Synchronization and Kron Reduction in Power Networks

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Today’s presentation is about a controversial topic.

“Well known” observations:
1. Power networks are coupled oscillators
2. Kuramoto oscillators synchronize for large coupling
3. Graph theory quantifies coupling in a network, e.g., $\lambda_2$
4. Hence, power networks synchronize for large $\lambda_2$

... anyways, transient stability is solved

Misunderstandings between physicists and control engineers:
1. Exact conditions for as realistic model as possible
2. Not aiming for the most insightful and concise model

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Intro: The North American Power Grid

“. . . the largest and most complex machine engineered by humankind.”
[P. Kundur ’94, V. Vittal ’03, . . . ]

“. . . the greatest engineering achievement of the 20th century.”
[National Academy of Engineering ’10]

1. large-scale, complex, nonlinear, and rich dynamic behavior
2. 100 years old and operating at its capacity limits
⇒ recent blackouts: New England ’03 + Italy ’03, Brazil ’09
THE BLACKOUT OF 2003: Failure Reveals Creaky System, Experts Believe

Energy is one of the top three national priorities.

Expected additional synergetic effects in future “smart grid”:

⇒ increasing complexity and renewable stochastic power sources

⇒ increasingly many transient disturbances to be detected and rejected

Transient Stability: Generators have to maintain synchronism in presence of large transient disturbances such as faults or loss of

- transmission lines and components,
- generation or load.
Power network topology:

1. \( n \) generators \( \blacksquare \), each connected to a generator terminal bus \( \blacklozenge \)
2. \( n \) generators terminal buses \( \blacklozenge \) and \( m \) load buses \( \bullet \) form connected graph
3. admittance matrix \( Y_{\text{network}} \in \mathbb{C}^{(2n+m) \times (2n+m)} \) characterizes the network
Structure-preserving DAE power network model:

1. Dynamic swing equations for generators:

\[ \frac{M_i}{\pi f_0} \ddot{\theta}_i = -D_i \dot{\theta}_i + P_{\text{mech.in},i} - P_{\text{electr.out},i} \]

- Active output power:

\[ P_{\text{electr.out},i} = \Re( \mathbf{V}_i \mathbf{Y}_{\text{network},i,j}(\mathbf{V}_i^* - \mathbf{V}_j^*)) \]

- \( \theta_i(t) \) is measured w.r.t. a 60Hz rotating frame

2. Passive interior network:

- Loads are modeled as shunt admittances

\[ \Rightarrow \text{Kirchhoff equations: } I = \mathbf{Y}_{\text{network}} \mathbf{V} \]

\[ \Rightarrow \text{Algebraic power flow equations for loads: } 0 = \mathbf{V}_i \sum_{j \in \text{loads}} \mathbf{Y}_{\text{network},i,j} \mathbf{V}_j^* \]
Intro: Mathematical Model of a Power Network

Structure-preserving DAE power network model:

1. **Dynamic swing equations** for generators ■:

\[
\frac{M_i}{\pi f_0} \ddot{\theta}_i = -D_i \dot{\theta}_i + P_{\text{mech.in},i} - P_{\text{electr.out},i}
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- Active output power ■ \(\rightarrow\) ♦:

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2. **Passive interior network** ♦ & ●:

- Loads are modeled as *shunt admittances*

  \(\Rightarrow\) **Kirchhoff equations**: \(I = Y_{\text{network}} V\)

  \(\Rightarrow\) **Algebraic power flow equations** for ♦ & ●:

\[
0 = V_i \sum_{j \in \Box \cup \Diamond \cup \bullet} Y^*_{\text{network},i,j} V_j^*
\]

Dörfler and Bullo (UCSB)
Synchronization & Kron Reduction
Smat Grids @ LANL
Network-Reduction to an ODE power network model:

- boundary nodes (□) & interior nodes (● & ◊)
- algebraic Kirchhoff equations \( I = Y_{\text{network}} V \):
  \[
  \begin{bmatrix}
  I_{\text{boundary}} \\
  0 
  \end{bmatrix}
  =
  \begin{bmatrix}
  Y_{\text{boundary}} & Y_{\text{bound-int}} \\
  Y_{\text{bound-int}}^T & Y_{\text{interior}} 
  \end{bmatrix}
  \begin{bmatrix}
  V_{\text{boundary}} \\
  V_{\text{interior}} 
  \end{bmatrix}
  \]

