Algebraic Methods in Power Grid Control and Optimization

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Overview

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Stability Analysis

- A power grid system is generically described by a set of DAEs:

$$\dot{x} = f(x, y, \mu)$$
$$0 = g(x, y, \mu)$$

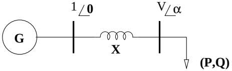
  where $x \in \mathbb{R}^n$ are the *state* variables and $y \in \mathbb{R}^m$ are the *algebraic* variables.

- We want to determine the stationary points of the system

$$0 = f(x_0, y_0, \mu)$$
$$0 = g(x_0, y_0, \mu)$$

- What are their properties: stability, bifurcation analysis, region of attraction, disturbance analysis, design controllers, etc.

Example: Voltage Collapse

- A model power system: 

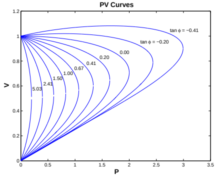
- The state variables are $x = (\alpha, V)$ and the bifurcation parameters are $\mu = (P, Q)$.
- The equations that determine the system equilibria are:

$$0 = -4V\sin(\alpha) - P$$
$$0 = -4V^2 + 4V\cos(\alpha) - Q$$

- What are the *safety margins* for the allowable variations in the loads?

Reference: Dobson, I., *Computing a closest bifurcation instability in multidimensional parameter space*, Nonlinear Science 3, 307-327, 1993.

Power-Voltage Relationships



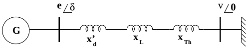
- For various *load power factors*

$$\cos(\phi) := \frac{P}{\sqrt{P^2 + Q^2}}$$

there is a maximum deliverable power to the load node.

- For a given load power below the maximum, there are two solutions to the load flow equations.

Example: Time domain Stability

- Consider this model: 

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = 10\lambda - 20\sin(x_1) - x_2$$

- The equilibrium points can be found from the steady-state (power flow) equations:

$$0 = x_2$$
$$0 = 10\lambda - 20\sin(x_{10}) - x_{20}$$

Equilibria

- The solutions are:

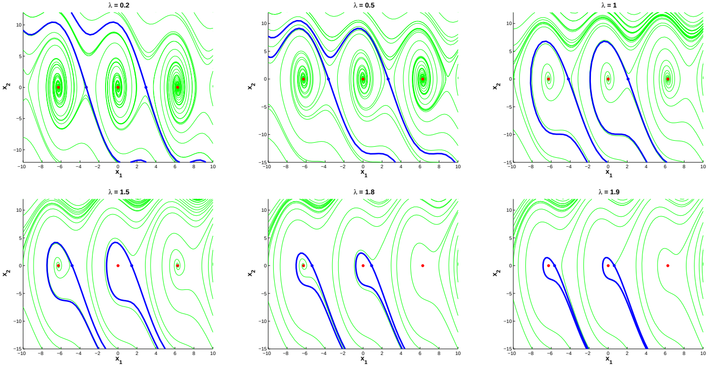
$$\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = \begin{bmatrix} \sin^{-1}(\lambda/2) \\ 0 \end{bmatrix} \tag{1}$$

- With two equilibrium points (and their periodic images):

$$x_{1s} = \sin^{-1}(\lambda/2)$$
$$x_{1u} = \pi - \sin^{-1}(\lambda/2)$$

Reference: Milano, F., *Power System Modelling and Scripting*, Springer, Heidelberg, in press.

Stability and Region of Attraction



Linear Matrix Inequalities

- $F \in \mathcal{S}^{n\times n}$ is *positive semidefinite* (denoted $F \succeq 0$) if $x^T F x \geq 0$ for all $x \in \mathbb{R}^n$.
- For $A, B \in \mathcal{S}^{n\times n}$, write $A \prec B$ if $A - B \prec 0$. Similar notation holds for $\preceq, \succ$, and $\succeq$.
- Given matrices $\{F_i\}_{i=0}^m \subset \mathcal{S}^{n\times n}$ a *Linear Matrix Inequality* (LMI) is a constraint on $\lambda \in \mathbb{R}^m$ of the form:

