_a), \;\; \vec\sigma_a = (\sigma_{ab}, \sigma_{ac}, \cdots)$$

$$\sigma_{ab} = \sigma_{ba} = \pm 1$$

$$f_a(\vec\sigma_a = (\sigma_{ab}, \cdots)) \to \sum_{\sigma'_{ab}} G_{ab}(\sigma_{ab}, \sigma'_{ab})\, f_a(\sigma'_{ab}, \cdots)$$

$$\sum_{\sigma_{ab}} G_{ab}(\sigma_{ab}, \sigma') G_{ba}(\sigma_{ab}, \sigma'') = \delta(\sigma', \sigma'')$$

The partition function is invariant under any G-gauge!

$$Z = \sum_{\vec\sigma} \prod_a f_a(\vec\sigma_a) = \sum_{\vec\sigma} \prod_a \left(\sum_{\vec\sigma'_a} f_a(\vec\sigma'_a) \prod_{b \in a} G_{ab}(\sigma_{ab}, \sigma'_{ab}) \right)$$

Belief Propagation as a Gauge Fixing

Chertkov, Chernyak '06

$$Z=\sum_{\vec\sigma}\prod_a f_a(\vec\sigma_a)=\sum_{\boldsymbol{\sigma}}\prod_a\left(\sum_{\vec\sigma'_a} f_a(\vec\sigma'_a)\prod_{b\in a}G_{ab}(\sigma_{ab},\sigma'_{ab})\right)$$

$$Z=\underbrace{Z_0(G)}_{\substack{\text{ground state}\\ \vec\sigma=+\vec 1}}+\underbrace{\sum_{\text{all possible colorings of the graph}} Z_c(G)}_{\vec\sigma\neq+\vec 1,\quad \text{excited states}}$$

Belief Propagation Gauge

$\underline{\forall a\ \&\ \forall b\in a:}$

$$\sum_{\vec\sigma'_a} f_a(\vec\sigma')G^{(bp)}_{ab}(\sigma_{ab}=-1,\sigma'_{ab})\prod_{c\in a}^{c\neq b}G^{(bp)}_{ac}(+1,\sigma'_{ac})=0$$

No loose BLUE=colored edges at any vertex of the graph!

Belief Propagation as a Gauge Fixing (II)

$\forall a$ & $\forall b \in a$:

$$\begin{cases} \sum_{\vec{\sigma'}_a} f_a(\vec{\sigma}')G^{(bp)}_{ab}(-1,\sigma'_{ab})\prod_{c\in a}^{c\neq b} G^{(bp)}_{ac}(+1,\sigma'_{ac})=0 \\ \sum_{\sigma_{ab}} G_{ab}(\sigma_{ab},\sigma')G_{ba}(\sigma_{ab},\sigma'')=\delta(\sigma',\sigma'') \end{cases} \Rightarrow \begin{cases} G^{(bp)}_{ba}(+1,\sigma'_{ab})=\rho_a^{-1}\overbrace{\sum_{\vec{\sigma'}_a\setminus\sigma'_{ab}} f_a(\vec{\sigma}')\prod_{c\in a}^{c\neq b} G^{(bp)}_{ac}(+1,\sigma'_{ac})}^{sum-product} \\ \rho_a=\sum_{\vec{\sigma'}_a} f_a(\vec{\sigma}')\prod_{c\in a} G^{(bp)}_{ac}(+1,\sigma'_{ac}) \end{cases}$$

Belief Propagation in terms of Messages

$$G^{(bp)}_{ab}(+1,\sigma)=\frac{\exp(\sigma\eta_{ab})}{\sqrt{2\cosh(\eta_{ab}+\eta_{ba})}},\quad G^{(bp)}_{ab}(-1,\sigma)=\sigma\frac{\exp(-\sigma\eta_{ba})}{\sqrt{2\cosh(\eta_{ab}+\eta_{ba})}}\Longrightarrow$$

$$\sum_{\vec{\sigma}_a\setminus\sigma_{ab}} f_a(\vec{\sigma}_a)\exp\left(\sum_{c\in a}\sigma_{ac}\eta_{ac}\right)\left(\sigma_{ab}-\tanh\left(\eta_{ab}+\eta_{ba}\right)\right)=0$$

$$b_a(\vec{\sigma}_a)=\frac{f_a(\vec{\sigma}_a)\exp\left(\sum_{b\in a}\sigma_{ab}\eta_{ab}\right)}{\sum_{\vec{\sigma}_a} f_a(\vec{\sigma}_a)\exp\left(\sum_{b\in a}\sigma_{ab}\eta_{ab}\right)},\quad b_{ab}(\sigma)=\frac{\exp(\sigma(\eta_{ab}+\eta_{ba}))}{\sum_\sigma \exp(\sigma(\eta_{ab}+\eta_{ba}))}$$

Variational Principe and Gauge Fixing

$$Z = \underbrace{Z_0(G)}_{\vec{\sigma}=+\vec{1}} + \sum_{\vec{\sigma}\neq+\vec{1}} Z_c(G), \quad Z_0(G) \Rightarrow \underbrace{Z_0(\epsilon), \quad \epsilon_{ab}(\sigma_{ab}) = G_{ab}(+1,\sigma_{ab})}_{\text{depends only on the ground state gauges}}$$

Variational formulation of Belief Propagation

$$\left.\frac{\partial Z_0(\epsilon)}{\partial \epsilon_{ab}(\sigma_{ab})}\right|^{(bp)} = 0 \quad \Leftrightarrow \quad \text{Belief Propagation Equations}$$

$\mathcal{F}_0(\epsilon) = -\ln Z_0(\epsilon)$ – Bethe Free Energy at the BP-extremum

