Random Matrix Theory and the Electric Grid

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Outline

- Complex systems: some basics
- Random Matrix Theory and nuclear physics
- Applications to complex systems
- Simple network models
- Possible extensions

The Electric Grid

Generating choices in context

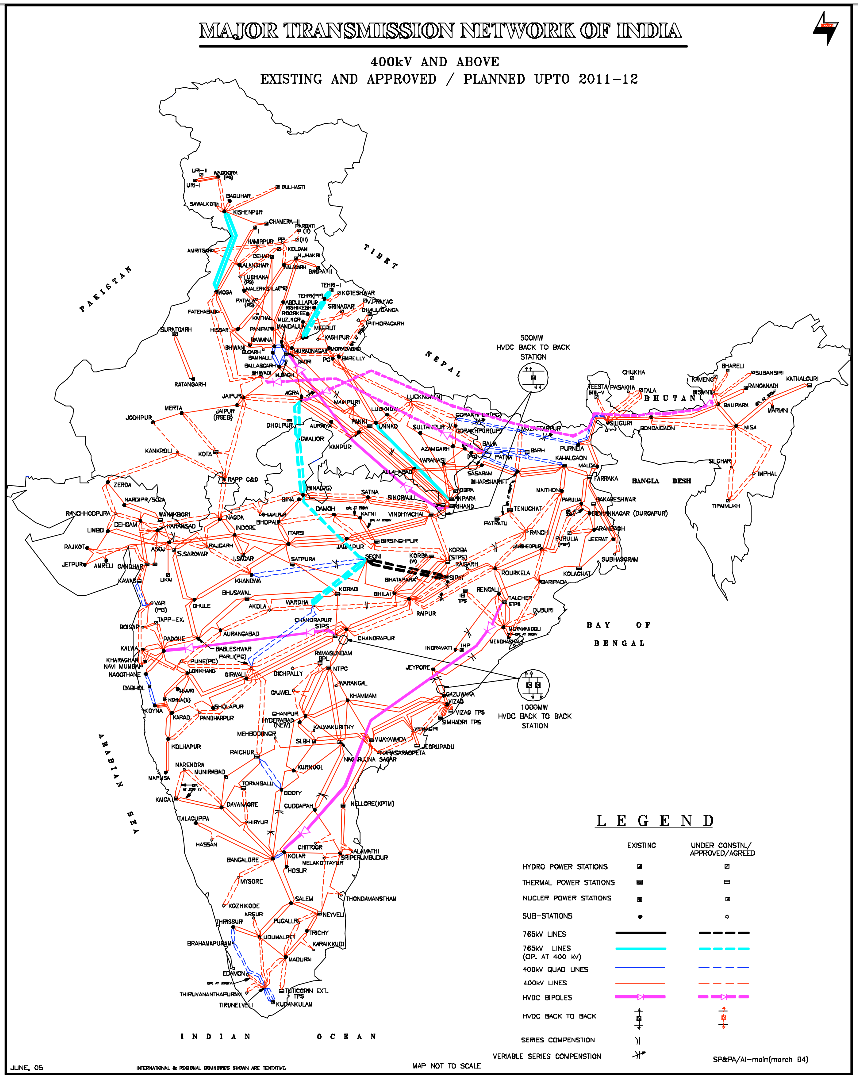
What is an electric grid?

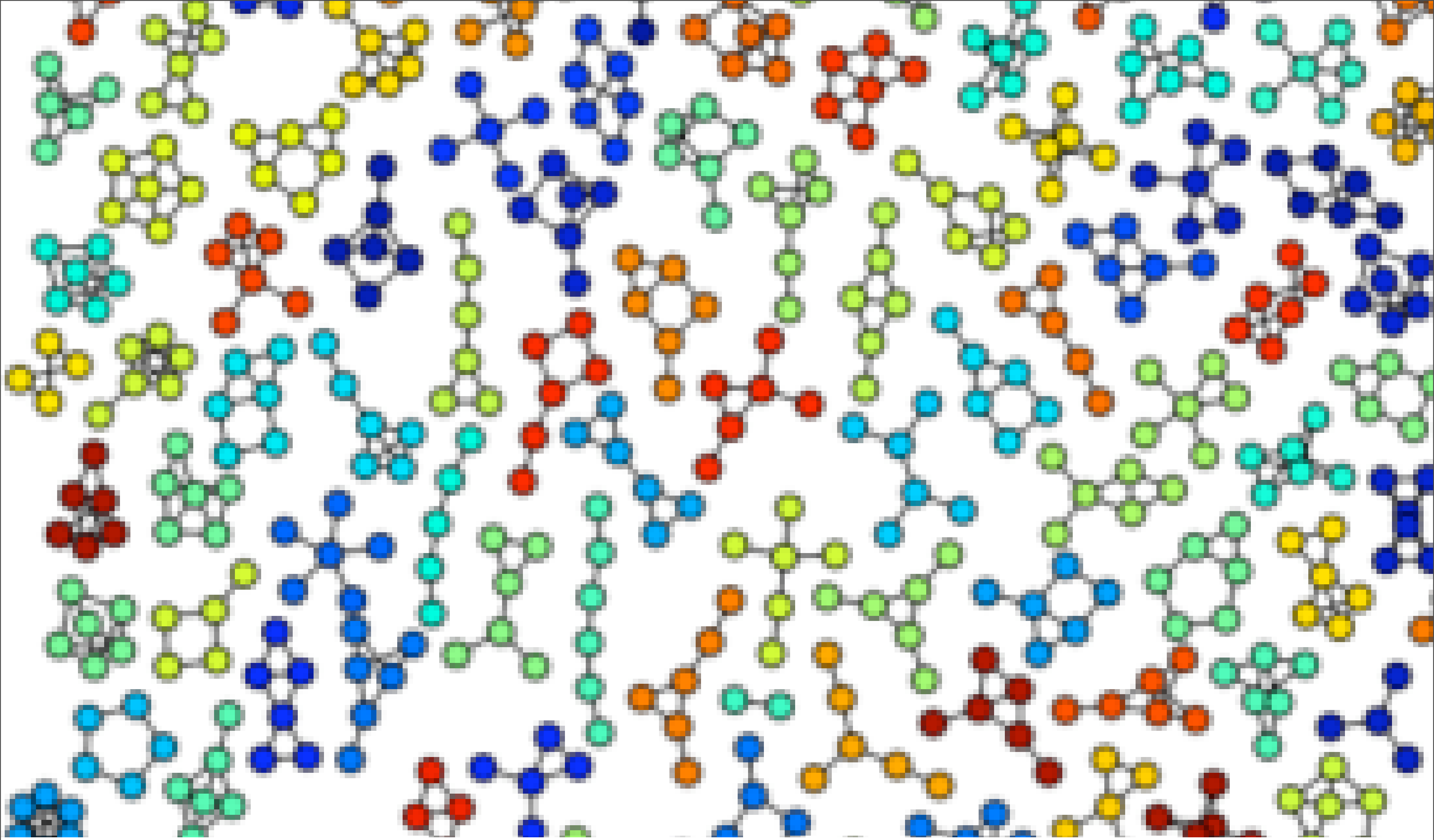
- NODES: generating stations, substations of various types

- EDGES: High-voltage transmission lines

- Interested in network topology, stability.

- Very simple model- no differentiation between sources and sinks, no load.