Map between boundary nodes \( I_{\text{boundary}} = Y_{\text{reduced}} V_{\text{boundary}} \) given by

Schur-complement: \( Y_{\text{reduced}} = Y_{\text{network}} / Y_{\text{interior}} \)

- network reduced to active nodes (generators)
- \( Y_{\text{reduced}} \) induces complete “all-to-all” coupling graph
- active output power □ → □:
  \[
  P_{\text{electr.out},i} = \Re \left( V_i \sum_{j=1}^{n} Y_{\text{reduced},i,j} V_j^* \right)
  \]
Network-Reduced ODE power network model:

classic model of interconnected swing equations

\[ \frac{M_i}{\pi f_0} \ddot{\theta}_i = -D_i \dot{\theta}_i + \omega_i - \sum_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \phi_{ij}) \]

defined on “all-to-all” reduced network \( Y_{\text{reduced}} \) with

\[ P_{ij} = |V_i| |V_j| |Y_{\text{reduced},i,j}| > 0 \]

is max. power transferred \( i \leftrightarrow j \)

\[ \phi_{ij} = \arctan(\Re(Y_{\text{reduced},i,j}) / \Im(Y_{\text{reduced},i,j})) \in [0, \pi/2) \]

reflects losses \( i \leftrightarrow j \)

\[ \omega_i = P_{\text{mech.in},i} - |V_i|^2 \Re(Y_{\text{reduced},i,i}) \]

is effective power input of \( i \)
Intro: Transient Stability Analysis in Power Networks

\[
\frac{M_i}{\pi f_0} \ddot{\theta}_i = -D_i \dot{\theta}_i + \omega_i - \sum_{j\neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij})
\]

**Classic problem setup** in transient stability analysis:
- *frequency equilibrium*: \((\dot{\theta}_i, \ddot{\theta}_i) = (0, 0)\) for all \(i\)

1. power network in stable frequency equilibrium
2. \(\rightarrow\) transient network disturbance and fault clearance
3. stability analysis of a new frequency equilibrium in post-fault network

**General synchronization problem:**
- *synchronous equilibrium*: \(|\theta_i - \theta_j|\) bounded & \(\dot{\theta}_i = \dot{\theta}_j\) for all \(\{i, j\}\)

synchronization in presence of transient network disturbances
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Classic analysis methods: Hamiltonian arguments

\[
\frac{M_i}{\pi f_0} \ddot{\theta}_i = -D_i \dot{\theta}_i - \nabla_i U(\theta)^T
\]

Energy function analysis or analysis of reduced gradient flow:

[N. Kakimoto et al. '78, H.-D. Chiang et al. '94 ]

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\dot{\theta}_i = -\nabla_i U(\theta)^T
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Key objective: compute domain of attraction via numerical methods
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Key objective: compute domain of attraction via numerical methods

⇒ Open problem [D. Hill and G. Chen ’06]: power system \(\leftrightarrow\) network:

transient stability, performance, and robustness of a power network \(\leftrightarrow\) graph properties of underlying network (topologic, spectral, etc)
Consensus protocol in $\mathbb{R}^n$:

\[ \dot{x}_i = - \sum_{j \neq i} a_{ij} (x_i - x_j) \]

- **n identical agents** with state variable $x_i \in \mathbb{R}$
- **objective**: state agreement: $x_i(t) - x_j(t) \to 0$
- **application**: agreement and coordination algorithms, . . .
- **references**: [M. DeGroot ’74, J. Tsitsiklis ’84, R. Olfati-Saber ’04, …]
### Consensus protocol in $\mathbb{R}^n$:

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### Kuramato model in $\mathbb{T}^n$:

$$\dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j \neq i} \sin(\theta_i - \theta_j)$$

- **n non-identical oscillators** with phase $\theta_i \in \mathbb{T}$ & frequency $\omega_i \in \mathbb{R}$
- **objective**: synchronization: $|\theta_i(t) - \theta_j(t)|$ bounded
- $\dot{\theta}_i(t) - \dot{\theta}_j(t) \to 0$
- **application**: sync phenomena
- **references**: [Y. Kuramoto '75, A. Winfree '80, S. Strogatz '00, ...]
Open problem in synchronization and transient stability in power networks: relation to underlying network state, parameters, and topology

\[ \frac{M_i}{\pi f_0} \ddot{\theta}_i = -D_i \dot{\theta}_i + \omega_i - \sum_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij}) \]

