Continuous Newton's Method for Power Flow Analysis

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Biography

⇒ Federico Milano received from the University of Genoa, Italy, the Electrical Engineering degree and the Ph.D. degree in 1999 and 2003, respectively.

⇒ In 2001 and 2002, he worked at the University of Waterloo, Canada, as a Visiting Scholar.

⇒ He is currently a tenure associate professor at the University of Castilla-La Mancha, Ciudad Real, Spain.

⇒ His research interests include voltage stability, electricity markets and computer-based power system modelling.

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Power Flow Problem

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Background (I)

⮕ A classical problem of circuit theory is to find all branch currents and all node voltages of an assigned circuit.

⮕ Typical input data are generator voltages as well as the impedances of all branches.

⮕ If all impedances are constant, the resulting set of equations that describe the circuit is **linear**.

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Example 1 (I)

⟶ Classical circuit problem.
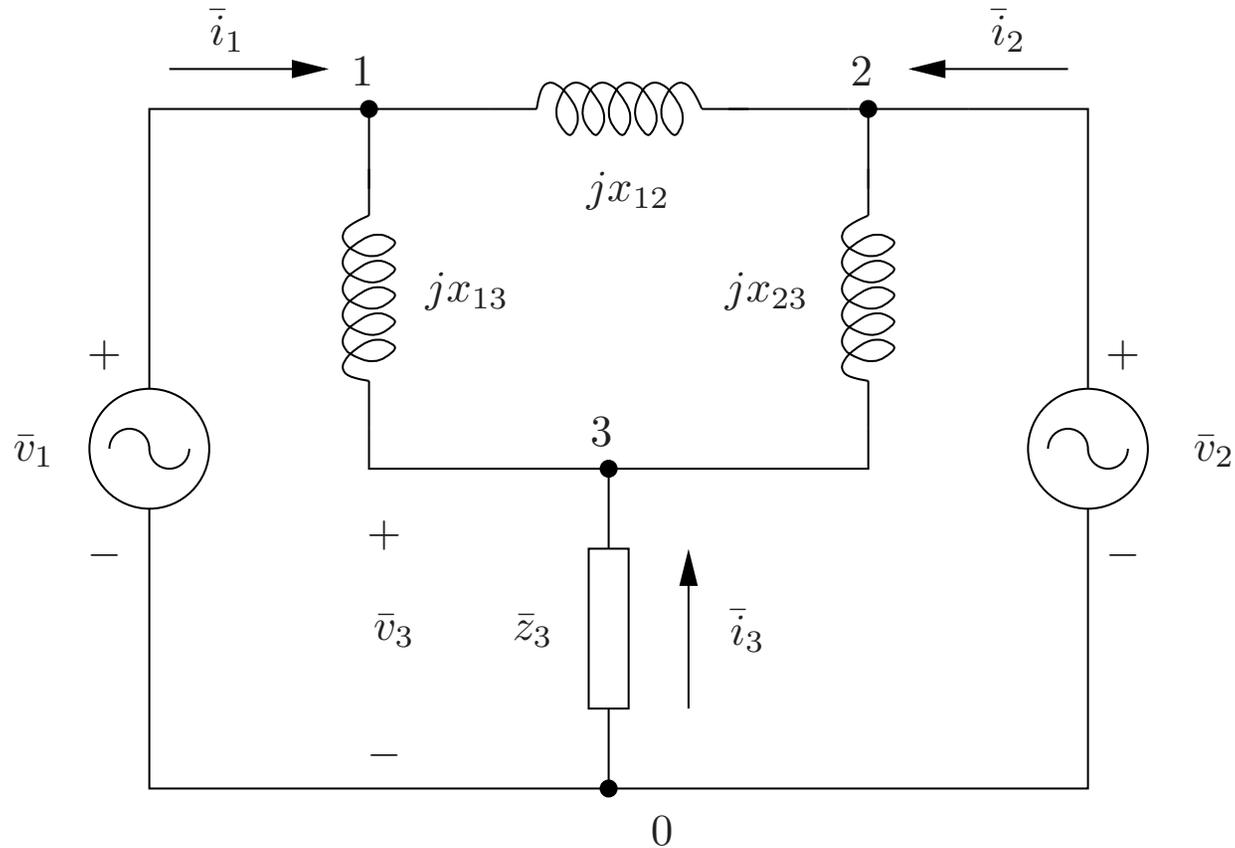
Example 1 (II)

➠ Using the branch current method, one has:

$$
\begin{aligned}
0 &= \frac{\bar{v}_1 - \bar{v}_2}{jx_{12}} + \frac{\bar{v}_1 - \bar{v}_3}{jx_{13}} - \bar{i}_1 \\
0 &= \frac{\bar{v}_2 - \bar{v}_1}{jx_{12}} + \frac{\bar{v}_2 - \bar{v}_3}{jx_{23}} - \bar{i}_2 \\
0 &= \frac{\bar{v}_3 - \bar{v}_1}{jx_{13}} + \frac{\bar{v}_3 - \bar{v}_2}{jx_{23}} - \bar{i}_3
\end{aligned}
\tag{1}
$$