$$F_0 + \sum_{k=1}^{m} \lambda_k F_k \succeq 0\,. \tag{2}$$

Semidefinite Programming

- A *Semidefinite Program* (SDP) is an optimization problem with a linear cost, LMI constraints, and matrix equality constraints.
- Given matrices $\{F_k\}_{k=1}^m \subset \mathcal{S}^{n\times n}$ and $c \in \mathbb{R}^m$, a SDP solves the following problem:

$$\begin{array}{ll} \min\limits_{\lambda\in\mathbb{R}^m} & c^T\lambda \\ \text{subject to:} & F_0 + \sum\limits_{k=1}^m \lambda_k F_k \succeq 0 \end{array}$$

Polynomials

- Given $\alpha \in \mathbb{N}^n$, a monomial in n variables is a function $m_\alpha : \mathbb{R}^n \to \mathbb{R}$ defined as $m_\alpha(x) := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.
- The degree of a monomial is defined as $\deg m_\alpha := \sum_{i=1}^{n} \alpha_i$.
- A polynomial is a function $p : \mathbb{R}^n \to \mathbb{R}$ defined as:

$$p := \sum_{\alpha \in \mathcal{A}} c_\alpha m_\alpha = \sum_{\alpha \in \mathcal{A}} c_\alpha x^\alpha \tag{3}$$

- The set of polynomials in n variables $\{x_1, \ldots, x_n\}$ will be denoted $\mathbb{R}[x_1, \ldots, x_n]$ or, more compactly, $\mathcal{R}_n$.
- Define a subset of $\mathcal{R}_n$ as $\mathcal{R}_{n,d} := \{p \in \mathcal{R}_n | \deg p \leq d\}$.

Gram Matrix Representation

- If $p \in \mathcal{R}_{n,2d}$ then there exists a $Q \in \mathcal{S}^{l_z \times l_z}$ such that $p = z_{n,d}^T Q z_{n,d}$ where $l_z = \binom{n+d}{d}$ and

$$z_{n,d} := [1, x_1, x_2, \ldots, x_n, x_1^2, x_1 x_2, \ldots, x_n^2, \ldots, x_n^d]^T \quad (4)$$

- All solutions to $p = z_{n,d}^T Q z_{n,d}$ can be expressed as the sum of a particular solution Q_0 and a homogeneous solution.
- There is a set of linearly independent homogeneous solutions $\{Q_i\}_{i=1}^h$ each of which satisfies $z_{n,d}^T Q_i z_{n,d} = \theta$.

Gram Matrix Example

- The polynomial $p = 2x_1^4 + 2x_1^3x_2 - x_1^2x_2^2 + 5x_2^4$ can be written as $p = z_{2,2}^T Q z_{2,2}$ where

$$z_{2,2} = \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix}, Q_0 = \begin{bmatrix} 2 & 1 & -0.5 \\ 1 & 0 & 0 \\ -0.5 & 0 & 5 \end{bmatrix}, Q_1 = \begin{bmatrix} 0 & 0 & -0.5 \\ 0 & 1 & 0 \\ -0.5 & 0 & 0 \end{bmatrix}$$

- We can define an affine subspace of symmetric matrices related to p as

$$\mathcal{S}_p = \{Q | z_{n,d}^T Q z_{n,d} = p(x)\} = \left\{ Q_0 + \sum_{i=1}^{h} \lambda_i Q_i \middle| \lambda_i \in \mathbb{R} \right\}$$

Positive Semidefinite Polynomials

- $p \in \mathcal{R}_n$ is *positive semi-definite* (PSD) if $p(x) \geq 0 \, \forall x$.
- The set of PSD polynomials in n variables $\{x_1, \ldots, x_n\}$ will be denoted $\mathcal{P}[x_1, \ldots, x_n]$ or $\mathcal{P}_n$. Also define $\mathcal{P}_{n,d} = \mathcal{P}_n \bigcap \mathcal{R}_{n,d}$.
- Our computational procedures will be based on constructing polynomials which are PSD.
- Objective: Given $p \in \mathcal{R}_n$, we would like a polynomial-time *sufficient* condition for testing if $p \in \mathcal{P}_n$.