General Remarks on Gauge Fixing

- Related to the Re-parametrization Framework of Wainwright, Jaakkola and Willsky '03
- Generalizable to q-ary alphabet Chernyak, Chertkov '07
- ... suggests Loop Series for the Partition Function $\Rightarrow$

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Loop Series:

Chertkov,Chernyak '06

Exact (!!) expression in terms of BP

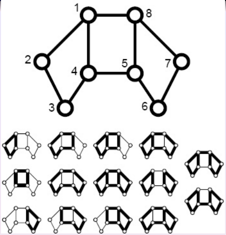
$$Z=\sum_{\vec{\sigma}_\sigma}\prod_a f_a(\vec{\sigma}_a)=Z_0\left(1+\sum_C r(C)\right)$$

$$r(C)=\frac{\prod_{a\in C}\mu_a}{\prod_{(ab)\in C}(1-m_{ab}^2)}=\prod_{a\in C}\tilde{\mu}_a$$

$C\in$ Generalized Loops = Loops without loose ends

$$m_{ab}=\sum_{\vec{\sigma}_a} b_a^{(bp)}(\vec{\sigma}_a)\sigma_{ab}$$

$$\mu_a=\sum_{\vec{\sigma}_a} b_a^{(bp)}(\vec{\sigma}_a)\prod_{b\in a,C}(\sigma_{ab}-m_{ab})$$



- The Loop Series is finite
- All terms in the series are calculated within BP
- BP is exact on a tree
- Different series for different BP-extrema
- Other (than BP) gauges lead to other representations
- Generalizes high-temperature expansion

Loop Calculus & Related

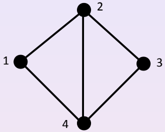
- Improving Decoding of LDPC codes in the Error-floor regime [MC, Chernyak '06]
- Loop Corrections for Approximate Inference on Factor Graphs [Gòmez, Mooij, Kappen '07]
- Low bound on Partition Function (Loop Series is positive): ordered state of FRFI [Sudderth, Wainwright, Willsky '08]
- Loop Calculus for Planar Problems [MC, Chernyak, Teodorescu '08]
- Counting and Learning in Matching
  Tracking Particles in Fluid Experiment [MC, Kroc, Vergassola '08]
- Counting Independent Sets [Chandrasekar, Gamarnik, Shah '08 and Shin, MC, Shah '08]
- Determinants, Fermions and Loops $\Rightarrow$

Outline

Matrix and Monomer-Dimer Model

H: $N \times N$ matrix
$\mathcal{G}(H)$: nodes - $a \in \mathcal{G}_0$
undirected edges - $\{a,b\} \in \mathcal{G}_1$
directed edges - $(a,b) \in \mathcal{G}_1$

$$H = \begin{pmatrix} H_{11} & H_{12} & 0 & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ 0 & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{pmatrix}$$



Monomer-Dimer Model (on the same graph)

$$Z_{MD}(H) \equiv \sum_{\pi} \left(\prod_{a \in \mathcal{G}_0} w_a^{\pi_a} \right) \left(\prod_{\{a,b\} \in \mathcal{G}_1} w_{ab}^{\pi_{ab}} \right) \left(\prod_{a \in \mathcal{G}_0} \delta\left(\pi_a + \sum_{b \sim a} \pi_{ab}, 1 \right) \right)$$

$\boldsymbol{\pi} \equiv \boldsymbol{\pi}_v \cup \boldsymbol{\pi}_e$, $\boldsymbol{\pi}_v \equiv (\pi_a = 0,1; a \in \mathcal{G}_0)$, and $\boldsymbol{\pi}_e \equiv (\pi_{ab} = 0,1; \{a,b\} \in \mathcal{G}_1)$
$w_{ab} \equiv -H_{ab}H_{ba}$ and $w_a \equiv H_{aa}$

- Do $\det(H)$ and $Z_{MD}(H)$ have anything in common?
- Any relation to Loops and Belief Propagation?

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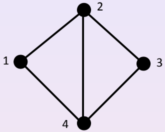

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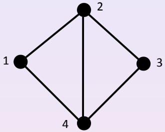
directed edges - $(a, b) \in \mathcal{G}_1$

$$H = \begin{pmatrix} H_{11} & H_{12} & 0 & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ 0 & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{pmatrix}$$



Monomer-Dimer Model (on the same graph)

$$Z_{MD}(H) \equiv \sum_{\boldsymbol{\pi}} \left(\prod_{a \in \mathcal{G}_0} w_a^{\pi_a} \right) \left(\prod_{\{a,b\} \in \mathcal{G}_1} w_{ab}^{\pi_{ab}} \right) \left(\prod_{a \in \mathcal{G}_0} \delta\left(\pi_a + \sum_{b \sim a} \pi_{ab}, 1 \right) \right)$$

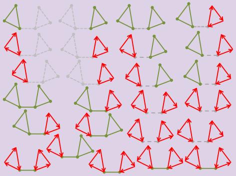
$$\boldsymbol{\pi} \equiv \boldsymbol{\pi}_v \cup \boldsymbol{\pi}_e,\ \boldsymbol{\pi}_v \equiv (\pi_a = 0, 1; a \in \mathcal{G}_0), \text{ and } \boldsymbol{\pi}_e \equiv (\pi_{ab} = 0, 1; \{a, b\} \in \mathcal{G}_1)$$
$$w_{ab} \equiv -H_{ab} H_{ba} \text{ and } w_a \equiv H_{aa}$$

- Do $\det(H)$ and $Z_{MD}(H)$ have anything in common?
- Any relation to Loops and Belief Propagation?