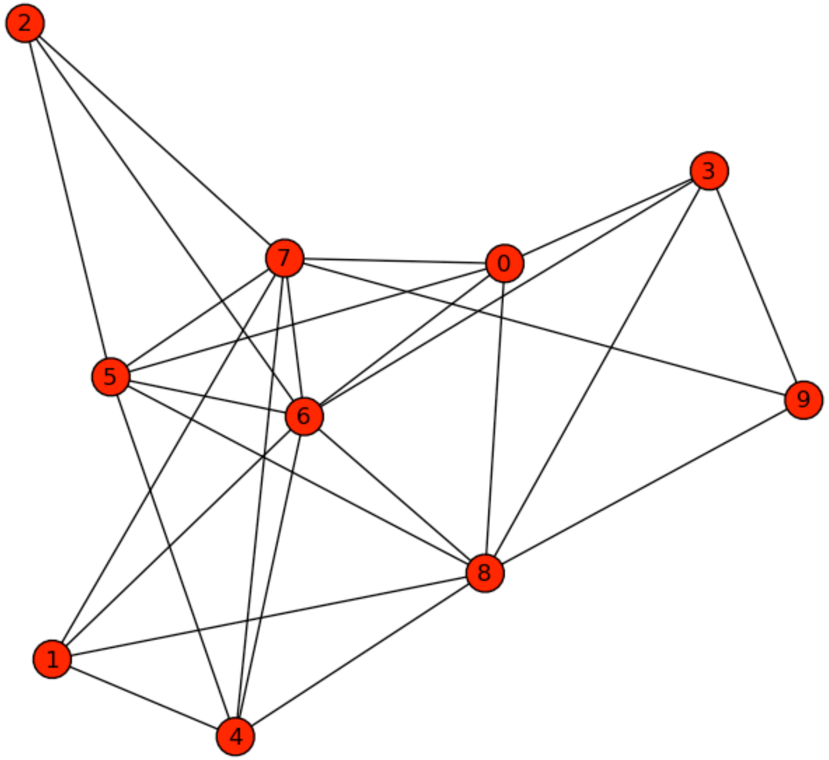


Complex networks

Some basics

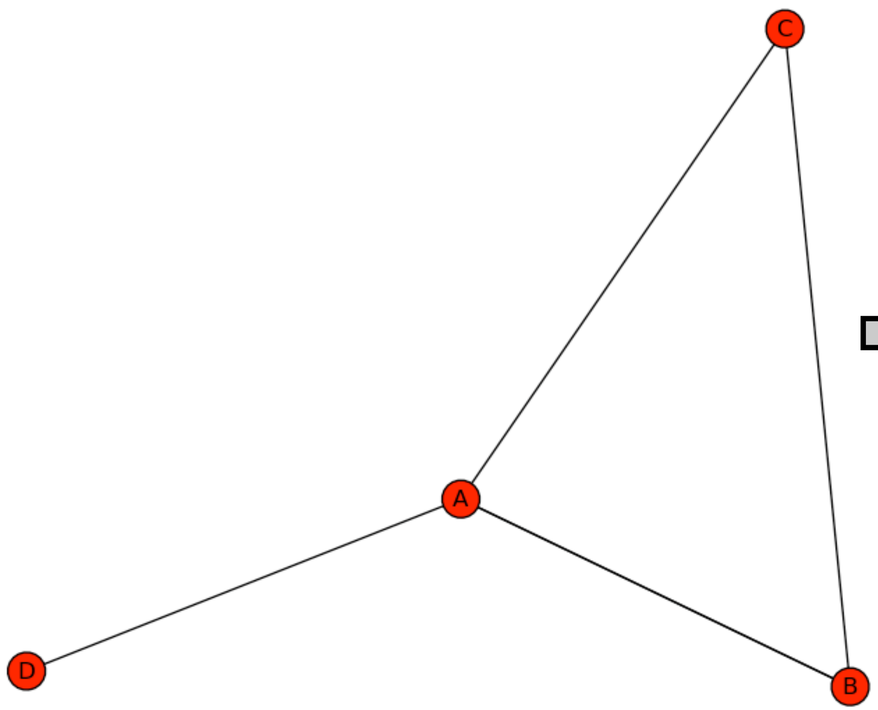
Properties of complex networks

- Number of nodes N
- Degree k: number of edges directly attached to a node
- Degree distribution p(k)
- Scale-free: $p(k) \sim k^{\alpha}$
- Exponential: $p(k) \sim e^{k/\kappa}$
- Often, this is what we can measure.



Adjacency matrices

- $A_{ij} = N$ if nodes i,j are connected by N lines
- 0 otherwise



$$
\begin{array}{c c}
 & \begin{array}{cccc} A & B & C & D \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}
\end{array}
$$

Eigenvalues of A

- Find spectrum of this matrix. Why?

- Eigenvalues and eigenvectors provide a label-independent way of measuring the properties of the network.

- Dm=number of paths that return to starting node after m steps.

- Can prove that $$D_m = \sum_{j=1}^{N} (\lambda_j)^m$$

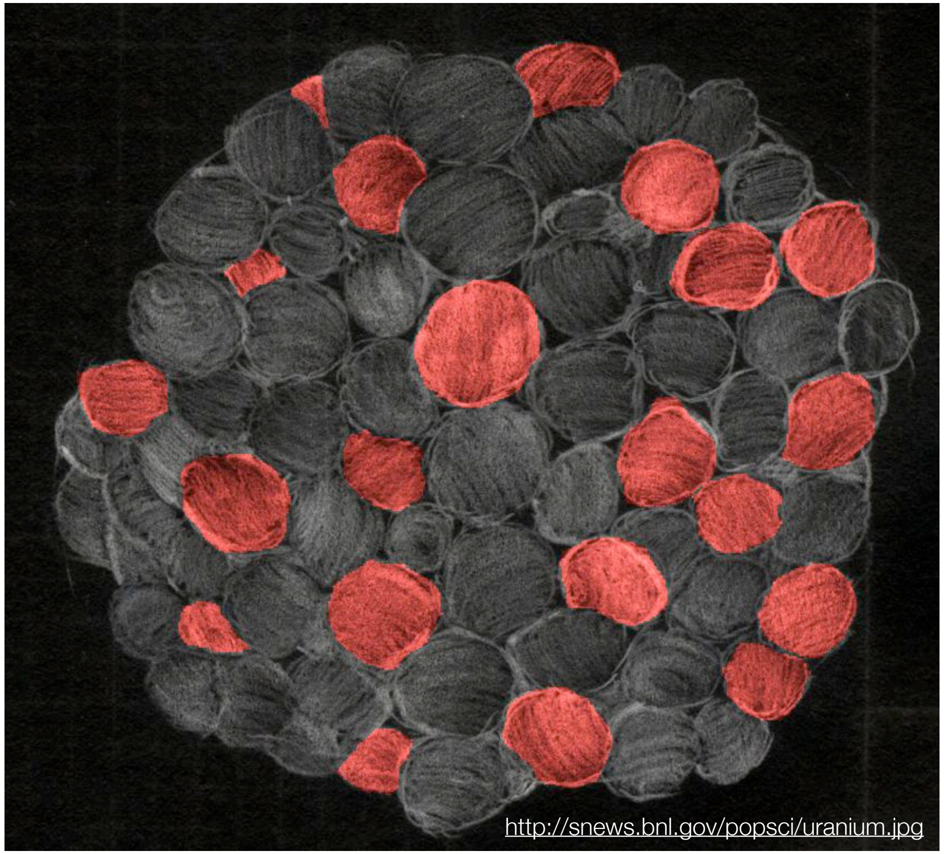
- Eigenvalues tell you something about graph topology.

Robustness and sensitivity

- In a complex, interacting network, faults or accidents in one part affect the entire network.
- The study of complex networks is the study of the ways in which perturbations propagate through a system.
- The field is very new. We are still developing tools to understand large systems.
- One important question: under what conditions are networks chaotic?

A short detour into nuclear physics

- Heavy nuclei are complex systems
- Interactions between nucleons are known, but in practice direct calculation is intractable.
- Need a statistical description.
- Energies of the system are eigenvalues of Hamiltonian H.
- H is a large random matrix.



Symmetry and probability distributions

$$H = \begin{pmatrix} H_{11} & H_{12} & \cdots & H_{1N} \\ H_{2,1} & H_{2,2} & \cdots & H_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ H_{N1} & H_{N2} & \cdots & H_{NN} \end{pmatrix}$$

Symmetry and probability distributions

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- Probability conservation: $H = H^\dagger$
- If time-reversal-invariant: $H = H^T$

Symmetry and probability distributions

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Gaussian-distributed independent random variables:

$$d[H] = \prod_{i \leq j} dH_{ij}$$

Joint probability distribution

- We can calculate the joint probability distribution of these matrix elements assuming O(N) symmetry:

$$P(H)d[H] = \mathcal{N}_0 \exp\left(\frac{-N}{4\gamma^2}\mathrm{tr}(\mathrm{H}^2)\right) d[H]$$

- N is the size of the matrix
- The parameter γ defines the mean level density and is determined empirically.
- For details see Mehta (1991).

