Synchronization and Kron Reduction in Power Networks

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Today's presentation is about a controversial topic

"Well known" observations:

1. power networks are coupled oscillators
2. Kuramoto oscillators synchronize for large coupling
3. graph theory quantifies coupling in a network, eg, λ_2
4. hence, power networks synchronize for large λ_2

... anyways, transient stability is solved

Misundestandings between physicists and control engineers:

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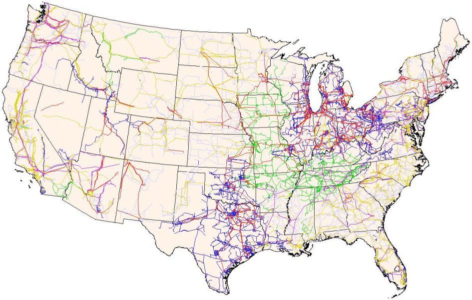
Intro: The North American Power Grid



"...the largest and most complex machine engineered by humankind."

[P. Kundur '94, V. Vittal '03, ...]

"...the greatest engineering achievement of the 20th century."

[National Academy of Engineering '10]

1. large-scale, complex, nonlinear, and rich dynamic behavior
2. 100 years old and operating at its capacity limits

⇒ recent blackouts: New England '03 + Italy '03, Brazil '09

Intro: Transient Stability in Power Networks

The New York Times

THE BLACKOUT OF 2003: Failure Reveals Creaky System, Experts Believe 8/15/2003

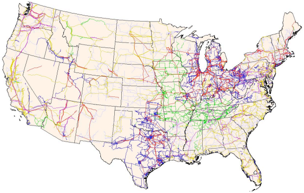
THE WHITE HOUSE WASHINGTON

Energy is one of the top three national priorities

Expected additional synergetic effects in future "**smart grid**":

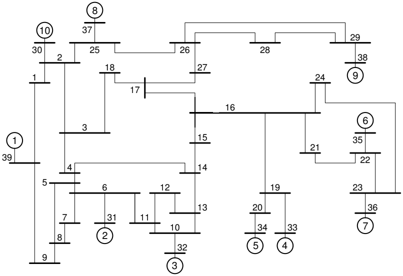
⇒ increasing complexity and renewable stochastic power sources

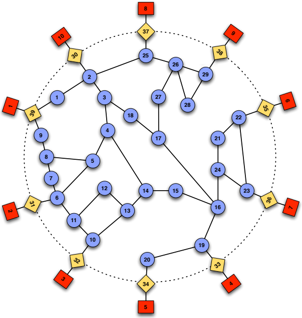
⇒ increasingly many transient disturbances to be detected and rejected



Transient Stability: Generators have to maintain synchronism in presence of large transient disturbances such as faults or loss of

- transmission lines and components,
- generation or load.





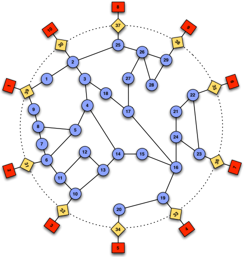
Power network topology:

1. n generators ■, each connected to a generator terminal bus ◆
2. n generators terminal buses ◆ and m load buses ● form connected graph
3. admittance matrix $\mathbf{Y}_{\mathbf{network}} \in \mathbb{C}^{(2n+m)\times(2n+m)}$ characterizes the network

Structure-preserving DAE power network model:

1. dynamic **swing equations** for generators ■:

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + P_{\text{mech.in},i} - P_{\text{electr.out},i}$$



- active output power ■ → ◇ :

$$P_{\text{electr.out},i} = \Re(V_i\,\mathbf{Y}^*_{\text{network},i,j}(V_i^* - V_j^*))$$

- $\theta_i(t)$ is measured w.r.t. a 60Hz rotating frame

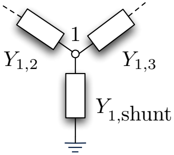
2. passive interior network ◇ & ● :