Highlights of Results [Chernyak, Chertkov '08]

Loop Series for Determinant [BP gauge]

$$\det(H) = Z_{\rm BP}\left(1 + \sum_{C\in GL(\mathcal{G})}\ \sum_{C'\in ODC(C)} r(C,C')\right)$$

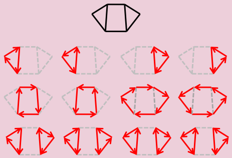


$GL(\mathcal{G})$ - set of generalized loops on $\mathcal{G}$
$ODC(C)$ - set of oriented disjoint cycles on C
Z_{BP}, $r(C,C')$ are expressed via BP solution

MD as Series of Determinants [non-BP gauge]

$$Z_{\rm MD} = \sum_{C\in ODC(\mathcal{G})} \bar{r}(C)$$

$$\bar{r}(C) = \det(H|_{\mathcal{G}\setminus C}) \prod_{(a,b)\in C} H_{ab}$$



Berezin/Grassman (super) Calculus

Berezin/Grassman (anti-commuting) Variables, Integrals

$$\{\bar{\boldsymbol{\theta}}, \boldsymbol{\theta}\} = \{\bar{\theta}_a, \theta_a\}_{a \in \mathcal{G}_0}, \quad a = 1, \cdots, N$$

$$\forall a, b \in \mathcal{G}_0 : \quad \theta_a \theta_b = -\theta_b \theta_a, \quad \bar{\theta}_a \theta_b = -\theta_b \bar{\theta}_a, \quad \bar{\theta}_a \bar{\theta}_b = -\bar{\theta}_b \bar{\theta}_a$$

<u>Berezin measure :</u> $\mathcal{D}\boldsymbol{\theta}\mathcal{D}\bar{\boldsymbol{\theta}} = \prod_{a \in \mathcal{G}_0} d\theta_a d\bar{\theta}_a$

$$\forall a, b \in \mathcal{G}_0 : \quad \int \theta_a d\theta_a = \int \bar{\theta}_a d\bar{\theta}_a = 1, \quad \int d\theta_a = \int d\bar{\theta}_a = 0$$

... and Determinants

via Grassmans on vertexes

$$\det H = \underbrace{\int \mathcal{D}\boldsymbol{\theta}\mathcal{D}\bar{\boldsymbol{\theta}} e^{S_0(\bar{\boldsymbol{\theta}},\boldsymbol{\theta})}}_{\text{Gaussian (fermi) Graphical Model}}, \quad S_0(\bar{\boldsymbol{\theta}},\boldsymbol{\theta}) = \sum_{a\in\mathcal{G}_0} H_{aa}\bar{\theta}_a\theta_a + \sum_{a,b\in\mathcal{G}_0}^{a\sim b} H_{ab}\bar{\theta}_a\theta_b$$

$$\{\bar{\boldsymbol{\theta}},\boldsymbol{\theta}\} = \{\bar{\theta}_a,\theta_a\}_{a\in\mathcal{G}_0} \text{ with } a = 1,\dots,N$$

via Grassmans on edges

$$\det H = \left(\prod_{\{a,b\}\in\mathcal{G}_1}(H_{ab}H_{ba})\right)\left(\prod_{a\in\mathcal{G}_0} H_{aa}\right)\int \mathcal{D}\boldsymbol{\chi}\mathcal{D}\bar{\boldsymbol{\chi}} \prod_{a\in\mathcal{G}_0} f_a(\bar{\boldsymbol{\chi}}_a,\boldsymbol{\chi}_a)\prod_{\alpha\in\mathcal{G}_1} g_\alpha(\bar{\boldsymbol{\chi}}_\alpha,\boldsymbol{\chi}_\alpha)$$

$$g_\alpha(\bar{\boldsymbol{\chi}}_\alpha,\boldsymbol{\chi}_\alpha) = \exp\left((H_{ab})^{-1}\bar{\chi}_{ab}\chi_{ba} + (H_{ba})^{-1}\bar{\chi}_{ba}\chi_{ab}\right)$$

$$\alpha = \{a,b\}, \quad \boldsymbol{\chi}_\alpha = \{\chi_{ab},\chi_{ba}\}, \quad \bar{\boldsymbol{\chi}}_\alpha = \{\bar{\chi}_{ab},\bar{\chi}_{ba}\}$$

$$f_a(\bar{\boldsymbol{\chi}}_a,\boldsymbol{\chi}_a) = \exp\left(-(H_{aa})^{-1}\sum_{b\in\mathcal{G}_0}^{b\sim a}\bar{\chi}_{ba}\sum_{b'\in\mathcal{G}_0}^{b'\sim a}\chi_{b'a}\right)$$

$$\boldsymbol{\chi}_a = \{\chi_{ba}\}_{b\in\mathcal{G}_0, b\sim a}, \quad \bar{\boldsymbol{\chi}}_a = \{\bar{\chi}_{ba}\}_{b\in\mathcal{G}_0, b\sim a}$$

Gauge Transformation

$$\det H = \left(\prod_{(a,b)\in\mathcal{G}_1}(H_{ab}H_{ba})\right)\left(\prod_{a\in\mathcal{G}_0}H_{aa}\right)\int \mathcal{D}\chi\mathcal{D}\bar{\chi}\prod_{a\in\mathcal{G}_0}f_a(\bar{\chi}_a,\chi_a)\prod_{\alpha\in\mathcal{G}_1}g_\alpha(\bar{\chi}_\alpha,\chi_\alpha)$$

$$g_\alpha(\bar{\chi}_\alpha,\chi_\alpha) = \exp\left(\frac{\bar{\chi}_{ab}\chi_{ba}}{H_{ab}}+\frac{\bar{\chi}_{ba}\chi_{ab}}{H_{ba}}\right)$$