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Example 1 (III)

➡ In vector form:

$$\begin{bmatrix} \bar{i}_1 \\ \bar{i}_2 \\ 0 \end{bmatrix} = \left(\bar{Y} + I_3 \begin{bmatrix} 0 \\ 0 \\ 1/\bar{z}_3 \end{bmatrix} \right) \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \bar{v}_3 \end{bmatrix} = \bar{Y}_{\text{tot}} \bar{v} \qquad (2)$$

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Example 1 (IV)

➡ where $\boldsymbol{I}_3$ is a 3×3 identity matrix and $\bar{\boldsymbol{Y}}$ is the so-called admittance matrix:

$$\bar{\boldsymbol{Y}} = \begin{bmatrix} 1/jx_{12} + 1/jx_{13} & -1/jx_{12} & -1/jx_{13} \\ -1/jx_{12} & 1/jx_{12} + 1/jx_{23} & -1/jx_{23} \\ -1/jx_{13} & -1/jx_{23} & 1/jx_{13} + 1/jx_{23} \end{bmatrix}$$

$$(3)$$

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Background (II)

➠ The power flow problem is conceptually the same problem as solving a steady-state ac circuit.

➠ The only, though substantial, difference is the set of input data.

➠ Loads are expressed in terms of consumed active and reactive powers (*PQ load*) and generators are defined in terms of constant voltage magnitude and active power injection (*PV generator*).

➠ Hence, the power flow problem is **nonlinear**.

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Example 2 (I)

➡ Classical power flow problem.
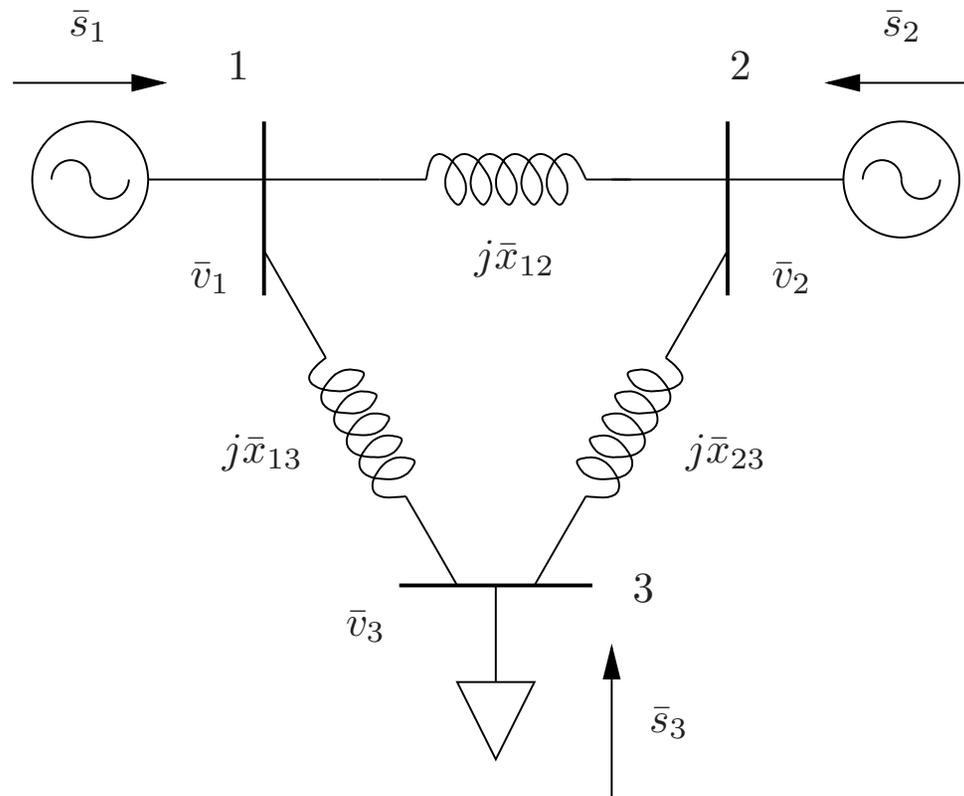
Example 2 (II)

➠ The power flow problem is formulated in order to determine unknown voltage magnitudes and angles.

$$0 = \frac{v_2 v_1}{x_{12}} \sin(\theta_2 - \theta_1) - p_2 \tag{4}$$

$$0 = \frac{v_3 v_1}{x_{13}} \sin(\theta_3 - \theta_1) + \frac{v_3 v_2}{x_{23}} \sin(\theta_3 - \theta_2) - p_3$$

$$0 = \frac{v_3^2}{x_{13}} + \frac{v_3^2}{x_{23}} - \frac{v_3 v_1}{x_{13}} \cos(\theta_3 - \theta_1) - \frac{v_3 v_2}{x_{23}} \cos(\theta_3 - \theta_2) - q_3$$

where the unknowns are v_3, θ_3 and θ_2.

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Rationales Behind Power Flow Problem Formulation

➠ Loads at high voltage level are modelled as constant PQ due to under-load tap changers.