- loads are modeled as *shunt admittances*

⇒ Kirchhoff equations: $I = \mathbf{Y}_{\text{network}} V$

⇒ algebraic **power flow equations** for ◇ & ● :

$$0 = V_i \sum\nolimits_{j \in ■ \cup ◇ \cup ●} \mathbf{Y}^*_{\text{network},i,j} V_j^*$$



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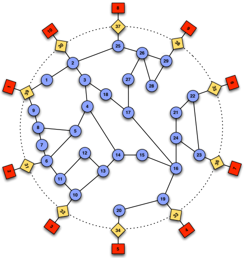

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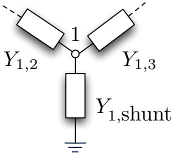
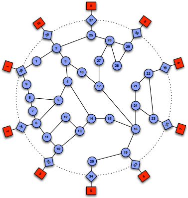


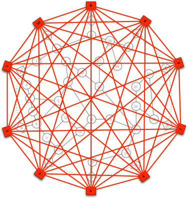
Network-Reduction to an ODE power network model:



- boundary nodes (■) & interior nodes (● & ◊)
- algebraic Kirchhoff equations $I = \mathbf{Y}_{\text{network}} V$:

$$\begin{bmatrix} I_{\text{boundary}} \\ \hline \mathbf{0} \end{bmatrix} = \left[\begin{array}{c|c} Y_{\text{boundary}} & Y_{\text{bound-int}} \\ \hline Y_{\text{bound-int}}^T & Y_{\text{interior}} \end{array}\right] \begin{bmatrix} V_{\text{boundary}} \\ \hline V_{\text{interior}} \end{bmatrix}$$

Map between boundary nodes $I_{\text{boundary}} = Y_{\text{reduced}} V_{\text{boundary}}$ given by **Schur-complement**: $Y_{\text{reduced}} = \mathbf{Y}_{\text{network}} / Y_{\text{interior}}$



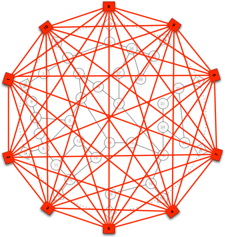
- network reduced to active nodes (generators)
- Y_{reduced} induces complete "all-to-all" coupling graph
- active output power ■ → ■ :

$$P_{\text{electr.out},i} = \Re\left(V_i \sum\nolimits_{j=1}^{n} Y^{*}_{\text{reduced},i,j} V_j^{*}\right)$$

Intro: Mathematical Model of a Power Network

Network-Reduced ODE power network model:

classic model of interconnected **swing equations** [Anderson et al. '77, M. Pai '89, P. Kundur '94, ...]:



$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij})$$

defined on "all-to-all" reduced network Y_{reduced} with

$P_{ij} = |V_i||V_j|\,|Y_{\text{reduced},i,j}| \; > 0$

is max. power transferred $i \leftrightarrow j$

$\varphi_{ij} = \arctan(\Re(Y_{\text{reduced},i,j})/\Im(Y_{\text{reduced},i,j})) \; \in [0, \pi/2)$

reflects losses $i \leftrightarrow j$

$\omega_i = P_{\text{mech.in},i} - |V_i|^2\Re(Y_{\text{reduced},i,i})$

is effective power input of i

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

Classic problem setup in transient stability analysis:

- *frequency equilibrium*: $(\dot{\theta}_i, \ddot{\theta}_i) = (0, 0)$ for all i

1. power network in stable frequency equilibrium
2. $\rightarrow$ transient network disturbance and fault clearance
3. stability analysis of a new frequency equilibrium in post-fault network

General synchronization problem:

- *synchronous equilibrium*: $|\theta_i - \theta_j|$ bounded & $\dot{\theta}_i = \dot{\theta}_j$ for all $\{i, j\}$

synchronization in presence of transient network disturbances

Intro: Transient Stability Analysis in Power Networks

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Classic analysis methods: Hamiltonian arguments