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Possible connection has often been hinted at in the literature

Power systems: [D. Subbarao et al., '01, G. Filatrella et al., '08, V. Fioriti et al., '09]
Networked control: [D. Hill et al., '06, M. Arcak, '07]
Dynamical systems: [H. Tanaka et al., '97, A. Arenas '08]
1. Introduction and Motivation

2. Singular perturbation analysis
   (to relate power network and Kuramoto model)

   \[ \frac{M_i}{\pi f_0} \ddot{\theta}_i = -D_i \dot{\theta}_i + \omega_i - \sum_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \phi_{ij}) \]

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3. Synchronization analysis (of non-uniform Kuramoto model)

4. Conclusions
Singular Perturbation Analysis

Time-scale separation in power network model:

\[
\frac{M_i}{\pi f_0} \ddot{\theta}_i = -D_i \dot{\theta}_i + \omega_i - \sum_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij})
\]

- singular perturbation parameter: \( \epsilon = \frac{M_{\text{max}}}{\pi f_0 D_{\text{min}}} \)

- reduced dynamics on slow time-scale (for \( \epsilon = 0 \))
  \[ D_i \dot{\theta}_i = \omega_i - \sum_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij}) \]

Tikhonov’s Theorem [H. Khalil ’02]:
Assume the non-uniform Kuramoto model synchronizes exponentially. Then \( \forall (\theta(0), \dot{\theta}(0)) \) there exists \( \epsilon^* > 0 \) such that \( \forall \epsilon < \epsilon^* \) and \( \forall t \geq 0 \)

\[
\theta_i(t)_{\text{power network}} - \theta_i(t)_{\text{non-uniform Kuramoto model}} = O(\epsilon).
\]
Singular Perturbation Analysis

**Time-scale separation** in power network model:

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  \( \Rightarrow \) non-uniform Kuramoto model:

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Discussion of the assumption $\epsilon = \frac{M_{\text{max}}}{\pi f_0 D_{\text{min}}}$ sufficiently small

1. **physical interpretation:** damping and sync on separate time-scales

2. **classic assumption** in literature on coupled oscillators: over-damped mechanical pendula and Josephson junctions

3. **physical reality:** with generator internal control effects $\epsilon \in O(0.1)$

4. **simulation studies** show accurate approximation even for large $\epsilon$

5. **topological equivalence independent of $\epsilon$:** 1st-order and 2nd-order models have the same equilibria, the Jacobians have the same inertia, and the regions of attractions are bounded by the same separatrices

6. **non-uniform Kuramoto model** corresponds to reduced gradient system $\dot{\theta}_i = -\nabla_i U(\theta)^T$ **used** successfully in academia and industry since 1978
Outline

1. Introduction and Motivation

2. Singular perturbation analysis
   (to relate power network and Kuramoto model)

3. Synchronization analysis (of non-uniform Kuramoto model)

4. Conclusions
Synchronization of Non-Uniform Kuramoto Oscillators

Non-uniform Kuramoto Model in $\mathbb{T}^n$:

$$D_i \dot{\theta}_i = \omega_i - \sum_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij})$$

- **Non-uniformity** in network: $D_i, \omega_i, P_{ij}, \varphi_{ij}$
- **Phase shift** $\varphi_{ij}$ induces lossless and lossy coupling:

$$\dot{\theta}_i = \frac{\omega_i}{D_i} - \sum_{j \neq i} \frac{P_{ij}}{D_i} \cos(\varphi_{ij}) \sin(\theta_i - \theta_j) + \frac{P_{ij}}{D_i} \sin(\varphi_{ij}) \cos(\theta_i - \theta_j)$$

- **Directed coupling** between oscillator $i$ and $j$
- **Synchronization analysis** in multiple steps:
  1. phase locking: $|\theta_i(t) - \theta_j(t)|$ becomes bounded
  2. frequency entrainment: $\dot{\theta}_i(t) - \dot{\theta}_j(t) \to 0$
  3. phase synchronization: $|\theta_i(t) - \theta_j(t)| \to 0$
Condition on network parameters:

\[
\text{network connectivity } > \text{ network's non-uniformity } + \text{ network's losses}
\]