➠ Generators are modelled as PV due to turbine and voltage regulators.

➠ Transmission lines and transformers are generally modelled as lumped π-circuits with constant parameters.

➠ Observe that one bus has to be the phase angle reference. Typically, one generator is used as *slack* bus.

➠ The *distributed* slack bus model is more physical.

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General Formulation of the Power Flow Problem (I)

⇒ The vector of currents injected at each node is:

$$\bar{i} = \bar{Y}\bar{v} \tag{5}$$

which leads to write the power flow problem as the complex power injections at buses:

$$\bar{s} = \bar{V}\bar{i}^* = \bar{V}\bar{Y}^*\bar{v}^* \tag{6}$$

where $\bar{V} = \text{diag}(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{n_b})$ and n_b is the number of network buses.

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General Formulation of the Power Flow Problem (II)

➠ Rewriting (6):

$$0 = \bar{s} - \bar{V}\bar{Y}^*\bar{v}^* \tag{7}$$

➠ Or, more in general:

$$0 = g(y) \tag{8}$$

where y and data per bus type are:

Bus type	Variables	Data
Slack generator	p, q	v, θ
PV generator	q, θ	p, v
PQ load	v, θ	p, q

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Relevant Issues of the Power Flow Problem

➥ The origins of the formulation of the power flow problem and the solution based on the Newton's method date back to the late sixties.

➥ Since then, a huge variety of studies have been presented about the solution of the power flow problem, addressing:

→ Starting initial guess

→ Computational efficiency

→ Ill-conditioned cases

→ Robustness

→ Multiple solutions

→ Unsolvable cases

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Taxonomy of the Power Flow Problem

⇒ It is relevant to classify the power flow problems into the following categories:

→ Well-conditioned case

→ Ill-conditioned case

→ Bifurcation point

→ Non-physical solution

→ Unsolvable case

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Well-conditioned case

➡ The power flow solution exists and is reachable using a flat initial guess (e.g., all load voltage magnitudes equal to 1 and all bus voltage angles equal to 0) and a standard Newton's method.

➡ This case is the most common situation.

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Ill-conditioned case

⏩ The solution of the power flow problem does exist, but standard solvers fail to get this solution starting from a flat initial guess.

⏩ This situation is due to the fact that the region of attraction of the power flow solution is narrow or does not contain the initial guess.

⏩ In this case, the failure of standard power flow procedure is due to the instability of the numerical method, not of the power flow equations.

⏩ Robust power flow methods have proved to be efficacious for solving ill-conditioned cases.

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Bifurcation Point

⟶ The solution of the power flow exists but it is either a saddle-node bifurcation or a limit-induced bifurcation.

→ Saddle-node bifurcations are associated with the maximum loading condition of a system. The solution cannot be obtained using standard or robust power flow methods, since the power flow Jacobian matrix is singular at the solution point.

→ Limited-induced bifurcations are associated with a physical limit of the system, such as the shortage of generator reactive power. The solution point is typically a well-conditioned case and does not show convergence issues.

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Non-physical Solution

▥➡ Solutions that cannot be accepted since some variable is out of its technical limits.

▥➡ Typically these solution are characterized by very low voltage levels.

▥➡ These solutions are also known as: *extraneous*, *false*, *lower*, or *unstable*.

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Unsolvable Case

▥➤ The solution of the power flow problem does not exist.

▥➤ Typically, the issue is that the loading level of the network is too high.

▥➤ As in the case of the bifurcation points, a continuation method or an optimal power flow problem allows defining the maximum loading level that the system can supply.

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Existing Power Flow Solution Methods

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Solution Methods (I)

➭ Methods that do not require the computation of the Jacobian matrix of g:

    → Jacobi's method.

    → Gauss-Seidel's method.

➭ Methods that require the computation of the Jacobian matrix of g:

    → Newton's (or Newton-Raphson's) method.

    → Robust Newton's methods.

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Solution Methods (II)

⇒ Methods that simplify the Jacobian matrix of g:

→ Inexact and dishonest Newton's methods.

→ Fast decoupled power flow.

⇒ Methods that simplify g:

→ DC power flow model.

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Newton-Raphson's Method (I)

➡ The i-th iteration of the classical Newton's method for (8) is as follows:

$$
\begin{aligned}
\Delta y^{(i)} &= -[g_y^{(i)}]^{-1} g^{(i)} \\
y^{(i+1)} &= y^{(i)} + \Delta y^{(i)}
\end{aligned}
\qquad (9)
$$

where $g^{(i)} = g(y^{(i)})$, $g_y^{(i)} = g_y(y^{(i)})$, and $g_y = \nabla_y^T g$ is the Jacobian matrix of the power flow equations.