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i - \nabla_i U(\theta)^T$$

Energy function analysis or analysis of reduced gradient flow:
[N. Kakimoto et al. '78, H.-D. Chiang et al. '94]

$$\dot{\theta}_i = -\nabla_i U(\theta)^T$$

Key objective: compute domain of attraction via numerical methods

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$\Longrightarrow$ **Open problem** [D. Hill and G. Chen '06]: power system $\overset{?}{\leftrightsquigarrow}$ network:

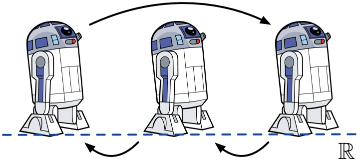
transient stability, performance, and robustness of a power network
$\overset{?}{\leftrightsquigarrow}$ graph properties of underlying network (topologic, spectral, etc)

Detour – Consensus Protocols & Kuramoto Oscillators

Consensus protocol in $\mathbb{R}^n$:

$$\dot{x}_i = -\sum_{j \neq i} a_{ij}(x_i - x_j)$$

- n identical **agents** with state variable $x_i \in \mathbb{R}$
- **objective**: state agreement: $x_i(t) - x_j(t) \to 0$
- **application:** agreement and coordination algorithms, . . .
- **references:** [M. DeGroot '74, J. Tsitsiklis '84, R. Olfati-Saber '04, ...]



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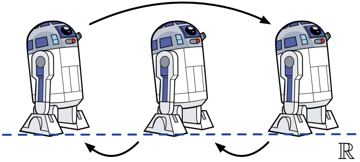
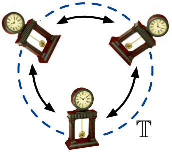
Kuramoto model in $\mathbb{T}^n$:

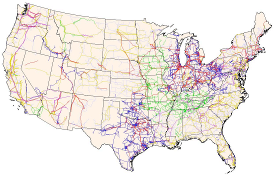
$$\dot{\theta}_i = \omega_i - \frac{K}{n} \sum\nolimits_{j \neq i} \sin(\theta_i - \theta_j)$$

- n non-identical **oscillators** with phase $\theta_i \in \mathbb{T}$ & frequency $\omega_i \in \mathbb{R}$
- **objective**: synchronization: $|\theta_i(t) - \theta_j(t)|$ bounded $\dot{\theta}_i(t) - \dot{\theta}_j(t) \to 0$
- **application:** sync phenomena
- **references:** [Y. Kuramoto '75, A. Winfree '80, S. Strogatz '00, ...]





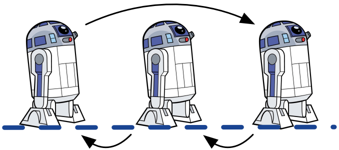
The Big Picture



Open problem in synchronization and transient stability in **power networks:** relation to underlying network state, parameters, and topology

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j \neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

Consensus Protocols:



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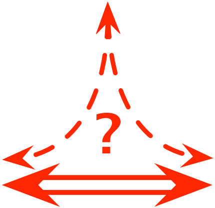
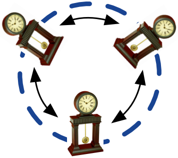


Kuramoto Oscillators:



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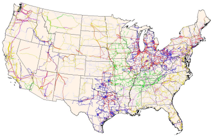

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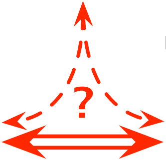

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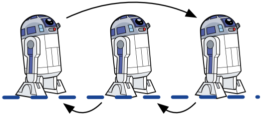

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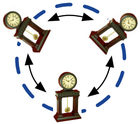

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Possible connection has often been hinted at in the literature

Power systems: [D. Subbarao et al., '01, G. Filatrella et al., '08, V. Fioriti et al., '09]
Networked control: [D. Hill et al., '06, M. Arcak, '07]
Dynamical systems: [H. Tanaka et al., '97, A. Arenas '08]

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2. **Singular perturbation analysis**
   (to relate power network and Kuramoto model)

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij})$$

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3. Synchronization analysis (of non-uniform Kuramoto model)