γ is the local Gauge variable $\quad \forall\alpha=\{a,b\}$:

$$g_\alpha{}^{(\gamma)} = \underbrace{\frac{e^{\gamma_{ab}\bar{\chi}_{ab}\chi_{ab}}e^{\gamma_{ba}\bar{\chi}_{ba}\chi_{ba}} - H_{ab}H_{ba}\gamma_{ab}\gamma_{ba}e^{\frac{\bar{\chi}_{ab}\chi_{ab}}{H_{ab}H_{ba}\gamma_{ba}}}e^{\frac{\bar{\chi}_{ab}\chi_{ab}}{H_{ab}H_{ba}\gamma_{ba}}}}{1-H_{ab}H_{ba}\gamma_{ab}\gamma_{ba}}}_{\text{ground state}=g_\alpha^{(0;\gamma)}} + \underbrace{\frac{\bar{\chi}_{ab}\chi_{ba}}{H_{ab}}+\frac{\bar{\chi}_{ba}\chi_{ab}}{H_{ba}}}_{\text{excited state}=g_\alpha^{(1;\gamma)}}$$

Determinant as a series – valid $\forall\gamma$!!

$\det H = \sum_{\sigma} Z_{\sigma}^{(\gamma)}, \quad \sigma = (\sigma_{ab}=0,1|\{a,b\}\in\mathcal{G}_1), \quad Z(\sigma)^{(\gamma)} \equiv$

$$\left(\prod_{(a,b)\in\mathcal{G}_1}(H_{ab}H_{ba})\right)\left(\prod_{a\in\mathcal{G}_0}H_{aa}\right)\int \mathcal{D}\chi\mathcal{D}\bar{\chi}\prod_{a\in\mathcal{G}_0}f_a(\bar{\chi}_a,\chi_a)\prod_{\alpha\in\mathcal{G}_1}g_\alpha^{(\sigma_{ab};\gamma)}(\bar{\chi}_\alpha,\chi_\alpha)$$

Belief Propagation (for fermi and bose)

$$\det H = \left(\prod_{(a,b)\in\mathcal{G}_1}(H_{ab}H_{ba})\right)\left(\prod_{a\in\mathcal{G}_0}H_{aa}\right)\int \mathcal{D}\boldsymbol{\chi}\mathcal{D}\bar{\boldsymbol{\chi}}\prod_{a\in\mathcal{G}_0}f_a(\bar{\boldsymbol{\chi}}_a,\boldsymbol{\chi}_a)\prod_{\alpha\in\mathcal{G}_1}g_\alpha(\bar{\boldsymbol{\chi}}_\alpha,\boldsymbol{\chi}_\alpha)$$

BP-fermi as γ-Gauge fixing

$$(a,b)\in\mathcal{G}_1:\quad H_{ab}H_{ba}\gamma_{ba}^{(bp)} = H_{bb}-\sum_{a'\sim b}^{a'\neq a}(\gamma_{a'b}^{(bp)})^{-1}$$

- No loose-end constraints: $\forall a\in\mathcal{G}_0$ and $\{a,c\}\in\mathcal{G}_1$:
  $$\int d\boldsymbol{\chi}_a d\bar{\boldsymbol{\chi}}_a f_a(\bar{\boldsymbol{\chi}}_a,\boldsymbol{\chi}_a)e^{\frac{\bar{\chi}_{ca}\chi_{ca}}{H_{ac}H_{ca}\gamma_{ac}}}\prod_{b\sim a}^{b\neq c}e^{\gamma_{ba}\bar{\chi}_{ba}\chi_{ba}}\Big|_{\gamma^{(bp)}}=0$$
- Variational constraints:
  $\forall a\in\mathcal{G}_0$ and $\{a,c\}\in\mathcal{G}_1$: $\quad \frac{\partial Z_0}{\partial\gamma_{ca}}\Big|_{\gamma^{bp}}=0,\quad Z_{BP;fermi}=Z_0(\gamma^{bp})$

Belief Propagation (for fermi and bose)

$$\det H = \left(\prod_{(a,b)\in\mathcal{G}_1}(H_{ab}H_{ba})\right)\left(\prod_{a\in\mathcal{G}_0}H_{aa}\right)\int \mathcal{D}\chi\mathcal{D}\bar\chi \prod_{a\in\mathcal{G}_0} f_a(\bar\chi_a,\chi_a)\prod_{\alpha\in\mathcal{G}_1} g_\alpha(\bar\chi_\alpha,\chi_\alpha)$$

BP-fermi as γ-Gauge fixing

$$(a,b)\in\mathcal{G}_1:\quad H_{ab}H_{ba}\gamma_{ba}^{(bp)} = H_{bb} - \sum_{a'\sim b}^{a'\neq a}(\gamma_{a'b}^{(bp)})^{-1}$$

- No loose-end constraints: $\forall a\in\mathcal{G}_0$ and $\{a,c\}\in\mathcal{G}_1$:
  $$\int d\chi_a d\bar\chi_a f_a(\bar\chi_a,\chi_a) e^{\frac{\bar\chi_{ca}\chi_{ca}}{H_{ac}H_{ca}\gamma_{ac}}}\prod_{b\sim a}^{b\neq c} e^{\gamma_{ba}\bar\chi_{ba}\chi_{ba}}\Big|_{\gamma^{(bp)}} = 0$$
- Variational constraints:
  $\forall a\in\mathcal{G}_0$ and $\{a,c\}\in\mathcal{G}_1$: $\quad \frac{\partial Z_0}{\partial\gamma_{ca}}\Big|_{\gamma^{bp}} = 0,\quad Z_{BP;fermi} = Z_0(\gamma^{bp})$