Sums of Squares Polynomials

- p is a *sum of squares* (SOS) if there exist polynomials $\{p_i\}_{i=1}^N$ such that $p = \sum_{i=1}^N p_i^2$.
- The set of SOS polynomials in n variables $\{x_1, \ldots, x_n\}$ will be denoted $\Sigma[x_1, \ldots, x_n]$ or Σ_n.
- If p is SOS then p is PSD. In general $\Sigma_{n,d} \subset \mathcal{P}_{n,d}$.
- **Theorem:** $p \in \Sigma_{n,2d}$ iff there exists $Q \succeq 0$ such that $p = z_{n,d}^T Q z_{n,d}$.

Reference: Parrilo, P., *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*, Caltech, 2000.

SOS Example

- $p = 2x_1^4 + 2x_1^3 x_2 - x_1^2 x_2^2 + 5x_2^4$ is SOS since $Q_0 + \lambda_1 Q_1 \succeq 0$ for $\lambda_1 = 5$.
- An SOS decomposition can be constructed from a Cholesky factorization:

$$Q + \lambda_1 Q_1 = L^T L$$

  where:

$$L = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 1 & -3 \\ 0 & 3 & 1 \end{bmatrix}$$

- Thus $p = (Lz)^T (Lz) == \frac{1}{2}(2x_1^2 - 3x_2^2 + x_1 x_2)^2 + \frac{1}{2}(x_3^2 + 3x_1 x_2)^2$

Connection to LMIs

Checking if a given polynomial is a SOS can be done by solving a LMI feasibility problem.

1. Let Q_0 be a particular solution of $p = z^T Qz$ and let $\{Q_i\}_{i=1}^h$ be a basis for the homegeneous solutions.
2. p is a SOS iff there exists $\lambda \in \mathbb{R}^h$ such that $Q_0 + \sum_{i=1}^h \lambda_i Q_i \succeq 0$

SOS programming

- Given $c \in \mathbb{R}^m$ and polynomials $\{p_k\}_{k=0}^m$ solve:

$$\begin{aligned} \min_{\alpha \in \mathbb{R}^m} \quad & c^T \alpha \\ \text{subject to:} \quad & p_0 + \sum_{k=1}^{m} \alpha_k p_k \in \Sigma[x] \end{aligned}$$

- This SOS programming problem is an SDP:
  - The cost is a linear function of α.
  - The SOS constrraint can be replaced with a LMI constraint.

Basic Algebraic Geometry

- Given $\{g_1, \dots, g_t\} \in \mathcal{R}_n$, the **Multiplicative Monoid** generated by g_j's is

$$\mathcal{M}(g_1, \dots, g_t) = \{g_1^{k_1} g_2^{k_2} \dots g_t^{k_t} \big| k_1, \dots, k_t \in \mathbb{Z}_+\}$$

- Given $\{f_1, \dots, f_s\} \in \mathcal{R}_n$, the **Cone** generated by f_j's is

$$\mathcal{P}(f_1, \dots, f_s) := \left\{ s_0 + \sum s_i b_i \Big| s_i \in \Sigma_n, b_i \in \mathcal{M}(f_1, \dots, f_s) \right\}$$

- Given $\{h_1, \dots, h_u\} \in \mathcal{R}_n$, the **Ideal** generated by h_k's is

$$\mathcal{I}(h_1, \dots, h_u) := \left\{ \sum h_k p_k \Big| p_k \in \mathcal{R}_n \right\}$$

The Positivstellensatz

Given polynomials $\{f_1, \ldots, f_s\}, \{g_1, \ldots, g_t\}$, and $\{h_1, \ldots, h_u\}$ in $\mathcal{R}_n$, the following are equivalent:

1. The set

$$\left\{ x \in \mathbb{R}^n \,\middle|\, \begin{array}{c} f_1(x) \geq 0, \ldots, f_s(x) \geq 0 \\ g_1(x) \neq 0, \ldots, g_t(x) \neq 0 \\ h_1(x) = 0, \ldots, h_u(x) = 0 \end{array} \right\} \tag{5}$$

   is empty.