4. Conclusions

Singular Perturbation Analysis

Time-scale separation in power network model:

$$\frac{M_i}{\pi f_0}\ddot{\theta}_i = -D_i\dot{\theta}_i + \omega_i - \sum\nolimits_{j \neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

- **singular perturbation parameter:** $\epsilon = \frac{M_{\text{max}}}{\pi f_0 D_{\text{min}}}$
- reduced dynamics on slow time-scale (for $\epsilon = 0$)
  $\Rightarrow$ **non-uniform Kuramoto model:**

$$D_i\dot{\theta}_i = \omega_i - \sum\nolimits_{j \neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

Tikhonov's Theorem [H. Khalil '02]**:**
Assume the non-uniform Kuramoto model synchronizes exponentially.
Then $\forall\, (\theta(0), \dot{\theta}(0))$ there exists $\epsilon^* > 0$ such that $\forall\, \epsilon < \epsilon^*$ and $\forall\, t \geq 0$

$$\theta_i(t)_{\text{power network}} - \theta_i(t)_{\text{non-uniform Kuramoto model}} = \mathcal{O}(\epsilon)\,.$$

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Singular Perturbation Analysis

Discussion of the assumption $\epsilon = \frac{M_{\max}}{\pi f_0 D_{\min}}$ sufficiently small

1. **physical interpretation:** damping and sync on separate time-scales
2. **classic assumption** in literature on coupled oscillators: over-damped mechanical pendula and Josephson junctions
3. **physical reality:** with generator internal control effects $\epsilon \in \mathcal{O}(0.1)$
4. **simulation studies** show accurate approximation even for large ϵ
5. **topological equivalence independent of ϵ:** 1st-order and 2nd-order models have the same equilibria, the Jacobians have the same inertia, and the regions of attractions are bounded by the same separatrices
6. non-uniform Kuramoto model corresponds to reduced gradient system $\dot{\theta}_i = -\nabla_i U(\theta)^T$ **used** successfully in academia and industry since 1978

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   (to relate power network and Kuramoto model)
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4. Conclusions

Synchronization of Non-Uniform Kuramoto Oscillators

Non-uniform Kuramoto Model in $\mathbb{T}^n$:

$$D_i\dot{\theta}_i = \omega_i - \sum\nolimits_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij})$$

- **Non-uniformity** in network: D_i, ω_i, P_{ij}, φ_{ij}
- **Phase shift** φ_{ij} induces lossless and lossy coupling:

$$\dot{\theta}_i = \frac{\omega_i}{D_i} - \sum\nolimits_{j \neq i} \frac{P_{ij}}{D_i} \cos(\varphi_{ij}) \sin(\theta_i - \theta_j) + \frac{P_{ij}}{D_i} \sin(\varphi_{ij}) \cos(\theta_i - \theta_j)$$

- **Directed coupling** between oscillator i and j
- **Synchronization analysis** in multiple steps:
  1. phase locking: $|\theta_i(t) - \theta_j(t)|$ becomes bounded
  2. frequency entrainment: $\dot{\theta}_i(t) - \dot{\theta}_j(t) \to 0$
  3. phase synchronization: $|\theta_i(t) - \theta_j(t)| \to 0$

Main Synchronization Result

Condition on network parameters:

network connectivity $>$ network's non-uniformity $+$ network's losses

1. **Non-Uniform Kuramoto Model:**

$\Longrightarrow$ exponential synchronization: phase locking & frequency entrainment

$\Longrightarrow$ guaranteed region of attraction: $|\theta_i(t_0) - \theta_j(t_0)| < \pi/2 - \varphi_{\text{max}}$

$\Longrightarrow$ gap in condition determines ultimate phase locking

2. **Power Network Model:**

$\Longrightarrow$ there exists ϵ sufficiently small such that for all $t \geq 0$

$$\theta_i(t)_{\text{power network}} - \theta_i(t)_{\text{non-uniform Kuramoto model}} = \mathcal{O}(\epsilon)\,.$$

$\Longrightarrow$ for ϵ and network losses φ_{ij} sufficiently small, $\mathcal{O}(\epsilon)$ error converges