1. **Non-Uniform Kuramoto Model:**
   - exponential synchronization: phase locking & frequency entrainment
   - guaranteed region of attraction: \(|\theta_i(t_0) - \theta_j(t_0)| < \pi/2 - \varphi_{\text{max}}\)
   - gap in condition determines ultimate phase locking

2. **Power Network Model:**
   - there exists \(\epsilon\) sufficiently small such that for all \(t \geq 0\)
     \[
     \theta_i(t)_{\text{power network}} - \theta_i(t)_{\text{non-uniform Kuramoto model}} = O(\epsilon).
     \]
   - for \(\epsilon\) and network losses \(\varphi_{ij}\) sufficiently small, \(O(\epsilon)\) error converges
Main Synchronization Result

Condition on network parameters:

network connectivity \(>\) network’s non-uniformity + network’s losses

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   \[\Rightarrow\] exponential synchronization: phase locking & frequency entrainment
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2. **Power Network Model:**

\[\Rightarrow\] there exists \( \epsilon \) sufficiently small such that for all \( t \geq 0 \)

\[ \theta_i(t)_{\text{power network}} - \theta_i(t)_{\text{non-uniform Kuramoto model}} = \mathcal{O}(\epsilon). \]

\[\Rightarrow\] for \( \epsilon \) and network losses \( \varphi_{ij} \) sufficiently small, \( \mathcal{O}(\epsilon) \) error converges
Non-uniform Kuramoto Model in $\mathbb{T}^n$ - rewritten:

$$\dot{\theta}_i = \frac{\omega_i}{D_i} - \sum_{j \neq i} \frac{P_{ij}}{D_i} \cos(\varphi_{ij}) \sin(\theta_i - \theta_j) + \frac{P_{ij}}{D_i} \sin(\varphi_{ij}) \cos(\theta_i - \theta_j)$$

Synchronization condition ($\star$):

$$n \frac{P_{\min}}{D_{\max}} \cos(\varphi_{\max}) > \max_{\{i,j\}} \left( \frac{\omega_i}{D_i} - \frac{\omega_j}{D_j} \right) + \max_i \sum_j \frac{P_{ij}}{D_i} \sin(\varphi_{ij}) \cdot$$

- worst lossless coupling
- worst non-uniformity
- worst lossy coupling

$\Rightarrow$ phase locking

$\Rightarrow$ frequency entrainment

Proof: contraction property and time-varying consensus protocols
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Classic (uniform) Kuramoto Model in $\mathbb{T}^n$:

$$\dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j \neq i} \sin(\theta_i - \theta_j)$$

Synchronization condition ($\star$):

$$K > \omega_{\text{max}} - \omega_{\text{min}}$$

Condition ($\star$) is necessary and sufficient when considering all distributions of $\omega \in [\omega_{\text{min}}, \omega_{\text{max}}]$.

Condition ($\star$) strictly improves existing bounds on Kuramoto model: [F. de Smet et al. '07, N. Chopra et al. '09, G. Schmidt et al. '09, A. Jadbabaie et al. '04, S.J. Chung et al. '10, J.L. van Hemmen et al. '93].

Necessary condition synchronization: $K > \frac{n}{2(n-1)}(\omega_{\text{max}} - \omega_{\text{min}})$

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$$K > \omega_{\text{max}} - \omega_{\text{min}}$$

Condition $(\star)$ is necessary and sufficient when considering all distributions of $\omega \in [\omega_{\text{min}}, \omega_{\text{max}}]$.

Condition $(\star)$ strictly improves existing bounds on Kuramoto model: [F. de Smet et al. '07, N. Chopra et al. '09, G. Schmidt et al. '09, A. Jadbabaie et al. '04, S.J. Chung et al. '10, J.L. van Hemmen et al. '93].