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Newton-Raphson's Method (II)

⇒ The geometrical interpretation of the Newton's method is well-known. For the actual value $y^{(i)}$, one computes the tangent of $g^{(i)}$ as:

$$\boldsymbol{\tau}(\boldsymbol{y}) = \boldsymbol{g}^{(i)} + \boldsymbol{g}_{\boldsymbol{y}}^{(i)}(\boldsymbol{y} - \boldsymbol{y}^{(i)}) \tag{10}$$

⇒ Imposing $\boldsymbol{\tau}(\boldsymbol{y}) = \boldsymbol{0}$ yields the value $y^{(i+1)}$ defined in (9).

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Geometrical Interpretation of the Newton-Raphson's Method

(a)

(b)

(c)

(d)
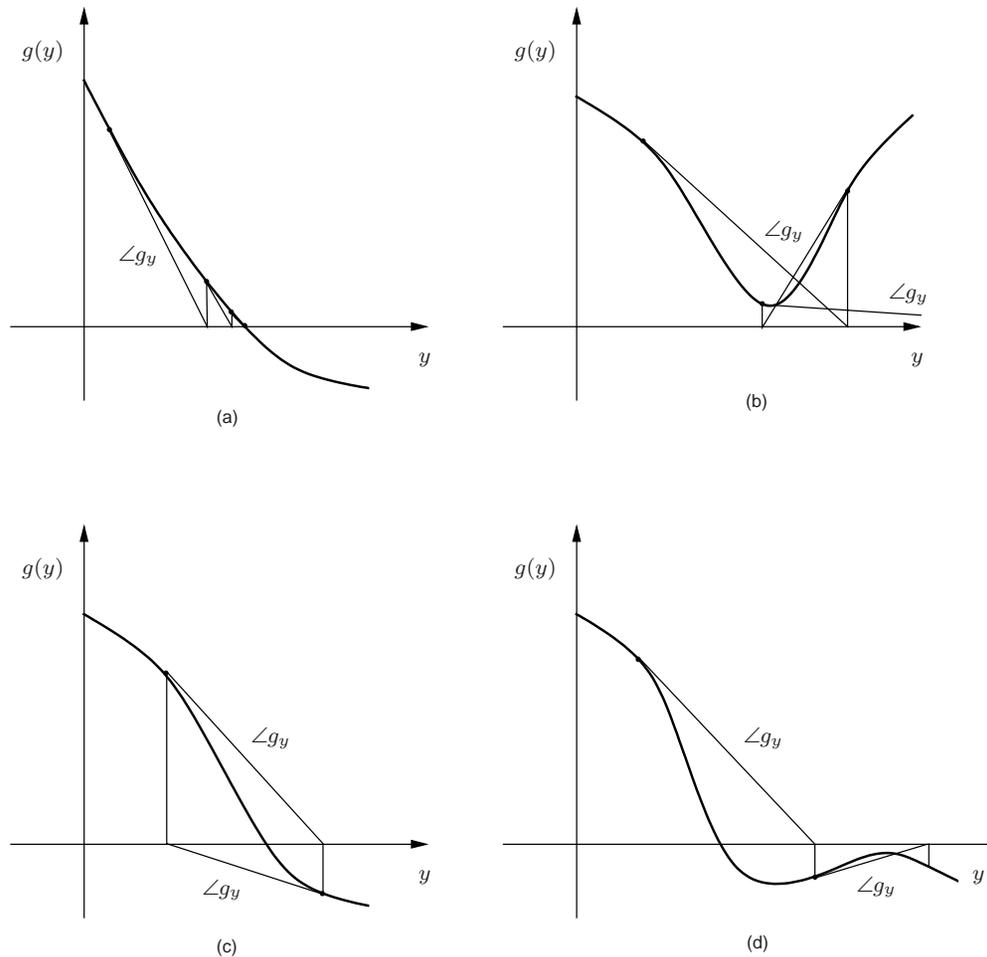
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Robust Newton's Method (I)

➠ There are idiosyncratic cases for which the Newton's technique fails to converge.

➠ A variety of *robust* variations of the basic Newton's method have been proposed in the literature for solving ill-conditioned cases.

➠ The majority of these techniques mainly consist in modifying the first equation of (9) as follows:

$$\Delta \boldsymbol{y}^{(i)} = -\alpha [\boldsymbol{g}_{\boldsymbol{y}}^{(i)}]^{-1} \boldsymbol{g}^{(i)} \qquad (11)$$

where α is a factor that improves the convergence properties of the iterative process.

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Robust Newton's Method (II)

➧ If α is the result of an optimization process, α is called *optimal multiplier* (e.g., Iwamoto's method).

➧ It is important not to confuse ill-conditioned cases with those that are unsolvable since the solution does not exist.

➧ Robust solvers are useful in case of ill-conditioned systems but do not generally work well for unsolvable cases.

➧ Unsolvable cases are better tackled using the continuation power flow technique.