Gaussian Orthogonal Ensemble

- Assuming time reversal symmetry, can diagonalize H:

$$H = \mathcal{O}^{-1}\Lambda\mathcal{O}$$

- So can write the probability distribution in terms of the eigenvalues:

$$P(H)d[H] = \tilde{N}_0 d\mathcal{O} \exp\left(\frac{-N}{4\gamma^2}\sum_i E_i^2\right)\prod_{i<j}|E_i - E_j|\prod_k dE_k$$

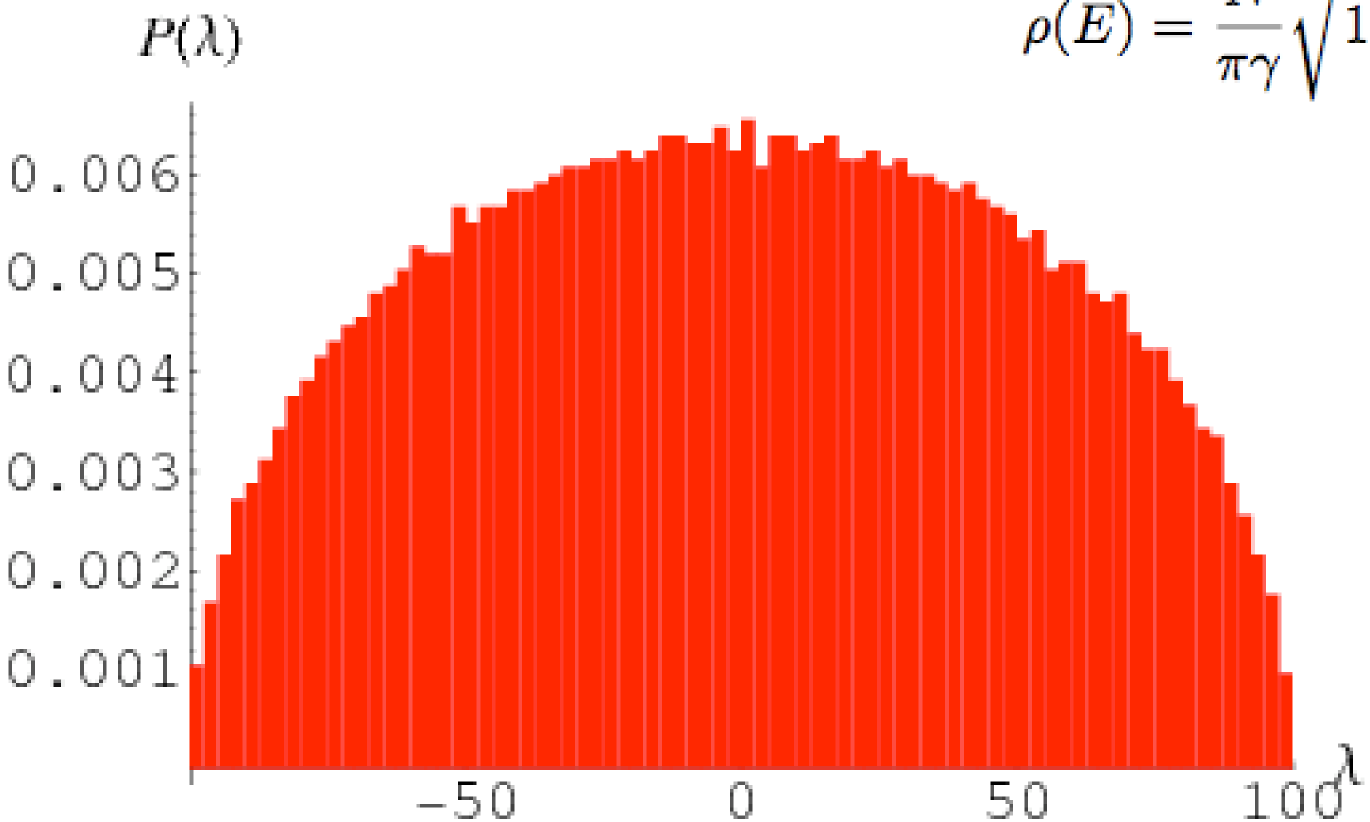
- Define level density:

$$\rho(E) = \sum_j \delta(E - E_j)$$

Wigner semicircle distribution

- For the GOE, can show eigenvalue density is described by a semicircle:

$$\rho(E) = \frac{N}{\pi\gamma}\sqrt{1 - \left(\frac{E}{2\gamma}\right)^2}$$



GOE, continued

- The symmetries of the system uniquely determine the behavior of the energy spectrum.

- The joint probability vanishes when Ei=Ej: level repulsion.

- “Unfold” spectrum so mean level spacing is one:

$$E \to \bar{E} = \int_0^{\bar{E}} \rho(E)dE$$

- Convenient to consider Nearest Neighbor Spacing Distribution:

$$P_{GOE}(s) = \frac{\pi}{2} \exp\left(\frac{-\pi s^2}{4}\right)$$

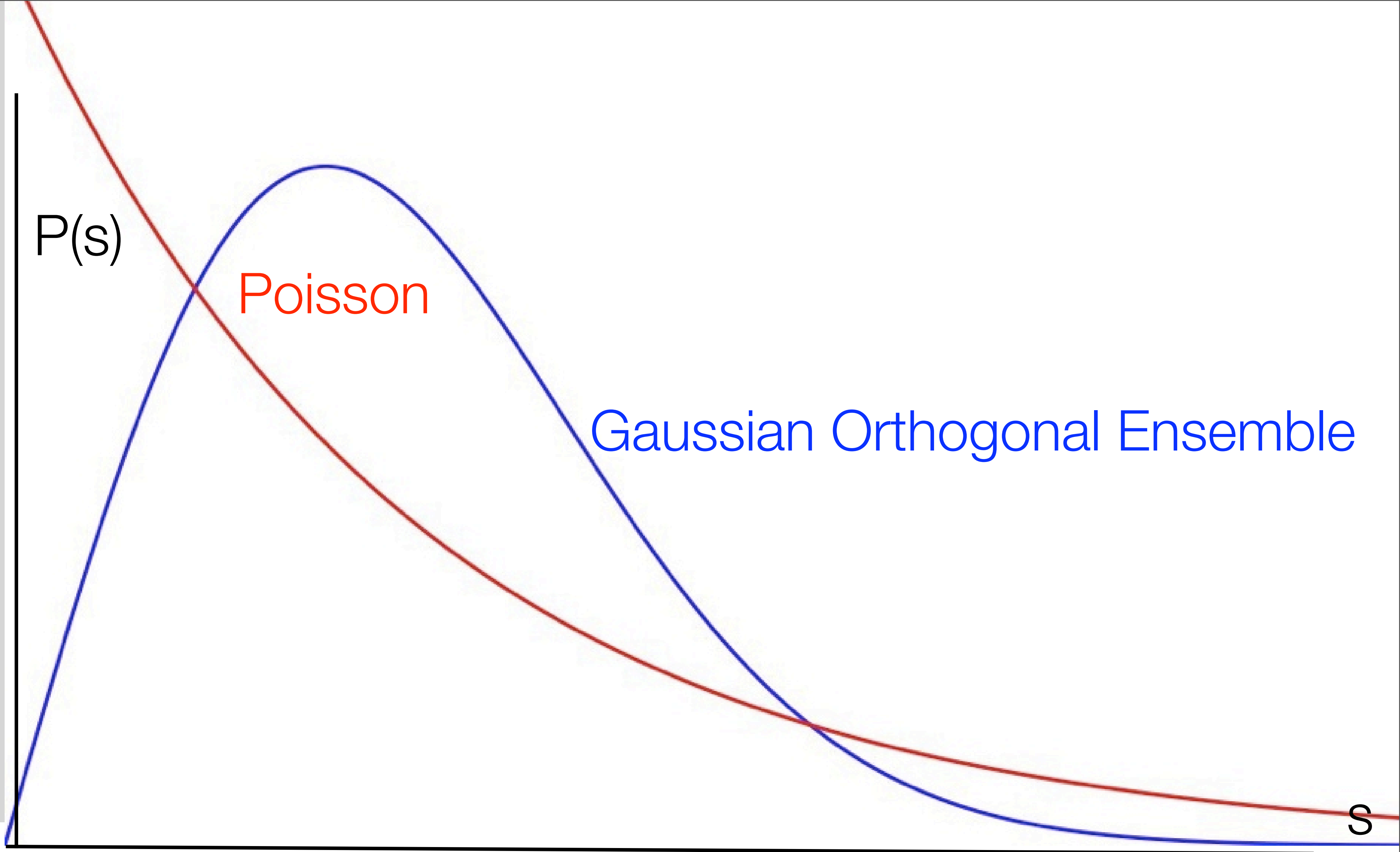
Contrast: noninteracting system

- Here, Hamiltonian is constrained to be diagonal (no interaction terms).