Necessary condition synchronization: $K > \frac{n}{2(n-1)}(\omega_{\text{max}} - \omega_{\text{min}})$ [J.L. van Hemmen et al. '93, A. Jadbabaie et al. '04, N. Chopra et al. '09].
Synchronization of Non-Uniform Kuramoto Oscillators

Non-uniform Kuramoto Model in $\mathbb{T}^n$ - rewritten:

$$
\dot{\theta}_i = \frac{\omega_i}{D_i} - \sum_{j \neq i} \frac{P_{ij}}{D_i} \cos(\varphi_{ij}) \sin(\theta_i - \theta_j) + \frac{P_{ij}}{D_i} \sin(\varphi_{ij}) \cos(\theta_i - \theta_j)
$$

Synchronization condition (⋆⋆):

$$
\lambda_2(L(P_{ij} \cos(\varphi_{ij}))) > f(D_i) \cdot \left(\frac{1}{\cos(\varphi_{\text{max}})}\right) \times

\left(\frac{\omega_i}{D_i} - \frac{\omega_j}{D_j}, \ldots\right)_{2} + \sqrt{\lambda_{\text{max}}(L)} \left(\sum_j \frac{P_{ij}}{D_i} \sin(\varphi_{ij}), \ldots\right)_{2}
$$

- **Lossless connectivity:** $\lambda_2(L(P_{ij} \cos(\varphi_{ij})))$
- **Non-uniform $D_i$'s necessary phase locking:** $f(D_i)$ \cdot \left(\frac{1}{\cos(\varphi_{\text{max}})}\right)
- **Non-uniformity:** $\left(\frac{\omega_i}{D_i} - \frac{\omega_j}{D_j}, \ldots\right)_{2}$
- **Lossy coupling:** $\sqrt{\lambda_{\text{max}}(L)} \left(\sum_j \frac{P_{ij}}{D_i} \sin(\varphi_{ij}), \ldots\right)_{2}$
Synchronization of Non-Uniform Kuramoto Oscillators

Non-uniform Kuramoto Model in $\mathbb{T}^n$ - rewritten:

$$\dot{\theta}_i = \frac{\omega_i}{D_i} - \sum_{j \neq i} \frac{P_{ij}}{D_i} \cos(\varphi_{ij}) \sin(\theta_i - \theta_j) + \frac{P_{ij}}{D_i} \sin(\varphi_{ij}) \cos(\theta_i - \theta_j)$$

Synchronization condition (⋆⋆):

$$\lambda_2(L(P_{ij} \cos(\varphi_{ij}))) > \begin{bmatrix} \frac{\omega_i}{D_i} - \frac{\omega_j}{D_j}, \cdots \end{bmatrix}_2$$

- Lossless connectivity
- Non-uniform $D_i$'s
- Necessary phase locking

$$\frac{f(D_i)}{\max(1/\cos(\varphi_{\text{max}}))} \times \begin{bmatrix} \cdots, \sum_j \frac{P_{ij}}{D_i} \sin(\varphi_{ij}), \cdots \end{bmatrix}_2$$

- Non-uniformity
- Lossy coupling

$\Rightarrow$ phase locking

$\Rightarrow$ frequency entrainment
Synchronization of Non-Uniform Kuramoto Oscillators

Non-uniform Kuramoto Model in $\mathbb{T}^n$ - rewritten:

$$\dot{\theta}_i = \frac{\omega_i}{D_i} - \sum_{j \neq i} \frac{P_{ij}}{D_i} \cos(\varphi_{ij}) \sin(\theta_i - \theta_j) + \frac{P_{ij}}{D_i} \sin(\varphi_{ij}) \cos(\theta_i - \theta_j)$$

Synchronization condition ($\star \star$):

$$\lambda_2(L(P_{ij} \cos(\varphi_{ij}))) > f(D_i) \cdot \frac{1}{\cos(\varphi_{\max})} \times$$

lossless connectivity

non-uniform $D_i$s necessary phase locking

non-uniformity

lossy coupling

Condition ($\star \star$) for regular Kuramoto model:

$$K > \| [\ldots, \omega_i - \omega_j, \ldots] \|_2$$
Non-uniform Kuramoto Model in $\mathbb{T}^n$ - rewritten:

$$\dot{\theta}_i = \frac{\omega_i}{D_i} - \sum_{j \neq i} \frac{P_{ij}}{D_i} \cos(\varphi_{ij}) \sin(\theta_i - \theta_j) + \frac{P_{ij}}{D_i} \sin(\varphi_{ij}) \cos(\theta_i - \theta_j)$$

Further interesting results:

1. explicit synchronization frequency
2. exponential rate of frequency entrainment
3. conditions for phase synchronization
4. results for general non-complete graphs

...to be found in our papers.
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   - The Big Picture

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   (to relate power network and Kuramoto model)

3. Synchronization analysis (of non-uniform Kuramoto model)
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4. Structure-preserving power network models
   - Kron Reduction of Graphs
   - Sufficient Conditions for Synchronization

5. Conclusions and Future Work
So far we considered a **network-reduced** power system model:

- network reduced to generator nodes
- synchronization conditions on $\lambda_2(P)$ and $P_{\text{min}}$
- all-to-all reduced admittance matrix $Y_{\text{reduced}} \sim P/V^2$
  (for uniform voltage levels $|V_i| = V$)

Topological non-reduced **structure-preserving** power system model:

- **boundary nodes** (generators) & **interior nodes** (buses)
- topological bus admittance matrix $Y_{\text{network}}$
  indicating transmission lines and loads (self-loops)
- **Schur-complement** relation: $Y_{\text{reduced}} = Y_{\text{network}}/Y_{\text{interior}}$
  c.f. “Kron reduction”, “Dirichlet-to-Neumann map”, “Schur contraction”, “Gaussian elimination”, ...
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Kron reduction of graphs

**Kron reduction** of a graph with
- boundary nodes ☐
- interior nodes ☮ with nonnegative self-loops \(\circ\)
- (loopy) Laplacian matrix \(Y_{\text{network}}\)

1. **Iterative Kron reduction** of a single interior node ☮:
   - Algebraic evolution of Laplacian matrix:
     \[ Y_{\text{reduced}}^{k+1} = Y_{\text{reduced}}^k / \text{●} \]
   - Topological evolution of the corresponding graph
**Kron reduction** of a graph with
- boundary nodes ■
- interior nodes • with nonnegative self-loops ◇
- (loopy) Laplacian matrix $Y_{\text{network}}$

1. Iterative Kron reduction of a single interior node •:
   
   $$Y_{\text{reduced}}^{k+1} = \frac{Y_{\text{reduced}}^k}{•}$$

2. Fully reduced Laplacian $Y_{\text{reduced}}$ is equivalent to Schur complement:
   
   $$Y_{\text{reduced}} = \frac{Y_{\text{network}}}{Y_{\text{interior}}}$$
Some properties of the **Kron reduction** process:

1. **Well-posedness:** Symmetric & irreducible (loopy) Laplacian matrices can be reduced and are closed under Kron reduction

2. **Topological properties:**
   - interior network connected $\Rightarrow$ reduced network complete
   - at least one node in interior network features a self-loop $\Rightarrow$ all nodes in reduced network feature self-loops

3. **Algebraic properties:** self-loops in interior network ...
   - decrease mutual coupling in reduced network
   - increase self-loops in reduced network
Some properties of the **Kron reduction** process:

1. **Well-posedness:** Symmetric & irreducible (loopy) Laplacian matrices can be reduced and are closed under Kron reduction.

2. **Topological properties:**
   - **interior network** connected $\Rightarrow$ **reduced network** complete
   - at least one node in **interior network** features a self-loop $\bigcirc$
   $\Rightarrow$ all nodes in **reduced network** feature self-loops $\bigcirc$

3. **Algebraic properties:** self-loops in **interior network**
   - decrease mutual coupling in **reduced network**
   - increase self-loops in **reduced network**
Some properties of the **Kron reduction** process:

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3. **Algebraic properties**: self-loops in **interior network**... 
   - decrease mutual coupling in **reduced network**
   - increase self-loops in **reduced network**

\[
Y_{\text{network}} \quad \xrightarrow{\text{Y_{\text{reduced}} = \frac{Y_{\text{network}}}{Y_{\text{interior}}}}} \quad Y_{\text{network}}
\]
Some properties of the **Kron reduction** process:

4 **Spectral properties:**
   - interlacing property: \( \lambda_i(Y_{\text{network}}) \leq \lambda_i(Y_{\text{reduced}}) \leq \lambda_{i+n-\Box}(Y_{\text{network}}) \)