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Geometrical Interpretation of the Robust Newton's Method

(a)

(b)
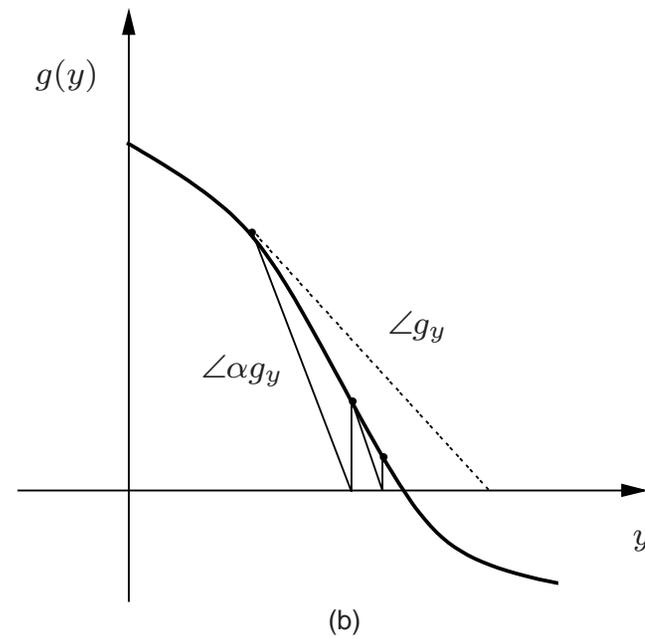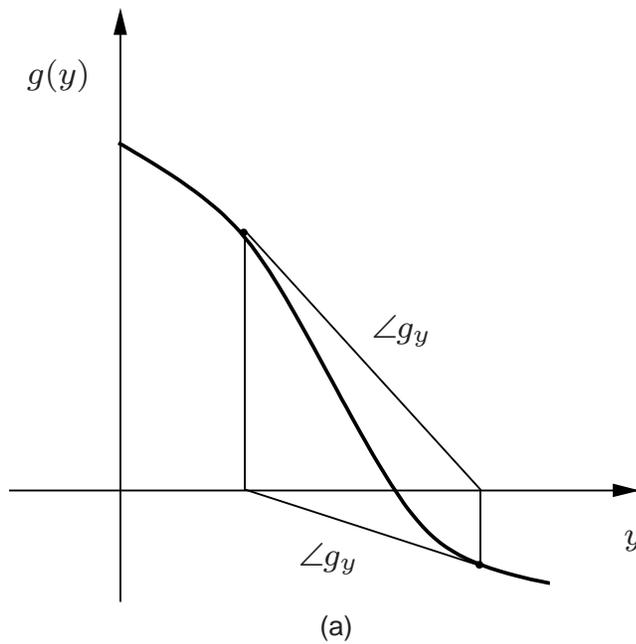
Main Issue of the Newton's Method

⇒ One of the most relevant drawbacks of the Newton's method is the need of factorizing the full Jacobian matrix at each iteration.

⇒ From the computational point of view, the factorization of a matrix is an order N^3 operation, i.e., the computational weight increases with the cube of the size N of the matrix.

⇒ The computational effort can be reduced to $N^{1.5}$ if using sparse matrices techniques, which allows saving a considerable time for large systems (e.g., thousands of buses).

⇒ However, the Jacobian matrix factorization remains the most critical issue of the Newton's method (about 85% of the total CPU time for networks with thousands of buses).

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Inexact and Dishonest Newton's Method

➠ Inexact methods aim to approximate the factorization of the Jacobian matrix.

➠ A family of inexact methods are based on the Generalized Minimal Residual (GMRES) method.

➠ The GMRES is a particular case of Krylov's subspace. The main issue is to properly pre-conditioning the Jacobian matrix.

➠ Dishonest methods compute the Jacobian matrix factorization only the first one or two iterations and then use the previous factorization for the remaining iterations.

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Geometrical Interpretation of the Dishonest Newton's Method
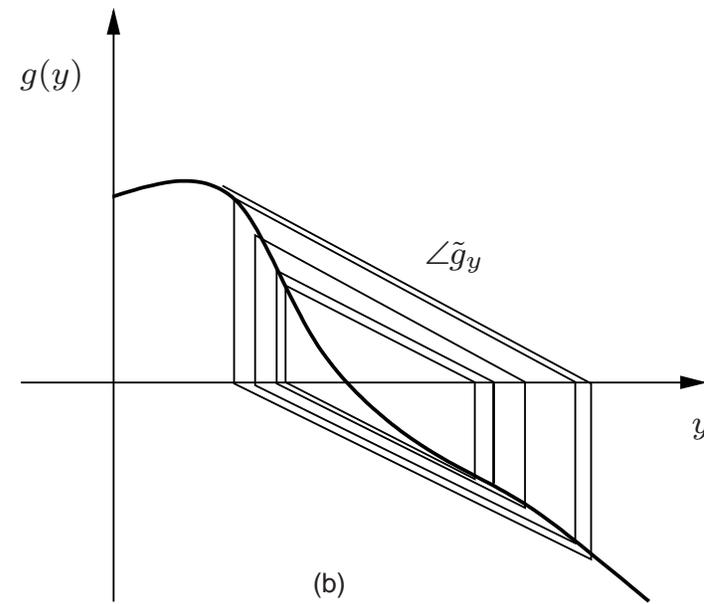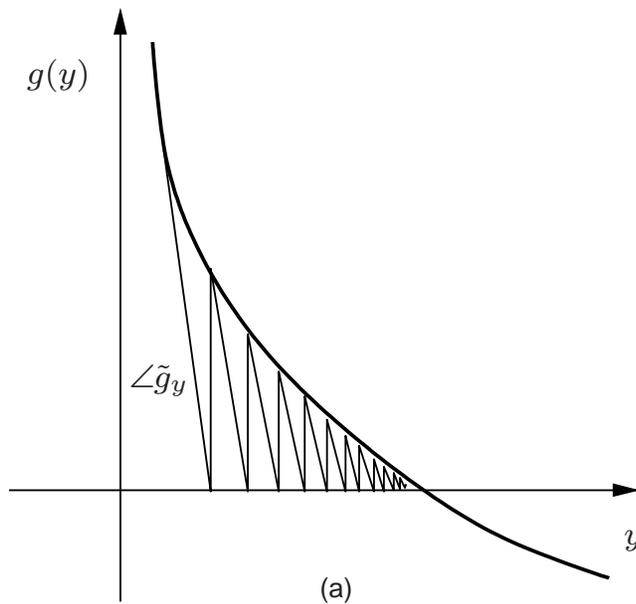
Fast Decoupled Power Flow (FDPF)