Gaussian (bose) Graphical Model [normal c-number integrations]

$$(\det H)^{-1} = \int\prod_a\left(\frac{d\bar\psi_a d\psi_a}{2\pi}\right)\exp\left(-\frac{1}{2}\sum_{a\in\mathcal{G}_0}H_{aa}\bar\psi_a\psi_a + \sum_{a,b\in\mathcal{G}_0}H_{ab}\bar\psi_a\psi_b\right)$$

- BP for "bose" is equivalent to BP for "fermi": $Z_{BP;fermi}Z_{BP;bose} = 1$

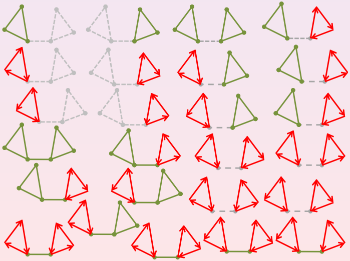
Loop Series for Determinant

$$\det(H) = Z_{BP;fermi}\left(1 + \sum_{C\in GL(\mathcal{G})}\ \sum_{C'\in ODC(C)} r(C,C')\right)$$

$GL(\mathcal{G})$ - set of generalized loops on $\mathcal{G}$

$ODC(C)$ - set of oriented disjoint cycles on C

Z_{BP}, $r(C,C')$ are expressed via BP solution



Details for BP-fermi Loop series

Gauge Theory

$\int det(H(\sigma))d\sigma =$ Gauge Theory

$=$ Monomer-Dimer

$=$ Disjoint Cycle Series

Inversion of Det-Series
in a simple (non-BP) gauge

$$\forall (a,b) \in \mathcal{G}_1: \quad H_{ab}(\boldsymbol{\sigma}) = \sigma_{ab} H_{ab}; \quad \forall a \in \mathcal{G}_0: \quad H_{aa}(\boldsymbol{\sigma}) = H_{aa}$$

$$Z = 2^{-|\mathcal{G}_1|} \sum_{\boldsymbol{\sigma} \in \mathcal{G}_1} \det(H(\boldsymbol{\sigma})) \equiv \int_{\mathcal{G}_1} \mathcal{D}\boldsymbol{\sigma} \det(H(\boldsymbol{\sigma}))$$

From Gauge Theory to Monomer-Dimer Model

From Graphical Gauge Model

$$Z = \int_{\mathcal{G}_1} \mathcal{D}\boldsymbol{\sigma}\, \det(H(\boldsymbol{\sigma})) = \int_{\mathcal{G}_1} \mathcal{D}\boldsymbol{\sigma} \int \mathcal{D}\boldsymbol{\theta}\mathcal{D}\bar{\boldsymbol{\theta}} \exp\left(\mathcal{S}_0(\bar{\boldsymbol{\theta}}, \boldsymbol{\theta}; \boldsymbol{\sigma})\right)$$

$$S_0(\bar{\boldsymbol{\theta}}, \boldsymbol{\theta}; \boldsymbol{\sigma}) = \sum_{a\in\mathcal{G}_0} H_{aa}\bar{\theta}_a\theta_a + \sum_{\{a,b\}\in\mathcal{G}_1} \sigma_{ab}(H_{ab}\bar{\theta}_a\theta_b + H_{ba}\bar{\theta}_b\theta_a)$$

To Monomer-Dimer Model

$$Z = \int \mathcal{D}\boldsymbol{\theta}\mathcal{D}\bar{\boldsymbol{\theta}} \prod_{a\in\mathcal{G}_0} (1 + w_a\bar{\theta}_a\theta_a) \prod_{\{a,b\}\in\mathcal{G}_1} (1 + w_{ab}\bar{\theta}_a\theta_a\bar{\theta}_b\theta_b)$$

$$= Z_{MD} \equiv \sum_{\boldsymbol{\pi}} \left(\prod_{a\in\mathcal{G}_0} w_a^{\pi_a}\right)\left(\prod_{\{a,b\}\in\mathcal{G}_1} w_{ab}^{\pi_{ab}}\right)\left(\prod_{a\in\mathcal{G}_0} \delta\left(\pi_a + \sum_{b\sim a}\pi_{ab}, 1\right)\right)$$

$$\boldsymbol{\pi} \equiv \boldsymbol{\pi}_v \cup \boldsymbol{\pi}_e, \quad \boldsymbol{\pi}_v \equiv (\pi_a = 0,1; a \in \mathcal{G}_0), \quad \boldsymbol{\pi}_e \equiv (\pi_{ab} = 0,1; \{a,b\} \in \mathcal{G}_1)$$

$$\forall (a,b) \in \mathcal{G}_1 : \quad w_{ab} = -H_{ab}H_{ba}; \quad \forall a \in \mathcal{G}_0 : \quad w_a = H_{aa}$$

From Gauge Theory to Determinants

From Graphical Gauge Model

$$
\begin{aligned}
Z &= \int_{\mathcal{G}_1} \mathcal{D}\boldsymbol{\sigma} \int \mathcal{D}\boldsymbol{\theta}\mathcal{D}\bar{\boldsymbol{\theta}} \exp\left(\mathcal{S}_0(\bar{\boldsymbol{\theta}}, \boldsymbol{\theta}; \boldsymbol{\sigma})\right) \\
&= \int \mathcal{D}\boldsymbol{\theta}\mathcal{D}\bar{\boldsymbol{\theta}} \prod_{a\in\mathcal{G}_0} e^{w_a\bar{\theta}_a\theta_a} \prod_{\{a,b\}\in\mathcal{G}_1} \left(e^{H_{ab}\bar{\theta}_a\theta_b + H_{ba}\bar{\theta}_b\theta_a} - [H_{ab}\bar{\theta}_a\theta_b + H_{ba}\bar{\theta}_b\theta_a]\right)
\end{aligned}
$$