   \( \Rightarrow \) algebraic connectivity \( \lambda_2 \) is non-decreasing

   - effect of self-loops \( \bigcirc \) on loop-less Laplacian matrices:
     \[
     \begin{align*}
     \lambda_2(L_{\text{reduced}}) + \max\{\bigcirc\} & \geq \lambda_2(L_{\text{network}}) + \min\{\bigcirc\} \\
     \lambda_2(L_{\text{reduced}}) + \min\{\bigcirc\} & \leq \lambda_{2+n-\Box}(L_{\text{network}}) + \max\{\bigcirc\}
     \end{align*}
     \]

5 **Effective resistance:**
   - Effective resistance \( R(i,j) \) among boundary nodes \( \Box \) is invariant

   - For boundary nodes \( \Box \): effective resistance \( R(i,j) \) uniform
     \( \Leftrightarrow \) coupling \( Y_{\text{reduced}}(i,j) \) uniform
     \( \Leftrightarrow \) \( 1/R(i,j) = \frac{n}{2} |Y_{\text{reduced}}(i,j)| \)
Some properties of the **Kron reduction** process:

4  **Spectral properties:**
   - interlacing property: $\lambda_i(Y_{\text{network}}) \leq \lambda_i(Y_{\text{reduced}}) \leq \lambda_{i+n-\text{}}(Y_{\text{network}})$
   - algebraic connectivity $\lambda_2$ is non-decreasing
   - effect of self-loops $\bigcirc$ on loop-less Laplacian matrices:
     
     $$\begin{align*}
     \lambda_2(L_{\text{reduced}}) + \max\{\bigcirc\} &\geq \lambda_2(L_{\text{network}}) + \min\{\bigcirc\} \\
     \lambda_2(L_{\text{reduced}}) + \min\{\bigcirc\} &\leq \lambda_{2+n-\text{}}(L_{\text{network}}) + \max\{\bigcirc\}
     \end{align*}$$

5  **Effective resistance:**
   - Effective resistance $R(i,j)$ among boundary nodes $\blacksquare$ is invariant
   - For boundary nodes $\blacksquare$: effective resistance $R(i,j)$ uniform
     $\Leftrightarrow$ coupling $Y_{\text{reduced}}(i,j)$ uniform $\Leftrightarrow 1/R(i,j) = \frac{n}{2} |Y_{\text{reduced}}(i,j)|$
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Assumption I: lossless network and uniform voltage levels $V$ at generators

**Spectral condition for synchronization:** $\lambda_2(P) \geq \ldots$ becomes

$$\lambda_2(i \cdot L_{\text{network}}) > \left\| \left( \frac{\omega_2}{D_2} - \frac{\omega_1}{D_1}, \ldots \right) \right\|_2 \cdot \frac{f(D_i)}{V^2} + \min\{\phi\}$$

Assumption II: effective resistance $R$ among generator nodes is uniform

**Resistance-based condition for synchronization:** $nP_{\text{min}} \geq \ldots$ becomes

$$\frac{1}{R} > \max_{\{i,j\}} \left\{ \frac{\omega_i}{D_i} - \frac{\omega_j}{D_j} \right\} \cdot \frac{D_{\text{max}}}{2V^2}$$
Assumption I: lossless network and uniform voltage levels $V$ at generators

1 Spectral condition for synchronization: $\lambda_2(P) \geq \ldots$ becomes

$$\lambda_2(i \cdot L_{\text{network}}) > \left\| \left( \frac{\omega_2}{D_2} - \frac{\omega_1}{D_1}, \ldots \right) \right\|_2 \cdot \frac{f(D_i)}{V^2} + \min \{ \varnothing \}$$

Assumption II: effective resistance $R$ among generator nodes is uniform

2 Resistance-based condition for synchronization: $nP_{\min} \geq \ldots$ becomes

$$\frac{1}{R} > \max_{\{i,j\}} \left\{ \frac{\omega_i}{D_i} - \frac{\omega_j}{D_j} \right\} \cdot \frac{D_{\max}}{2V^2}$$
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Conclusions

Open problem in synchronization and transient stability in **power networks**: relation to underlying network state, parameters, and topology

- **Singular perturbations and graph theory**
- **Time-varying Consensus Protocols**
- **Non-uniform Kuramoto Oscillators**
- **Kuramoto, consensus, and nonlinear control tools**

- **Details and papers** can be found at: http://motion.me.ucsb.edu
- **Ongoing work**: more detailed models, inverters and power electronics...