2. There exist polynomials $f \in \mathcal{P}(f_1, \ldots, f_s)$, $g \in \mathcal{M}(g_1, \ldots, g_t)$, and $h \in \mathcal{I}(h_1, \ldots, h_u)$ such that

$$f + g^2 + h = 0.$$

Positivstellensatz Certificates

- The LMI based tests for SOS polynomials can be used to prove that the set emptyness condition from the P-satz holds, by finding specific f, g and h such that $f + g^2 + h = 0$.
- These f, g and h are known as P-satz certificates since they certify that the equality holds.

Theorem:

Given polynomials $\{f_1, \ldots, f_s\}, \{g_1, \ldots, g_t\}$, and $\{h_1, \ldots, h_u\}$ in $\mathcal{R}_n$, if the set

$$\{x \in \mathbb{R}^n \big| f_i(x) \geq 0, g_j(x) \neq 0, h_k(x) = 0\}$$

is empty then the search for bounded degree P-satz certificates can be done using SDP. If the degree bound is chosen large enough the SDP will be feasible and give the refutation certificates.

Robust Bifurcation Analysis

- In power systems voltage collapse has its origin in a *saddle-node bifurcation*.
- There are few systematic approaches to the problem of computing bifurcation margins.
- These methods only compute the *locally closest* bifurcations to a given set of nominal parameters.
- We need more powerful methods *guaranteeing a minimum distance* to a singularity.

- The condition for a vector field $f(x, \mu)$ to have a saddle-node bifurcation at (x_0, μ_0) are:

$$\begin{aligned} f &= 0 \qquad & w^* D_\mu f &\neq 0 \\ w^* D_x f &= 0 \qquad & w^* D_x^2 f(v, v) &\neq 0 \end{aligned}$$

- In the polynomial case, the set where bifurcation occur is semialgebraic, since it can be described in the form described by the P-satz Theorem.
- If the problem contains *nonalgebraic* elements, it might be possbile to convert a non-polynomial system into a ational system.

Reference: Parrilo, P., *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*, Caltech, 2000.

- The system operates at $(P_0, Q_0, \alpha_0, V_0) = (0.5, 0.3, -0.13, 0.90)$
- Define $x := \sin(\alpha)$ and $y = \cos(\alpha)$.
- We want to minimize the function:

$$J(P, Q) = (P - 0.5)^2 + (Q - 0.3)^3$$

subject to the conditions:

$$\begin{aligned}
f1 &:= x^2 + y^2 - 1 = 0 \\
f2 &:= -4Vx - P = 0 \\
f3 &:= -4V^2 + 4Vy - Q = 0 \\
f4 &:= \det J = -16V(x^2 + y^2 - 2Vy) = 0
\end{aligned}$$

- Consider the problem of veryfing the implication

$$\{f_1(x) = 0, f_2(x) = 0, f_3(x) = 0, f_4(x) = 0\} \Rightarrow b(x) \geq 0$$

- The implication is true iff the following set is empty:

$$\{x \big| f_1(x) = 0, f_2(x) = 0, f_3(x) = 0, f_4(x) = 0, -b(x) \geq 0, b(x) \neq 0\}$$

- By the P-satz theorem this is true iff there exists polynomials $s_1, s_2 \in \Sigma_4$ and $p_1, \ldots, p_4 \in \mathcal{R}_4$ such that:

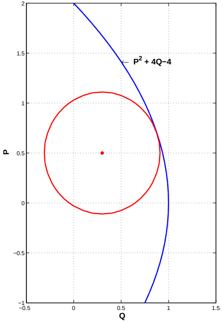
$$s_1 - s_2 b + \sum_{i=1}^{4} p_i f_i + b^{2k} = 0$$

$$s_1 - s_2 b + \sum_{i=1}^{4} p_i f_i + b^{2k} = 0$$

- Take $s_1(x) = 0, k = 1$,
  and $p_i(x) = b(x) r_i(x), i = 1, \ldots, 4$,
  in which case:

$$b(x) + \sum_{i=1}^{4} r_i f_i \in \Sigma_n$$

- Take
  $b(x) = J(P, Q) - \gamma$ and maximize over γ!