➠ The FDPF is a particular case of dishonest Newton's method.

➠ The Jacobian matrix is approximated so that it becomes block diagonal and all non-zero elements are constant. Hence only one factorization is needed.

➠ The FDPF requires much more iterations than the NR method but proved to be more robust.

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Comparison of Power Flow Solution Methods (I)

	Newton		Jacobi		Gauss-Seidel	
Bus #	Iter. #	time [s]	Iter. #	time [s]	Iter. #	time [s]
14	4	0.0050	76	0.0217	56	0.0288
118	5	0.0287	580	0.505	388	2.738
1228	5	0.210	454	5.120	224	112.4
11856	4	3.15	340	399.0	173	9112

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Comparison of Power Flow Solution Methods (II)

Bus #	GMRES		Dishonest		FDPF	
	Iter. #	time [s]	Iter. #	time [s]	Iter. #	time [s]
14	4	0.4339	7	0.0040	6	0.0053
118	7	53.53	15	0.0183	6	0.0117
1228	n.a.	n.a.	26	0.207	12	0.160
11856	n.a.	n.a.	10	3.820	5	5.174

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Region of Attraction (I)

➡ A key issue of any iterative technique is the initial guess.

➡ The only way to know if a given initial guess is adequate for obtaining a solution y_0 of the power flow problem is to determine the region of attraction of y_0. At this regard, the initial guess can be of two types:

→ The initial guess is inside the region of attraction of the solution y_0 and the numerical method converges.

→ The initial guess is outside the region of attraction of the solution y_0. By definition, a numerical method diverges or converges to a non-physical solution if one starts with such initial guess.

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Example 3 (I)

➡ Consider the following simple 2-bus system.
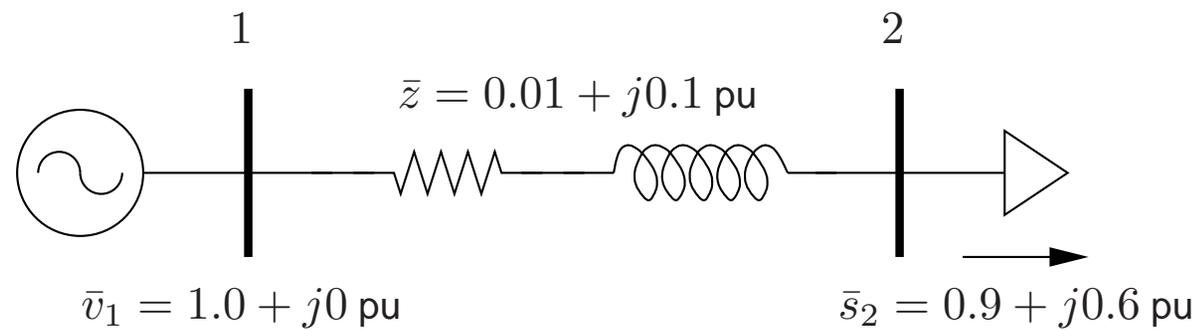
Example 3 (II)

➠ Region of attraction of the Newton's method for a 2-bus system.

Region of Attraction (II)

➠ Different methods have different region of attractions.

➠ Unfortunately defining the region of attraction is extremely costly.

➠ In practice it is virtually impossible to define the region of attraction for a real-world system.

➠ *Robust* methods can be thought as a way of *enlarging* the region of attraction.

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Continuous Newton's Method

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Ordinary Differential Equations (ODE)

➠ Let us consider a set of autonomous ODE:

$$\dot{y} = \tilde{f}(y) \tag{12}$$

The simplest method of numerical integrating (12) is the explicit Euler's method, as follows:

$$\Delta y^{(i)} = \Delta t \tilde{f}(y^{(i)}) \tag{13}$$
$$y^{(i+1)} = y^{(i)} + \Delta y^{(i)}$$

where Δt is a given step length.