$$P(H)d[H] = \bar{\mathcal{N}}_0 \prod_i \exp\left(\frac{-N}{4\gamma^2} H_{ii}^2\right) dH_{ii}.$$

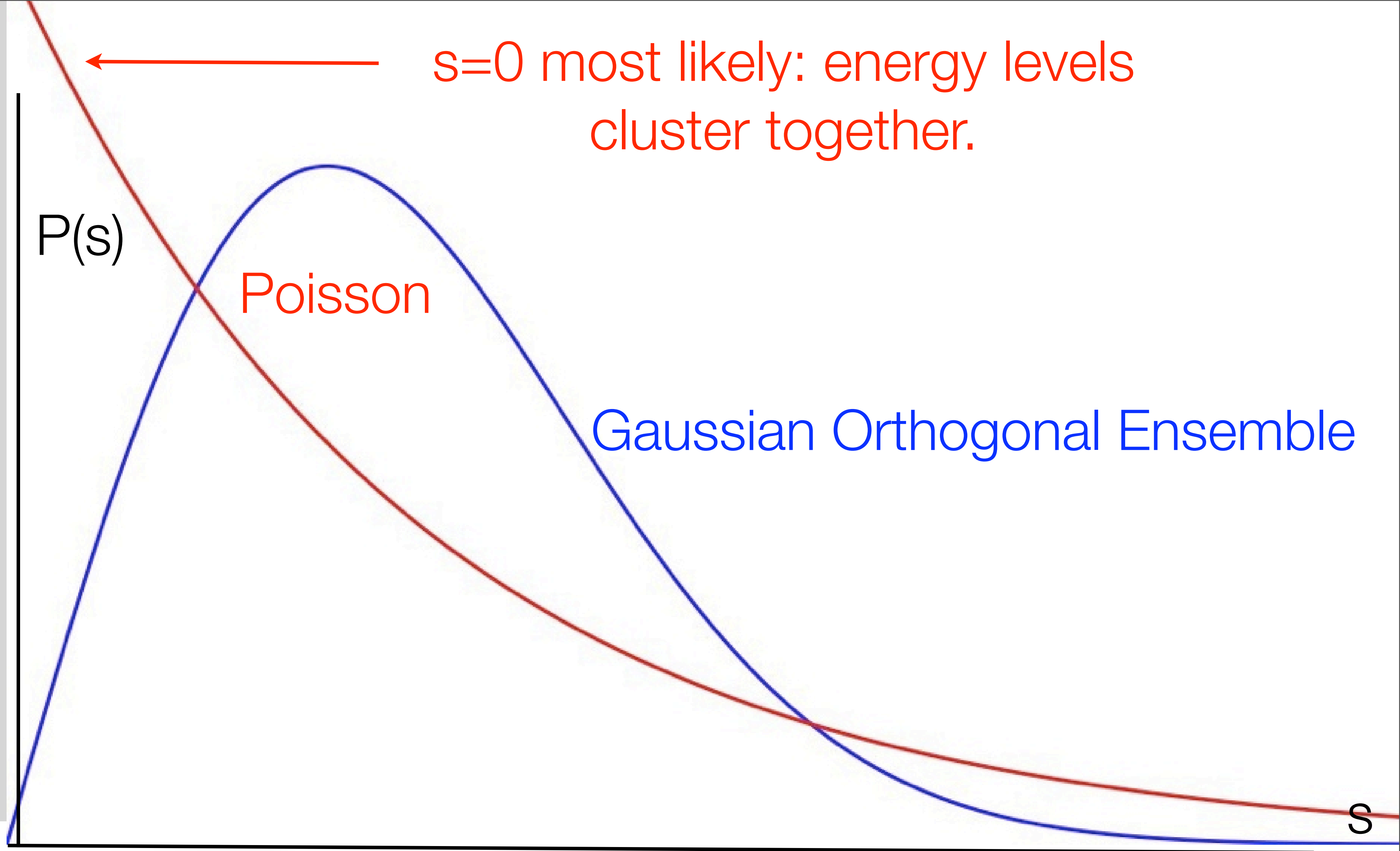
- No level repulsion.
- Eigenvalues are uncorrelated random variables.
- NNSD is Poisson:

$$P(s) = e^{-s}$$



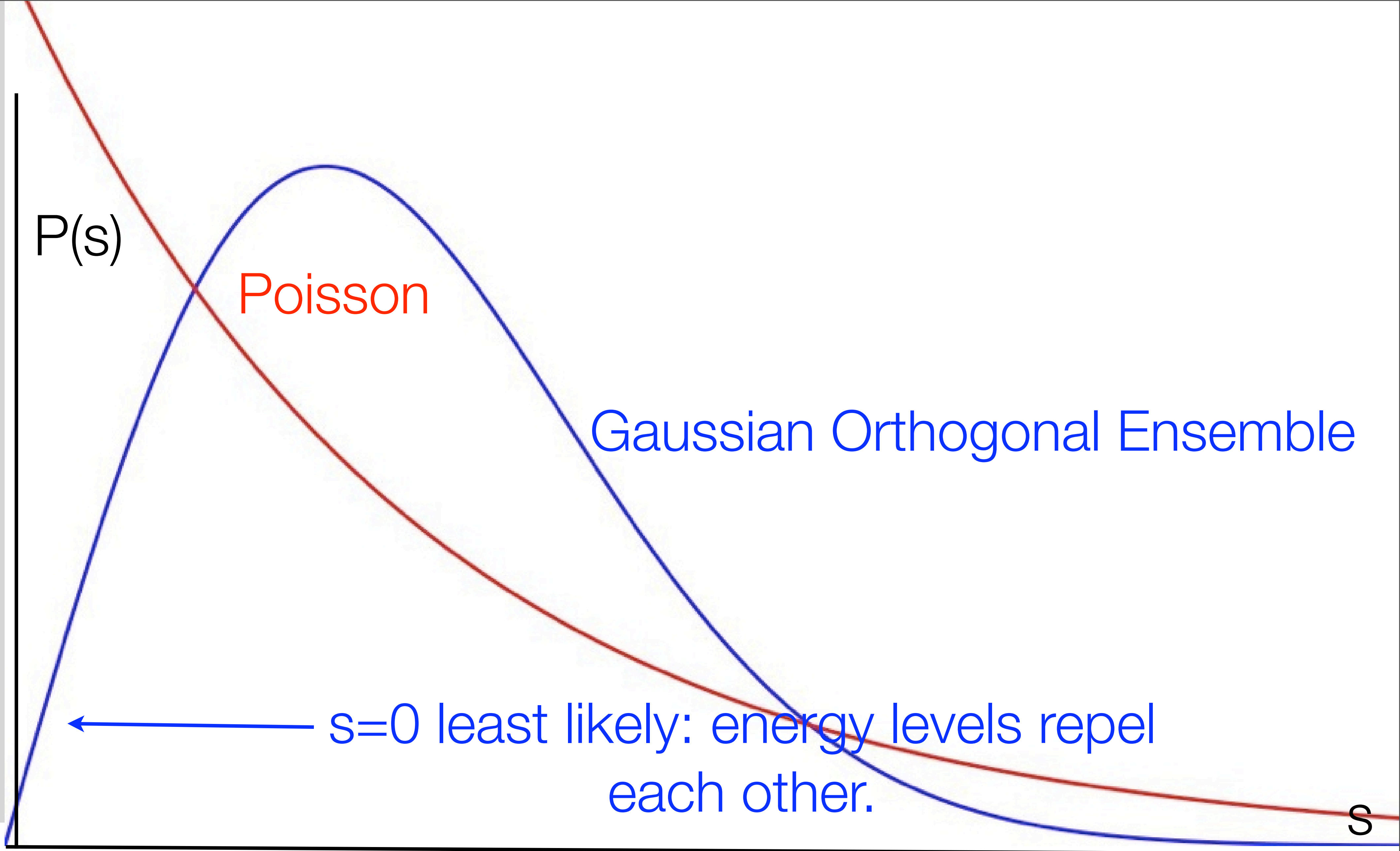
NNSD

Level repulsion vs. clustering



NNSD

Level repulsion vs. clustering



NNSD

Level repulsion vs. clustering

Quantum Chaos

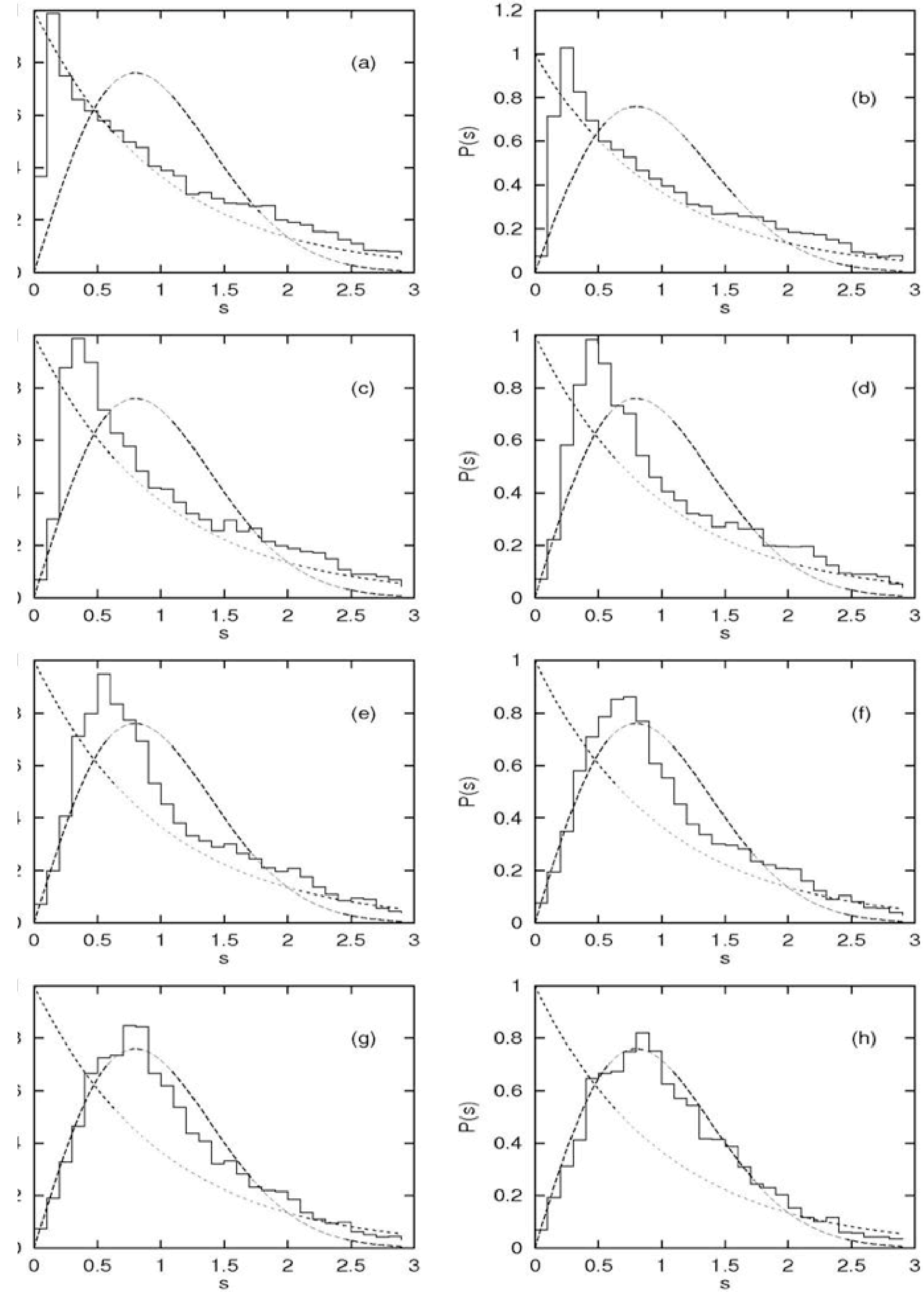
Conjectures

Chaos vs. Regularity

- These distributions appear in the study of quantum chaos.
- Two main conjectures underly the field:
- **Bohigas-Giannoni-Schmit:** Spectra of systems whose classical analogues are fully chaotic show correlation properties consistent with the Gaussian ensembles.
- **Berry-Tabor:** Spectra of systems whose classical analogues are fully regular show correlation properties best described by Poisson statistics.
- Intuitively, independent variables behave in a regular way. Large correlations induce chaos.

Order to Chaos

- Real nuclear data marks integrability to chaos transition.
- Chaos here is induced by the breaking of dynamical symmetries.
- Many more degrees of freedom than conserved quantities.
- Clear Poisson to GOE transition.

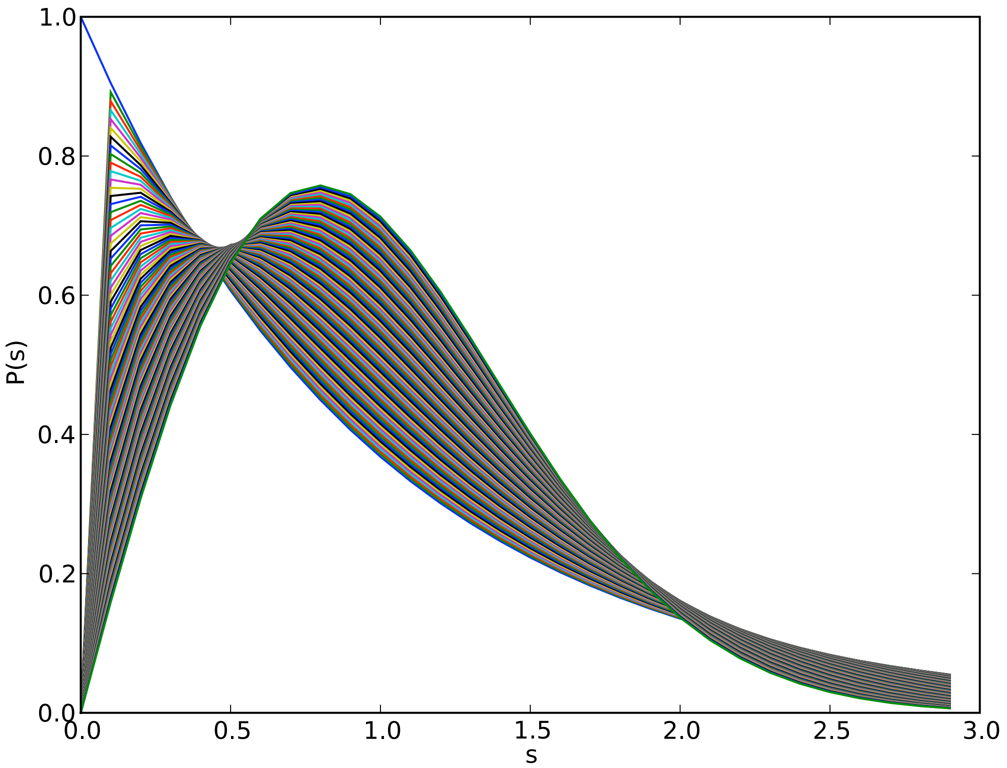


Measuring intermediate distributions

- Want distribution that interpolates between GOE and Poisson.

$$p(s) = (1+\beta)\alpha^{\beta+1}\exp(-\alpha s^{\beta+1})$$

$$\alpha = \Gamma\left(\frac{\beta+2}{\beta+1}\right)$$



- Brody parameter=1 for GOE, 0 for Poisson.