To Determinants

$$
Z = \sum_{C\in \mathrm{ODC}(\mathcal{G})} \bar{r}(C), \quad \bar{r}(C) = \det(H|_{\mathcal{G}\setminus C}) \prod_{(a,b)\in C} H_{ab}
$$

Established expanding into sum over $[\cdots]$, integrating and collecting H-monoms

Monomer-Dimer Models and Determinants [Cycle Series]

Determinant as Series over MD on subgraphs [non-BP gauge]

$\forall(a,b)\in\mathcal{G}_1:\quad \gamma_{ab}=\pm(H_{ab})^{-1}$ [no need to solve BP equations]

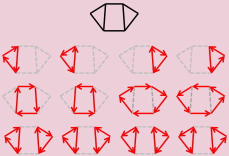
Series over "odd" states Det as Series over non-BP gauges:

$$\det(H)=\sum_{C\in ODC(\mathcal{G})} r(C),\quad r(C)=(-1)^{|C|}\prod_{(a,b)\in C} H_{ab} Z_{\rm MD}(\mathcal{G}\setminus C)$$

MD as Series of Determinants [non-BP gauge]

$$Z_{\rm MD}=\sum_{C\in ODC(\mathcal{G})}\bar{r}(C)$$

$$\bar{r}(C)=\det(H|_{\mathcal{G}\setminus C})\prod_{(a,b)\in C} H_{ab}$$



Future Challenges (Loop Calculus +)

- Disorder Average, Relation to Cavity, Replica Calculations
- Efficient Approximate Algorithms
- Relation & Complementarity to MCMC, Mixing Time
- Phase Transitions in Glasses (Physics) and SAT (CS, IT)
- Dynamical Graphical Models (non-equilibrium)

Fermion-Gauge Approach

- Extension to Surface Graphs (embedded into surfaces with nonzero genus)
- Easy planar and surface models
- From loops,cycles to walks/paths (e.g. in relation with Walk-Sum and ζ-functions)
- Super-Symmetric (fermions and bosons) Graphical Models

My papers on ... (A)

Loop Calculus, Series, Tower

- M. Chertkov, *Exactness of Belief Propagation for Some Graphical Models with Loops*, JSTAT to appear, `arxiv.org/abs/0801.0341`
- V. Chernyak and M. Chertkov, "Loop Calculus and Belief Propagation for q-ary Alphabet: Loop Tower," *Proceedings of IEEE ISIT 2007*, June 2007, Nice, `arXiv:cs.IT/0701086`.
- M. Chertkov and V. Chernyak, "Loop series for discrete statistical models on graphs," JSTAT/2006/P06009, `arXiv:cond-mat/0603189`.
- M. Chertkov and V. Chernyak, "Loop Calculus in Statistical Physics and Information Science," *Phys. Rev. E*, **73**, 065102(R) (2006), `arXiv:cond-mat/0601487`.

Loop Calculus for Graphical Codes

- M. Chertkov, "Reducing the Error Floor", invited talk at the *Information Theory Workshop '07 on "Frontiers in Coding"*, September 2-6, 2007.
- M. Chertkov and V. Chernyak, "Loop Calculus Helps to Improve Belief Propagation and Linear Programming Decodings of Low-Density-Parity-Check Codes," invited talk at 44^{th} *Allerton Conference*, September 27-29, 2006, Allerton, IL, `arXiv:cs.IT/0609154`.

All papers are available at http://cnls.lanl.gov/∼chertkov/pub.htm

My papers on ... (B)

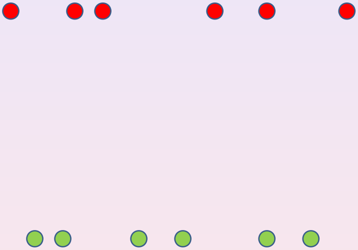
Particle Tracking, BP and Loops

- M. Chertkov, L. Kroc, M. Vergassola, "Belief Propagation and Beyond for Particle Tracking", arXiv,org/abs/0806.1199.

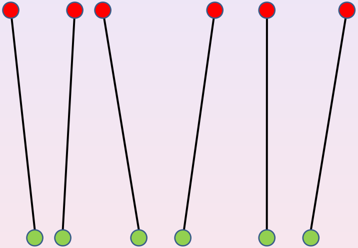
Fermions, Loops & Gauges, Planar & Surface Graphs

- V. Chernyak, M. Chertkov, "Fermions and Loops on Graphs. II. Monomer-Dimer Model as Series of Determinants", submitted to JSTAT, arXiv.org/abs/0809.3481.
- V. Chernyak, M. Chertkov, "Fermions and Loops on Graphs. I. Loop Calculus for Determinant", submitted to JSTAT, arXiv.org/abs/0809.3479.
- M. Chertkov, V. Chernyak, R. Teodorescu, "Belief Propagation and Loop Series on Planar Graphs", JSTAT/2008/P05003, arxiv.org/abs/0802.3950.