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$\Longrightarrow$ there exists ϵ sufficiently small such that for all $t \geq 0$

$$\theta_i(t)_{\text{power network}} - \theta_i(t)_{\text{non-uniform Kuramoto model}} = \mathcal{O}(\epsilon)\,.$$

$\Longrightarrow$ for ϵ and network losses φ_{ij} sufficiently small, $\mathcal{O}(\epsilon)$ error converges

Main Synchronization Result

Condition on network parameters:

network connectivity $>$ network's non-uniformity $+$ network's losses

1. **Non-Uniform Kuramoto Model:**

$\Longrightarrow$ exponential synchronization: phase locking & frequency entrainment

$\Longrightarrow$ guaranteed region of attraction: $|\theta_i(t_0) - \theta_j(t_0)| < \pi/2 - \varphi_{\text{max}}$

$\Longrightarrow$ gap in condition determines ultimate phase locking

2. **Power Network Model:**

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Non-uniform Kuramoto Model in $\mathbb{T}^n$ - rewritten:

$$\dot{\theta}_i = \frac{\omega_i}{D_i} - \sum\nolimits_{j \neq i} \frac{P_{ij}}{D_i} \cos(\varphi_{ij}) \sin(\theta_i - \theta_j) + \frac{P_{ij}}{D_i} \sin(\varphi_{ij}) \cos(\theta_i - \theta_j)$$

Synchronization condition $(\star)$:

$$\underbrace{n \frac{P_{\min}}{D_{\max}} \cos(\varphi_{\max})}_{\text{worst lossless coupling}} > \underbrace{\max_{\{i,j\}} \left(\frac{\omega_i}{D_i} - \frac{\omega_j}{D_j} \right)}_{\text{worst non-uniformity}} + \underbrace{\max_i \sum\nolimits_j \frac{P_{ij}}{D_i} \sin(\varphi_{ij})}_{\text{worst lossy coupling}} .$$

$\Rightarrow$ **phase locking**

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Proof: contraction property and time-varying consensus protocols

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$$K > \omega_{\max} - \omega_{\min}$$

Condition $(\star)$ is **necessary and sufficient**
when considering all distributions of $\omega \in [\omega_{\min}, \omega_{\max}]$.

Condition $(\star)$ strictly improves existing bounds on Kuramoto model:
[F. de Smet et al. '07, N. Chopra et al. '09, G. Schmidt et al. '09,
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Synchronization condition $(\star\star)$:

$$\underbrace{\lambda_2(L(P_{ij}\cos(\varphi_{ij})))}_{\text{lossless connectivity}} > \underbrace{f(D_i)}_{\text{non-uniform } D_i\text{s}} \cdot \underbrace{(1/\cos(\varphi_{\max}))}_{\text{necessary phase locking}} \times$$

$$\Big(\underbrace{\Big\|\Big[\dots, \frac{\omega_i}{D_i} - \frac{\omega_j}{D_j}, \dots\Big]\Big\|_2}_{\text{non-uniformity}} + \sqrt{\lambda_{\max}(L)} \underbrace{\Big\|\Big[\dots, \sum_j \frac{P_{ij}}{D_i} \sin(\varphi_{ij}), \dots\Big]\Big\|_2}_{\text{lossy coupling}} \Big)$$

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Condition $(\star\star)$ for regular Kuramoto model: $K > \|[\dots, \omega_i - \omega_j, \dots]\|_2$

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Further interesting results:

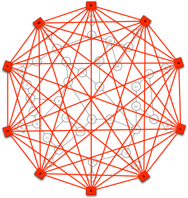
1. explicit synchronization frequency
2. exponential rate of frequency entrainment
3. conditions for phase synchronization
4. results for general non-complete graphs

. . . to be found in our papers.