Back to the grid

- For a complex network, the analogue of the Hamiltonian H is the adjacency matrix A.

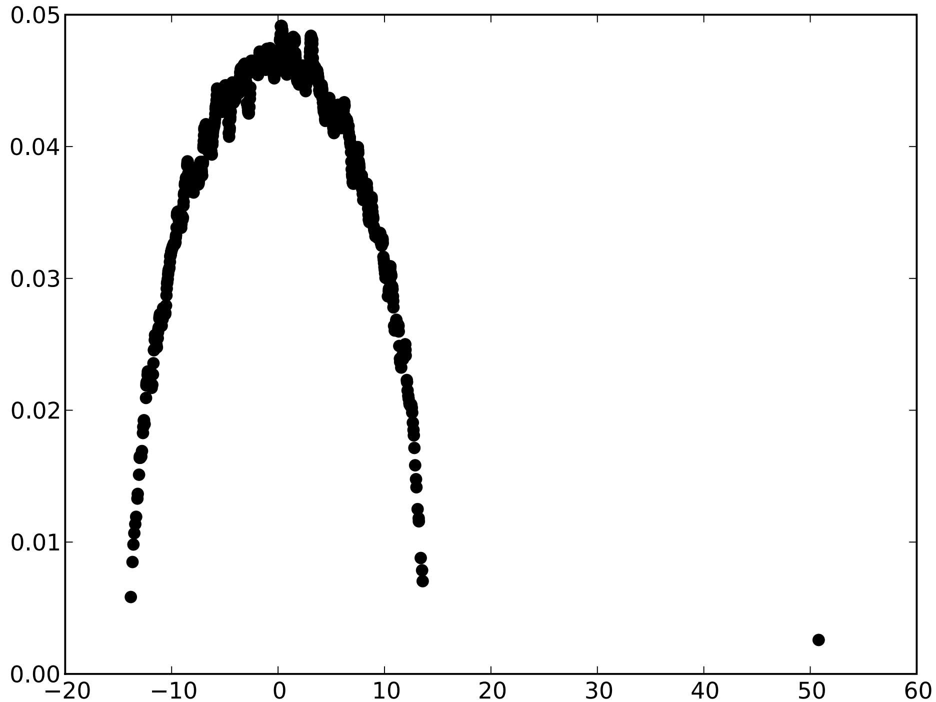
- Often, the only thing we can measure about a grid is its degree distribution.

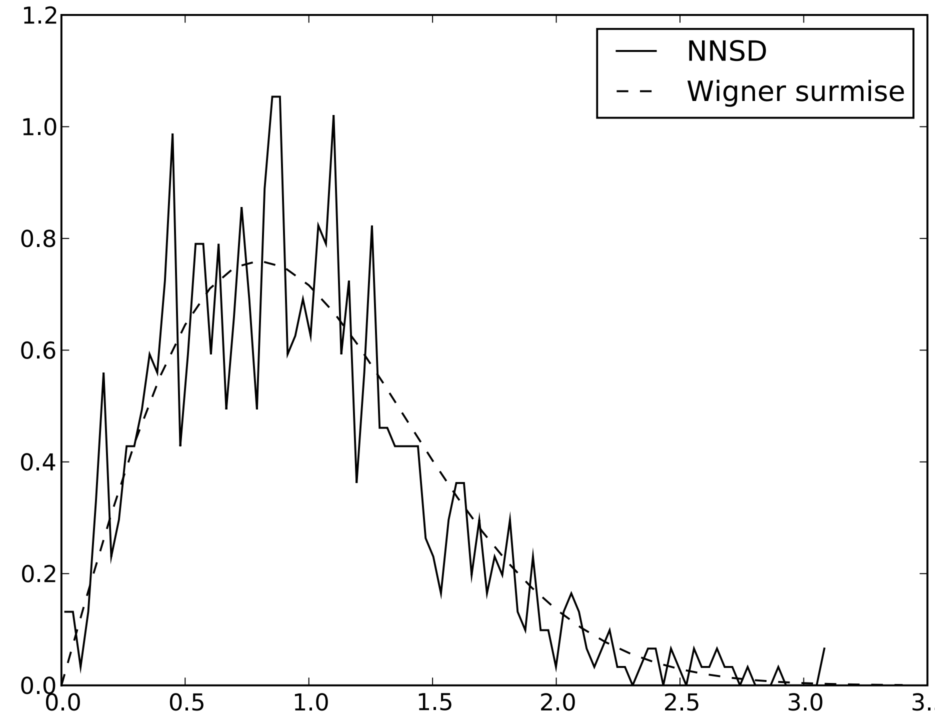
- In most circumstances, this is exponential:

$$p(k) \sim e^{-k/\kappa}$$

- Given this information, what can we say about the underlying grid statistics?

- Under what circumstances should we expect chaos?



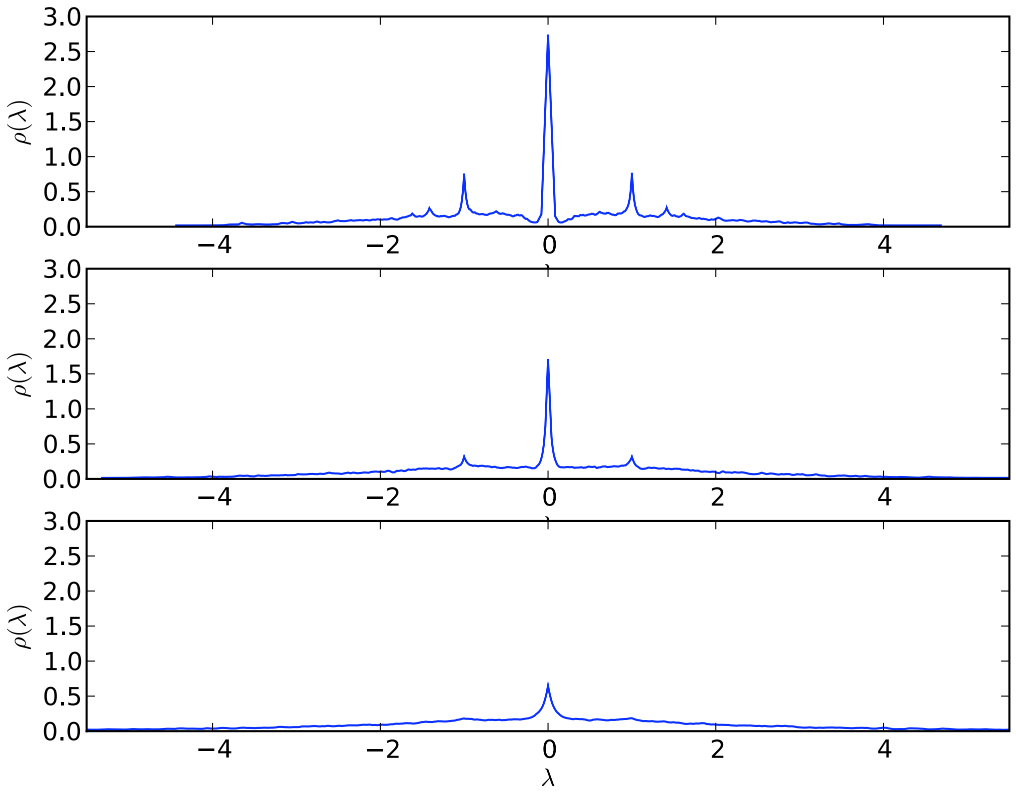


$$p_k = \binom{N}{k} p^k (1-p)^{N-k}$$

Erdos-Renyi random graph: nodes are connected with probability p.