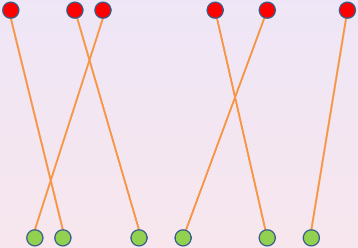
Particle Tracking in Fluid Mechanics



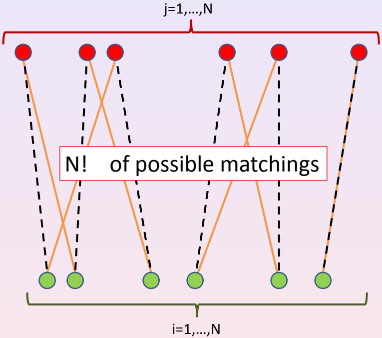
Particle Tracking in Fluid Mechanics



Particle Tracking in Fluid Mechanics



Particle Tracking in Fluid Mechanics



$d = 1$ advection and diffusion [example]

$$p_i^j = p(y^j|x_i) = \underbrace{\frac{\exp(-(y^j - e^S x_i)^2/(\kappa(e^{2S} - 1))}{\sqrt{\pi\kappa(e^{2S} - 1))}}}_{\text{probability of the i-j matching}}$$

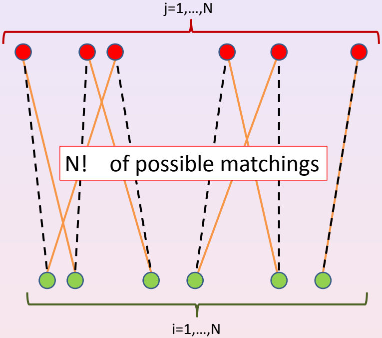
$$p\left(\hat{\sigma}|\vec{x};\vec{y}\right) = Z^{-1}\prod_{(i,j)}\left(p_i^j\right)^{\sigma_i^j} F(\hat{\sigma})$$

$$F(\hat{\sigma}) \equiv \prod_i \delta\left(\sum_j \sigma_i^j, 1\right)\prod_j \delta\left(\sum_i \sigma_i^j, 1\right)$$

$$Z(\kappa, S) \equiv \sum_{\hat{\sigma}\equiv(0,1)^{N^2}}\prod_{(i,j)}\left(p_i^j\right)^{\sigma_i^j} F(\hat{\sigma})$$

- Matching: $\text{argmax}_{\hat{\sigma}}\, p\left(\hat{\sigma}|\vec{x};\vec{y}\right)$ – [EASY, e.g. Bayati, Shah and Sharma '06]
- "Learning": $\text{argmax}_{\kappa,S} Z(\kappa, S)$ – [DIFFICULT]

"Learning" the environment [MC, Kroc, Vergassola '08]



$d=1$ advection and diffusion [example]

$$p_i^j = p(y^j|x_i) = \underbrace{\frac{\exp(-(y^j - e^S x_i)^2/(\kappa(e^{2S}-1))}{\sqrt{\pi\kappa(e^{2S}-1))}}}_{\text{probability of the i-j matching}}$$

$$p\left(\hat{\sigma}|\vec{x};\vec{y}\right) = Z^{-1}\prod_{(i,j)}\left(p_i^j\right)^{\sigma_i^j} F(\hat{\sigma})$$

$$F(\hat{\sigma}) \equiv \prod_i \delta\left(\sum_j \sigma_i^j, 1\right)\prod_j \delta\left(\sum_i \sigma_i^j, 1\right)$$

$$Z(\kappa, S) \equiv \sum_{\hat{\sigma}\equiv(0,1)^{N^2}}\prod_{(i,j)}\left(p_i^j\right)^{\sigma_i^j} F(\hat{\sigma})$$

$\text{argmax}_{\kappa,S} Z(\kappa, S)$

BP = bare approximation (heuristics)+ Loop Series

Loop Calculus for Matching

$$Z = Z_{\mathcal{BP}} * z, \quad z \equiv 1 + \sum_C r_C, \quad r_C = \left(\prod_{i\in C}(1-q_i)\right)\left(\prod_{j\in C}(1-q^j)\right)\prod_{(i,j)\in C}\frac{\beta_i^j}{1-\beta_i^j}$$

- $\forall C: \quad |r_C| \le 1$

Mixed Derivative

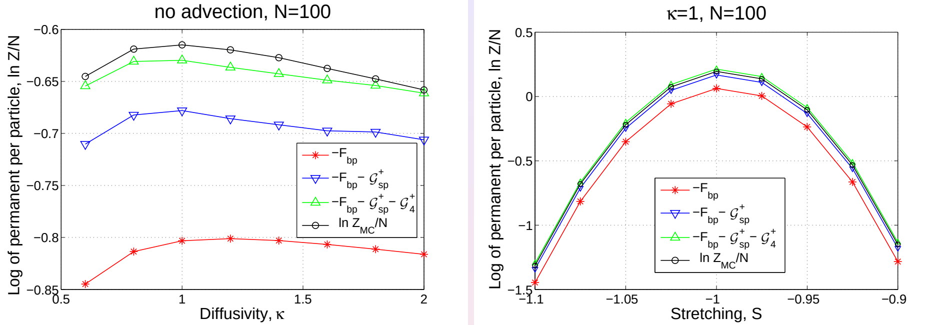
$$z = \left.\frac{\partial^{2N}\mathcal{Z}(\rho_1,\cdots,\rho_N,\rho^1,\cdots,\rho^N)}{\partial\rho_1\cdots\partial\rho_N\partial\rho^1\cdots\partial\rho^N}\right|_{\rho_1=\cdots=\rho_N=\rho^1\cdots=\rho^N=0}$$

$$\mathcal{Z}(\vec{\rho}) \equiv \exp\left(\sum_i \rho_i + \sum_j \rho^j\right)\prod_{(i,j)}\left(1 + \frac{\beta_i^j}{(1-\beta_i^j)}\exp\left(-\rho_i - \rho^j\right)\right)$$

Cauchy Integral

$$z = \oint_{\Gamma_\rho} \exp\left(-\mathcal{G}(\vec{\rho})\right)\frac{\prod_i d\rho_i \prod_j d\rho^j}{(2\pi i)^{2N}}, \quad \mathcal{G}(\vec{\rho}) \equiv \sum_i 2\ln\rho_i + \sum_j 2\ln\rho^j - \ln\mathcal{Z}$$

"Learning": Cauchy-based heuristics vs FPRAS



BP, Loop Series = Mixed Derivatives = Cauchy Integral
$\approx$ Saddle Point (heuristics) + Determinant + 4th order corr.