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Newton-Raphson's Method (reprise)

➭ Let's recall the i-th iteration of the classical Newton's method:

$$\Delta y^{(i)} \;=\; -[g_y^{(i)}]^{-1} g^{(i)} \tag{14}$$
$$y^{(i+1)} \;=\; y^{(i)} + \Delta y^{(i)}$$

where $g^{(i)} = g(y^{(i)})$, $g_y^{(i)} = g_y(y^{(i)})$, and $g_y = \nabla_y^T g$ is the Jacobian matrix of the power flow equations.

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Analogy between Euler's and Newton's Methods

▐▶ The analogy between (14) and (13) is straightforward if one defines:

$$\tilde{\boldsymbol{f}}(\boldsymbol{y}) = -[\boldsymbol{g_y}]^{-1}\boldsymbol{g}(\boldsymbol{y}) \tag{15}$$

Equations (14) can thus be viewed as the ith step of the Euler's forward method where $\Delta t = 1$.

▐▶ Furthermore, robust Newton's methods are nothing but the ith step of the Euler's integration method where $\Delta t = \alpha$.

▐▶ In other words, the computation of the optimal multiplier α corresponds to a variable step forward Euler's method.

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Continuous Newton's Method

➠ Equations (12) and (15) leads to:

$$\dot{y} = -[g_y]^{-1}g(y) \tag{16}$$

which is known as *continuous Newton's method*.

➠ The equilibrium point y_0 of (16) is

$$0 = \tilde{f}(y_0) = -[g_y|_0]^{-1}g(y_0) \tag{17}$$

Thus, assuming that g_y is not singular, y_0 is also the solution of the power flow problem.

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Stability of the Continuous Newton's Method (I)

▥➡ Differentiating (15) with respect to y leads to:

$$
\begin{aligned}
\tilde{f}_{\boldsymbol{y}} &= \nabla_{\boldsymbol{y}}^T \tilde{f}(\boldsymbol{y}) && (18) \\
&= -[\boldsymbol{g_y}]^{-1}\boldsymbol{g_y} - (\nabla_{\boldsymbol{y}}^T([\boldsymbol{g_y}]^{-1}))\boldsymbol{g}(\boldsymbol{y}) \\
&= -\boldsymbol{I}_{n_y} - (\nabla_{\boldsymbol{y}}^T([\boldsymbol{g_y}]^{-1}))\boldsymbol{g}(\boldsymbol{y})
\end{aligned}
$$

where $\boldsymbol{I}_{n_y}$ is the identity matrix of order n_y. Since the equilibrium point $\boldsymbol{y}_0$ is a solution for $\boldsymbol{g}(\boldsymbol{y}_0) = 0$, one has:

$$
\tilde{f}_{\boldsymbol{y}}|_0 = -\boldsymbol{I}_{n_y} \tag{19}
$$

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Stability of the Continuous Newton's Method (II)

⇒ Equation (19) implies that all eigenvalues of $\tilde{f}_y$ at the solution point are equal to -1.

⇒ (19) means that the solution of (16), if exists, is asymptotically stable.

⇒ The reachability of this solution depends on the starting point $y(t_0) = y^{(0)}$, which has to be within the *region of attraction* or *stability region* of the equilibrium point y_0.

⇒ The continuous Newton's method is expected to show better ability to converge than other methods previously discussed if the initial guess is within the region of attraction.

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Example 4

➠ The previous result is straightforward for a scalar $g(y)$, i.e. for $y \in \mathbb{R}$ and $g \in \mathbb{R}$, as follows:

$$\dot{y} \quad = \quad \tilde{f}(y) = -\frac{g(y)}{g_y(y)} \tag{20}$$

$$\Rightarrow \quad \tilde{f}_y(y) \quad = \quad -\frac{g_y(y)}{g_y(y)} + \frac{g_{yy}(y)}{g_y^2(y)} g(y) \tag{21}$$

$$= \quad -1 + \frac{g_{yy}(y)}{g_y^2(y)} g(y)$$

thus $\tilde{f}_y(y_0) = -1$ if $g(y_0) = 0$ and $g_y(y_0) \neq 0$.

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A General Framework for Power Flow Solvers

➠ It is well-known that the forward Euler's method, even with variable time step, can be numerically unstable.

➠ Given the analogy between the power flow equations and an ODE system, any well-assessed numerical method can be used for integrating (16).

➠ It is thus intriguing to use some efficient integration method for solving (16).

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Runge-Kutta Formulas

⇒ Since computing the Jacobian matrix of (16) is complex, explicit ODE integration methods should be preferred.

⇒ Runge-Kutta formulas are good candidates.