Exponential distributions

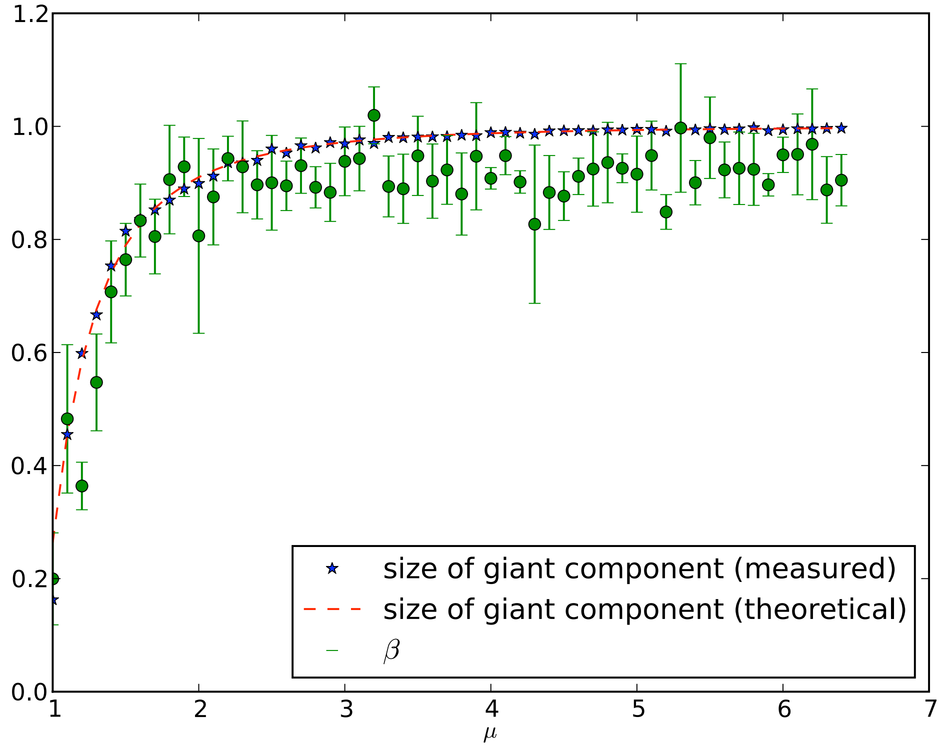
- Eigenvalue density for networks with varying mean degree.
- Peak at 0: nodes with connectivity 1 interacting with highly connected node.
- Peaks at +=1: connected pairs disconnected from greater network.
- Peaks decrease with increasing mean degree.



GOE vs. Poisson for exponential networks

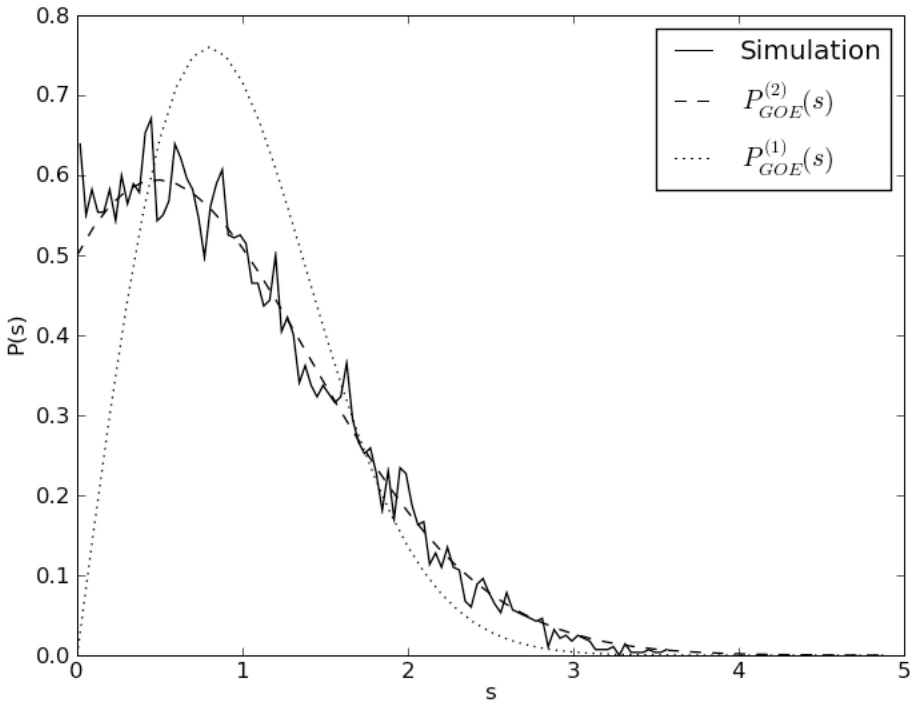
- Consistent with GOE for mean degree greater than 2.
- Consistent with Poisson otherwise.
- Giant component size:

$$S = \frac{1}{2}e^{1/\mu}\left(3 - \sqrt{4e^{1/\mu} - 3}\right)$$



Linking networks: superposition

- Consider two copies of the same network. If they are completely disconnected, their NNSD should be given by superposed GOEs:



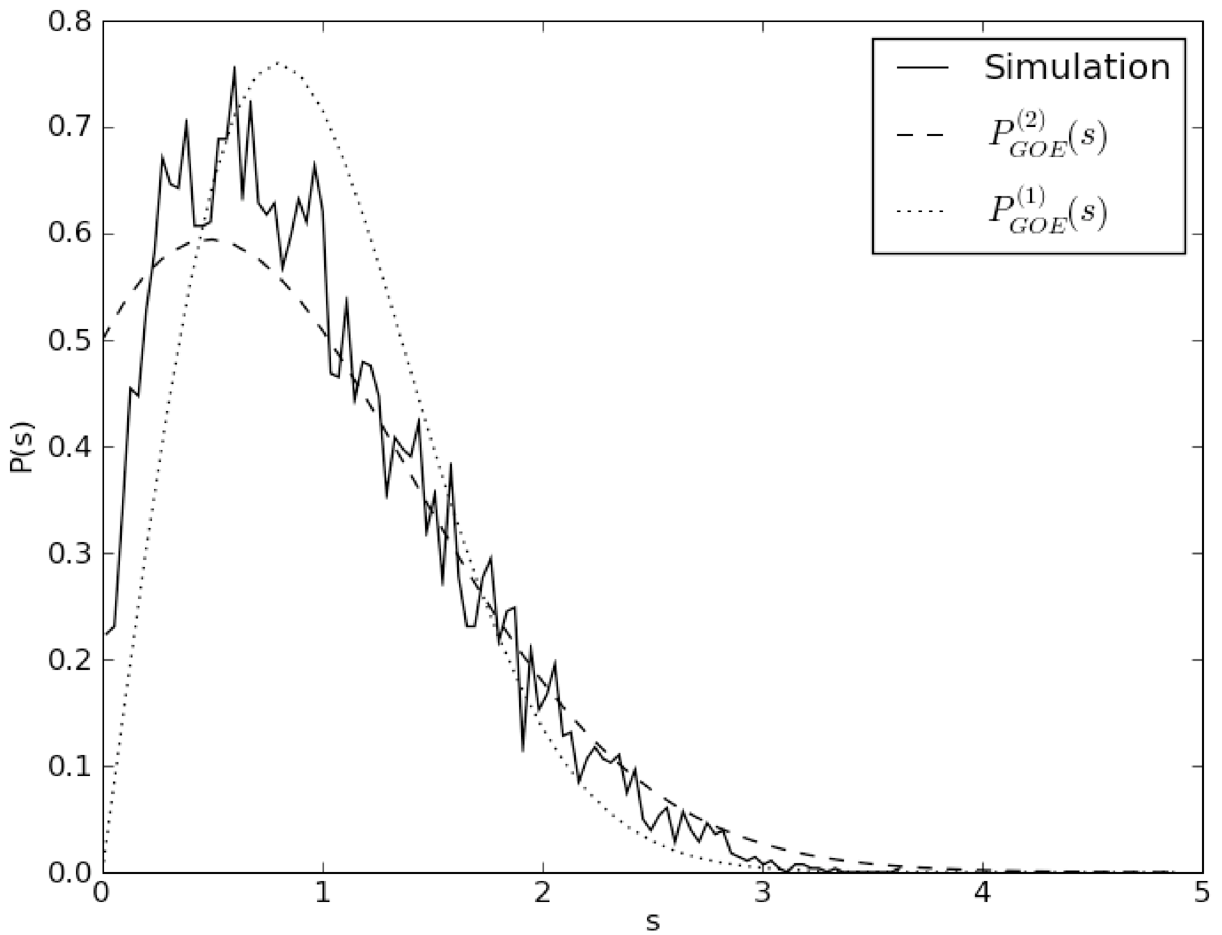
$$P^{(2)}_{GOE}(s) = \frac{\mathrm{d}}{\mathrm{d}s^2}\left\{E_1\left(\frac{s}{2}\right)^2\right\}$$

$$E_1(x) = \int_x^\infty (1 - F(t))\,dt$$

$$F(t) = \int_0^t P^{(1)}_{GOE}(s)ds.$$

Connecting with a single edge

- Linking the two networks with a single edge produces a strange distribution.