Dynamic Stability Framework

- Assume an autonomous nonlinear system of the form

$$\dot{z} = f(z, \mu)\,, \tag{6}$$

  where $z \in \mathbb{R}^n$ and for which we assume $f(0, \mu) = 0$.
- We want to assess the stability of its equlibrium fixed points and to estimate their region of attraction.
- **Idea:** Cast the Lyapunov stability arguments into SOS programming problems.
- Design controllers, perform disturbance analysis, etc.

Local Lyapunov Stability

Theorem For an open set $\mathcal{D} \subset \mathbb{R}^n$ with $0 \in \mathcal{D}$, suppose there exists a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$ such that

$$
\begin{aligned}
V(0) &= 0\,, \\
V(z) &> 0 \qquad \forall z \in \mathcal{D}\,, \\
\frac{\partial V}{\partial z} f(z) &\leq 0 \qquad \forall z \in \mathcal{D}\,.
\end{aligned}
$$

Then $z = 0$ is a stable equilibrium point of (6). Moreover, any region $\Omega_\beta := \{x \in \mathbb{R}^n \big| V(x) \leq \beta\}$ such that $\Omega_\beta \subseteq \mathcal{D}$ describes an positively invariant region contained in the equilibrium point's domain of attraction.

SOS relaxation

- Suppose that for the system (6) there exists a polynomial function $V(z)$ such that

$$V(0) = 0\,,$$
$$V(z) - \phi(z) \in \Sigma_n\,,$$
$$-\frac{\partial V}{\partial z} f(z) \in \Sigma_n$$

  where $\phi(z) > 0$ for $z \neq 0$. Then the zero equilibrium of (6) is stable.

- Choose $\phi(z) = \sum_{i=1}^{n} \epsilon_i z_i^2$, where $\sum \epsilon_i > \gamma$ with γ a positive number and $\epsilon_i \geq 0$.

Reference: Papachristodoulou, A. and Prajna, S., *Analysis of Non-polynomial systems Using the Sum of Squares Decomposition*, Positive Polynomials in Control, pp. 23-43, 2005.

Recasting Methodology for Non-polynomial vector fields

- Consider again the one-machine infinite-bus system:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= 10\lambda(1-\cos(x_1)) - 20\cos(x_{1s})\sin(x_1) - x_2\end{aligned}$$

- Define $x_3 = \sin(x_1)$ and $x_4 = 1 - \cos(x_1)$.

$$\dot{x}_1 = x_2 \tag{7}$$
$$\dot{x}_2 = 10\lambda x_4 - 20\cos(x_{1s})x_3 - x_2 \tag{8}$$
$$\dot{x}_3 = (1 - x_4)x_2 \tag{9}$$
$$\dot{x}_4 = x_3 x_2 \tag{10}$$

and introduce an equality constraint $x_3^2 + (1 - x_4)^2 = 1$.

- Generally, for a non-polynomial system $\dot{z} = f(z, \mu)$ the recasted system is written as:

$$\dot{\tilde{x}}_1 = f_1(\tilde{x}_1, \tilde{x}_2)\,,$$
$$\dot{\tilde{x}}_2 = f_2(\tilde{x}_1, \tilde{x}_2)\,,$$

  where $\tilde{x}_1 = (x_1, \ldots, x_n) = z$ are the original state variables, $\tilde{x}_2 = (x_{n+1}, \ldots, x_{n+m})$ are the new variables.
- The constraints arising directly from the recasting process are

$$\tilde{x}_2 = F(\tilde{x}_1)$$

  and those arising indirectly

$$G_1(\tilde{x}_1, \tilde{x}_2) = 0\,,$$
$$G_2(\tilde{x}_1, \tilde{x}_2) \geq 0\,.$$

Extension of Lyapunov Stability Theorem

- Let $\mathcal{D}_1 \subset \mathbb{R}^n$ and $\mathcal{D}_1 \subset \mathbb{R}^n$ be open sets such that $0 \in \mathcal{D}_1$ and $F(\mathcal{D}_1) \subseteq \mathcal{D}_2$.
- Assume that $\mathcal{D}_1 \times \mathcal{D}_2$ is a semialgebraic set defined by the following inequalities:

$$\mathcal{D}_1 \times \mathcal{D}_2 = \{(\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}^n \times \mathbb{R}^m : G_{\mathcal{D}}(\tilde{x}_1, \tilde{x}_2) \geq 0\}.$$

Reference: Papachristodoulou, A. and Prajna, S., *Analysis of Non-polynomial systems Using the Sum of Squares Decomposition*, Positive Polynomials in Control, pp. 23-43, 2005.