Outline

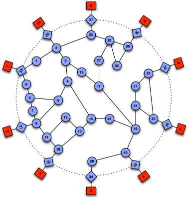
Structure-preserving power network models

So far we considered a **network-reduced** power system model:



- network reduced to generator nodes ■
- synchronization conditions on $\lambda_2(P)$ and $P_{\min}$
- all-to-all reduced admittance matrix $Y_{\text{reduced}} \sim P/V^2$ (for uniform voltage levels $|V_i| = V$)

Topological non-reduced **structure-preserving** power system model:



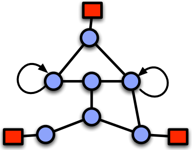
- boundary nodes (generators) & interior nodes (buses)
- topological bus admittance matrix $\mathbf{Y}_{\text{network}}$ indicating transmission lines and loads (self-loops)
- Schur-complement relation: $Y_{\text{reduced}} = \mathbf{Y}_{\text{network}} / Y_{\text{interior}}$
  c.f. "Kron reduction", "Dirichlet-to-Neumann map", "Schur contraction", "Gaussian elimination", ...

Outline

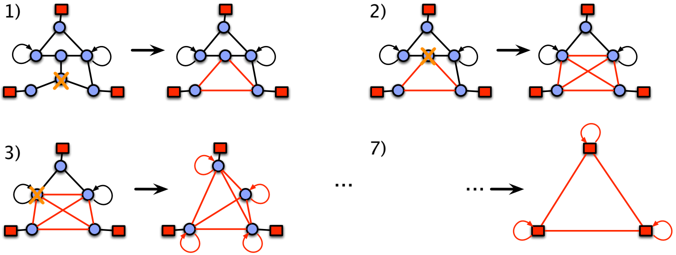
Kron reduction of graphs

Kron reduction of a graph with

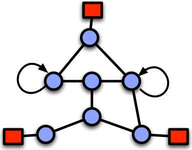
- boundary nodes ■
- interior nodes ● with nonnegative self-loops ⟲
- (loopy) Laplacian matrix $\mathbf{Y}_{\text{network}}$



1. Iterative Kron reduction of a single interior node ● :
   - Algebraic evolution of Laplacian matrix: $\mathbf{Y}^{k+1}_{\text{reduced}} = \mathbf{Y}^{k}_{\text{reduced}} / \bullet$
   - Topological evolution of the corresponding graph



Kron reduction of graphs



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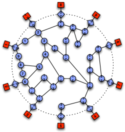
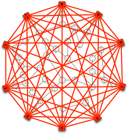
1. Iterative Kron reduction of a single interior node ● :

$$\mathbf{Y}^{k+1}_{\text{reduced}} = \mathbf{Y}^{k}_{\text{reduced}} / \bullet$$

2. Fully reduced Laplacian Y_{reduced} is equivalent to Schur complement:



$$\mathbf{Y}_{\text{network}} \curvearrowright \; Y_{\text{reduced}} = \mathbf{Y}_{\text{network}} / Y_{\text{interior}}$$



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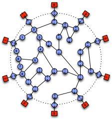

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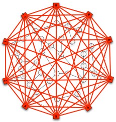
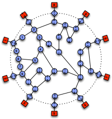
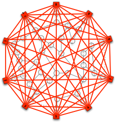


Some properties of the **Kron reduction** process:

1. **Well-posedness:** Symmetric & irreducible (loopy) Laplacian matrices can be reduced and are closed under Kron reduction
2. **Topological properties:**
   - interior network connected ⇒ reduced network complete
   - at least one node in interior network features a self-loop ⟲
     ⇒ all nodes in reduced network feature self-loops ⟲
3. **Algebraic properties:** self-loops in interior network . . .
   - decrease mutual coupling in reduced network
   - increase self-loops in reduced network

Kron reduction of graphs

 $\mathbf{Y}_{\text{network}}$ → $Y_{\text{reduced}} = \mathbf{Y}_{\text{network}}/Y_{\text{interior}}$ 

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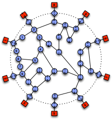

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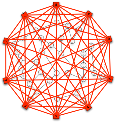

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Some properties of the **Kron reduction** process:

⋮

4. **Spectral properties**:
   - interlacing property: $\lambda_i(\mathbf{Y}_{\text{network}}) \leq \lambda_i(Y_{\text{reduced}}) \leq \lambda_{i+n-|\blacksquare|}(\mathbf{Y}_{\text{network}})$
   - $\Rightarrow$ algebraic connectivity λ_2 is non-decreasing
   - effect of self-loops $\circlearrowleft$ on loop-less Laplacian matrices:

$$\lambda_2(L_{\text{reduced}}) + \max\{\circlearrowleft\} \geq \lambda_2(L_{\text{network}}) + \min\{\circlearrowleft\}$$
$$\lambda_2(L_{\text{reduced}}) + \min\{\circlearrowleft\} \leq \lambda_{2+n-|\blacksquare|}(L_{\text{network}}) + \max\{\circlearrowleft\}$$

5. **Effective resistance:**
   - Effective resistance $R(i,j)$ among boundary nodes $\blacksquare$ is invariant
   - For boundary nodes $\blacksquare$: effective resistance $R(i,j)$ uniform
     $\Leftrightarrow$ coupling $Y_{\text{reduced}}(i,j)$ uniform $\Leftrightarrow 1/R(i,j) = \frac{n}{2}|Y_{\text{reduced}}(i,j)|$

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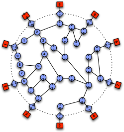
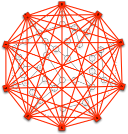


$$\mathbf{Y}_{\text{network}} \curvearrowright Y_{\text{reduced}} = \mathbf{Y}_{\text{network}} / Y_{\text{interior}}$$



Assumption I: lossless network and uniform voltage levels V at generators

1. **Spectral condition for synchronization:** $\lambda_2(P) \geq$ ... becomes

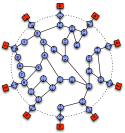
$$\lambda_2(i \cdot \mathbf{L}_{\text{network}}) > \left\| \left(\frac{\omega_2}{D_2} - \frac{\omega_1}{D_1}, \dots \right) \right\|_2 \cdot \frac{f(D_i)}{V^2} + \min\{\circlearrowleft\}$$

Assumption II: effective resistance R among generator nodes is uniform

2. **Resistance-based condition for synchronization:** $nP_{\min} \geq$ ... becomes

$$\frac{1}{R} > \max_{\{i,j\}} \left\{ \frac{\omega_i}{D_i} - \frac{\omega_j}{D_j} \right\} \cdot \frac{D_{\max}}{2V^2}$$

Synchronization in structure-preserving models



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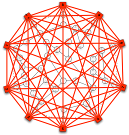

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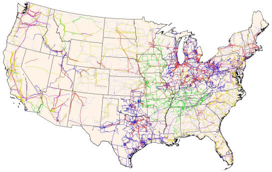
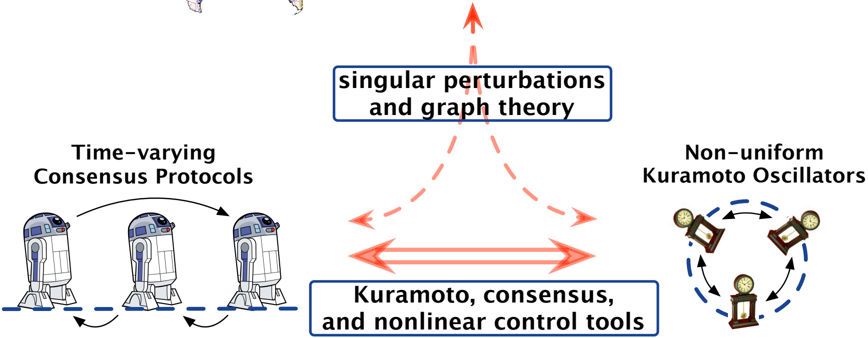
Outline

Conclusions



Open problem in synchronization and transient stability in **power networks:** relation to underlying network state, parameters, and topology



- **Details** and **papers** can be found at: `http://motion.me.ucsb.edu`
- **Ongoing work**: more detailed models, inverters and power electronics. . .