Fully Polynomial Randomized Algorithmic Scheme (Monte Carlo)

Shopping List for Matching & Learning +

- Analytical control of the saddle approximation quality
- Matching-Reconstruction as a Phase Transition
- $d = 2, 3$, realistic flows (multi scale)
- Multiple snapshots (a movie)
- ... technology for other reconstr. problems (e.g. phylogeny)

Loop Calculus +

Details for BP-fermi Loop Series

$$Z_{BP;fermi} = \prod_{\{a,b\}\in\mathcal{G}_1} \frac{H_{ab}H_{ba}\gamma_{ab}^{(bp)}\gamma_{ba}^{(bp)}}{1-H_{ab}H_{ba}\gamma_{ab}^{(bp)}\gamma_{ba}^{(bp)}} \prod_{c\in\mathcal{G}_0}\left(H_{cc}-\sum_{a'\sim c}(\gamma_{a'c}^{(bp)})^{-1}\right)$$

$$r(C,C') = (-1)^{\deg(C')} \prod_{a\in C_0} r_a(C,C') \prod_{\alpha\in C_1} r_\alpha(C,C')$$

$$a\in C_0\setminus C_0': \quad r_a(C,C') = \frac{H_{aa}-\sum_{a'\sim a}^{a'\in C_0} H_{a'a}H_{aa'}\gamma_{aa'}^{(bp)} - \sum_{a'\sim a}^{a'\in\mathcal{G}_0\setminus C_0}(\gamma_{a'a}^{(bp)})^{-1}}{H_{aa}-\sum_{a'\sim a}(\gamma_{a'a}^{(bp)})^{-1}}$$

$$a\in C_0': \quad r_a(C,C') = \frac{1}{H_{aa}-\sum_{a'\sim a}(\gamma_{a'a}^{(bp)})^{-1}},$$

$$\alpha\in C_1\setminus C_1',\ c=\partial_0\alpha,\ d=\partial_1\alpha: \quad r_\alpha(C,C') = -\frac{1}{H_{cd}H_{dc}\gamma_{cd}^{(bp)}\gamma_{dc}^{(bp)}},$$

$$\alpha\in C_1',\ c=\partial_0\alpha,\ d=\partial_1\alpha: \quad r_\alpha(C,C') = \frac{1-H_{cd}H_{dc}\gamma_{cd}^{(bp)}\gamma_{dc}^{(bp)}}{H_{cd}\gamma_{cd}^{(bp)}\gamma_{dc}^{(bp)}}$$

◂ BP-fermi Loop Series

Cycle Series for Determinant

$$\det H = \left(\prod_{(a,b)\in\mathcal{G}_1}(H_{ab}H_{ba})\right)\left(\prod_{a\in\mathcal{G}_0}H_{aa}\right)\int \mathcal{D}\boldsymbol{\chi}\mathcal{D}\bar{\boldsymbol{\chi}}\prod_{a\in\mathcal{G}_0}f_a(\bar{\boldsymbol{\chi}}_a,\boldsymbol{\chi}_a)\prod_{\alpha\in\mathcal{G}_1}g_\alpha(\bar{\boldsymbol{\chi}}_\alpha,\boldsymbol{\chi}_\alpha)$$

$$g_\alpha(\bar{\boldsymbol{\chi}}_\alpha,\boldsymbol{\chi}_\alpha)=\exp\left(\frac{\bar{\chi}_{ab}\chi_{ba}}{H_{ab}}+\frac{\bar{\chi}_{ba}\chi_{ab}}{H_{ba}}\right)$$

Simple [non-BP] choice of the γ-Gauge

$$g_\alpha = \underbrace{1-\frac{\bar{\chi}_{ab}\chi_{ab}\bar{\chi}_{ba}\chi_{ba}}{H_{ab}H_{ba}}}_{\text{even [ground] state}=g_\alpha^{(0)}}+\underbrace{\frac{\bar{\chi}_{ab}\chi_{ba}}{H_{ab}}+\frac{\bar{\chi}_{ba}\chi_{ab}}{H_{ba}}}_{\text{odd [excited] state}=g_\alpha^{(1)}}$$

Determinant as a series

$\det H=\sum_{\boldsymbol{\sigma}} Z_{\boldsymbol{\sigma}},\quad \sigma=(\sigma_{ab}=0,1|\{a,b\}\in\mathcal{G}_1),\quad Z(\sigma)\equiv$

$$\left(\prod_{(a,b)\in\mathcal{G}_1}(H_{ab}H_{ba})\right)\left(\prod_{a\in\mathcal{G}_0}H_{aa}\right)\int \mathcal{D}\boldsymbol{\chi}\mathcal{D}\bar{\boldsymbol{\chi}}\prod_{a\in\mathcal{G}_0}f_a(\bar{\boldsymbol{\chi}}_a,\boldsymbol{\chi}_a)\prod_{\alpha\in\mathcal{G}_1}g_\alpha^{(\sigma_{ab};\gamma)}(\bar{\boldsymbol{\chi}}_\alpha,\boldsymbol{\chi}_\alpha)$$

Only Oriented Disjoint Cycles survive [give non-zero contribution]

◂ Monomer-Dimer Models and Determinants