Proposition

Suppose that for the system (7) and the functions $F(\tilde{x}_1)$, $G_1(\tilde{x}_1, \tilde{x}_2)$, $G_2(\tilde{x}_1, \tilde{x}_2)$, and $G_{\mathcal{D}}(\tilde{x}_1, \tilde{x}_2)$ there exists polynomial functions $\lambda_{1,2}(\tilde{x}_1, \tilde{x}_2)$, and SOS polynomials $\sigma_i(\tilde{x}_1, \tilde{x}_2)$, such that

$$\begin{aligned}
&V(0, \tilde{x}_{2,0}) = 0\,, \\
&V - \lambda_1^T G_1 - \sigma_1^T G_2 - \sigma_2^T G_{\mathcal{D}} - \phi \in \Sigma_n\,, \\
&-\left(\frac{\partial V}{\partial \tilde{x}_1} f_1 + \frac{\partial V}{\partial \tilde{x}_2} f_2\right) - \lambda_2^T G_1 - \sigma_3^T G_2 - \sigma_4^T G_{\mathcal{D}} \in \Sigma_n\,,
\end{aligned}$$

where $\phi(\tilde{x}_1, F(\tilde{x}_2) > 0$ for $\forall \tilde{x}_1 \in \mathcal{D}_1 \setminus 0$, then $z = 0$ is a stable equilibrium of (6).

Example: one-machine infinite-bus system

- Define an equality constraint: $G_1 := x_3^2 + x_4^2 - 2x_4$.
- Define $\mathcal{D}_1 \times \mathcal{D}_2$ as:

$$G_{\mathcal{D}}(1) = \beta^2 - (x_1^2 + x_2^2) \geq 0$$
$$G_{\mathcal{D}}(2) = (x_3 - \sin(\beta))(x_3 + \sin(\beta)) \geq 0$$

- Define $\phi(\tilde{x}_1, \tilde{x}_2) = \sum_{i=1}^{4} \epsilon_i x_i^2$ with $\epsilon_i \geq 0$.

Thank you Antonis!

- Solve the following optimization problem:

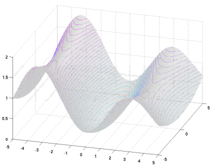
$$\max_{\epsilon,\lambda\in\mathcal{R}_4,\sigma\in\Sigma_4} \beta$$

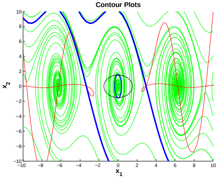
$$\text{subject to:} \qquad V - \lambda_1 G_1 - \sigma_1 G_{\mathcal{D}}(1) - \sigma_1 G_{\mathcal{D}}(1) - \phi \succeq 0$$

$$-\frac{dV}{dt} - \lambda_2 G_1 - \sigma_3 G_{\mathcal{D}}(1) - \sigma_4 G_{\mathcal{D}}(1) \succeq 0$$

- We find for $\beta = 1.5$

$$\begin{aligned} V = {} & 0.0020275x_1^2 - 0.0042255x_1 \sin(x_1) - 0.04157x_1(1-\cos(x_1)) \\ & - 0.0001238x_1 + 0.014573x_2^2 + 0.0029823x_2 \sin(x_1) \\ & - 0.00034485x_2(1-\cos(x_1)) + 0.20613 \sin(x_1)^2 \\ & + 0.016014 \sin(x_1)(1-\cos(x_1)) + 0.2033(1-\cos(x_1))^2 \\ & + 0.17784(1-\cos(x_1)) \end{aligned}$$




Contour Plots
x_2
x_1