⇒ For example the classic RK4 formula is as follows:

$$
\begin{aligned}
\boldsymbol{k}_1 &= \boldsymbol{f}(\boldsymbol{x}^{(i)}) \\
\boldsymbol{k}_2 &= \boldsymbol{f}(\boldsymbol{x}^{(i)} + 0.5\Delta t \boldsymbol{k}_1) \\
\boldsymbol{k}_3 &= \boldsymbol{f}(\boldsymbol{x}^{(i)} + 0.5\Delta t \boldsymbol{k}_2) \\
\boldsymbol{k}_4 &= \boldsymbol{f}(\boldsymbol{x}^{(i)} + \Delta t \boldsymbol{k}_3) \\
\boldsymbol{x}^{(i+1)} &= \boldsymbol{x}^{(i)} + \Delta t(\boldsymbol{k}_1 + 2\boldsymbol{k}_2 + 2\boldsymbol{k}_3 + \boldsymbol{k}_4)/6
\end{aligned}
\tag{22}
$$

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Example 5 (I)

▶ Let consider the 1254-bus model of the UCTE system.

Example 5 (II)

➠ Comparison of methods for solving the power flow of the UCTE system.

Method	# Iter. $\epsilon = 10^{-3}$	# Iter. $\epsilon = 10^{-4}$	# Iter. $\epsilon = 10^{-5}$
Standard NR	-	-	-
Fast Decoupled PF	-	-	-
Iwamoto's method	99	320	1021
Simple robust method	31	39	47
Runge-Kutta method	10	13	16

where ϵ is the required convergence error tolerance.

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Example 5 (III)

➠ Comparison of convergence errors obtained with different robust power flow solution methods for the UCTE system.

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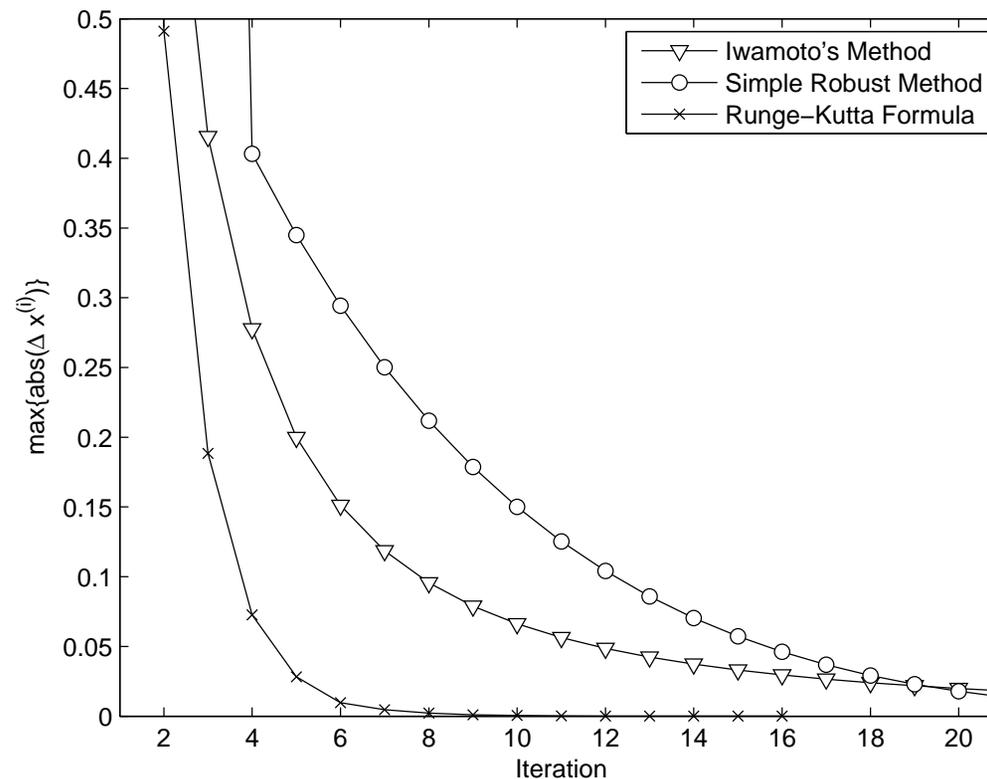
Example 5 (IV)

�➡ Comparison of the computational burden of different solution techniques for the UCTE system.

Method	CPU time (s)
Iwamoto's method	106.5
Simple robust method	3.5
Runge-Kutta method	3.4

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Conclusions

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Contributions

⇒ This talk has presented a continuous version of the Newton's method for solving the power flow problem.

⇒ The main contributions discussed in the talk are:

→ A general framework for applying efficient numerical integration techniques for solving ill-conditioned power flow cases.

→ Proposal of a formal taxonomy of the existing numerical methods for solving the power flow problem.

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Future Work

➡ Future work will concentrate on further developing the analogy between the power flow problem, ODE systems and homotopy methods.

➡ The stability and region of attraction of the continuous Newton's method are promising fields of research.

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References

⟱➡ Further insights and references to all topics covered in the talk can be found in:

F. Milano, Continuous Newton's Method for Power Flow Analysis, IEEE Transactions on Power Systems, Vol. 24, No. 1, pp. 50-57, February 2009.

Available at:

http://www.uclm.es/area/gsee/Web/Federico/publications.htm

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Thanks for your attention!

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Questions?

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