Describing interconnection

- In the absence of connections between networks, adjacency matrix is block diagonal:

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

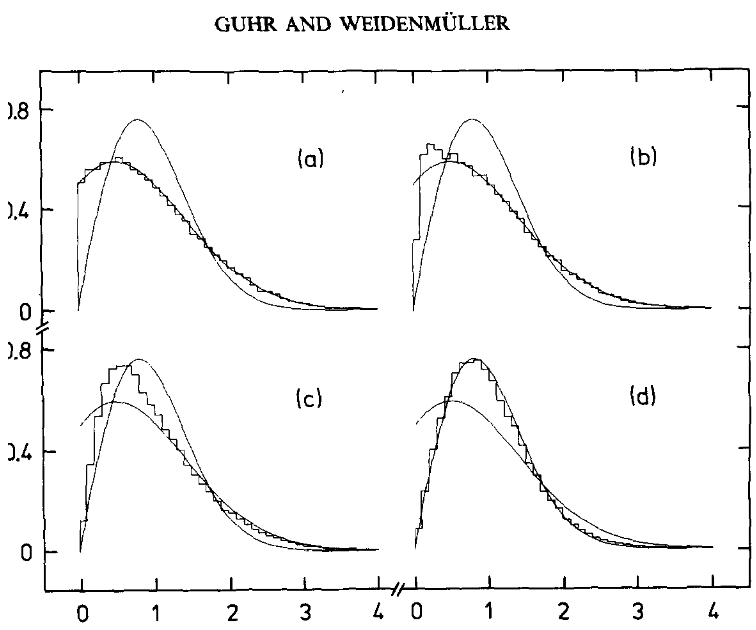
- Connections add off-diagonal elements:

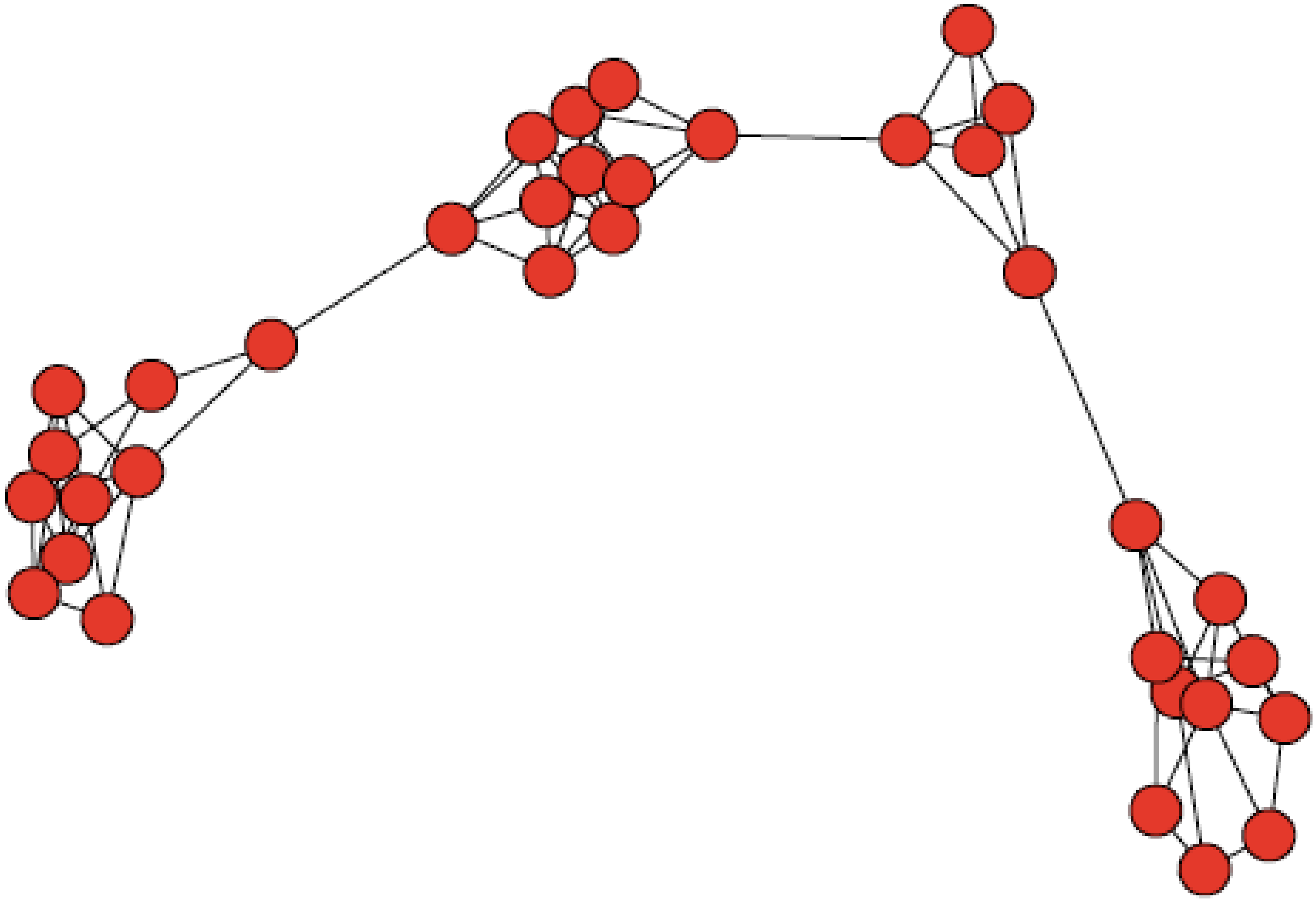
$$A = A_0 + A_{int}$$

-

Nuclear symmetry breaking

- Same formalism is used to describe symmetry breaking in nuclear systems.
- If symmetry is exact, Hamiltonian is block-diagonal, with blocks labeled by values of the good quantum number.
- Weak symmetry breaking mixes eigenstates.
- Insensitive to relative size of blocks.



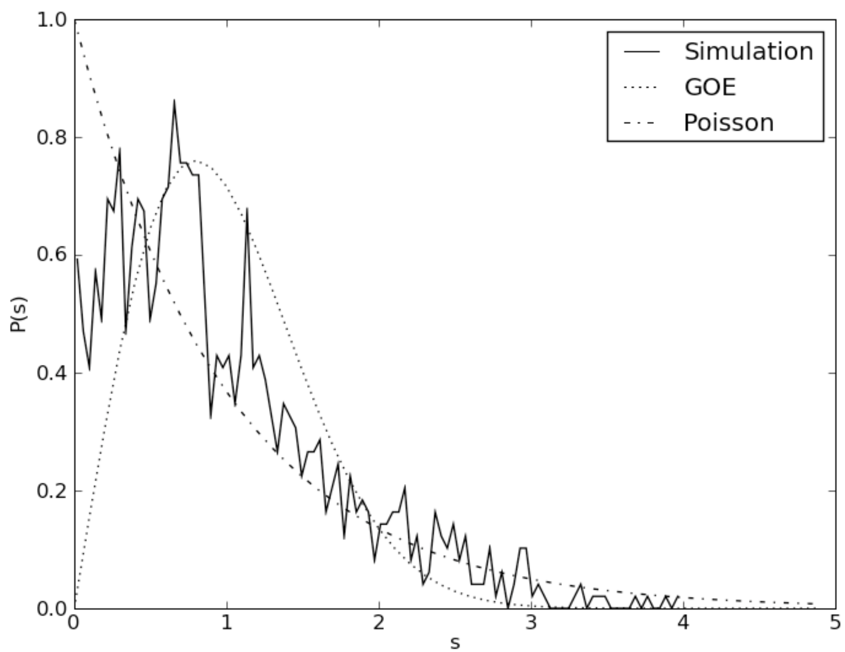


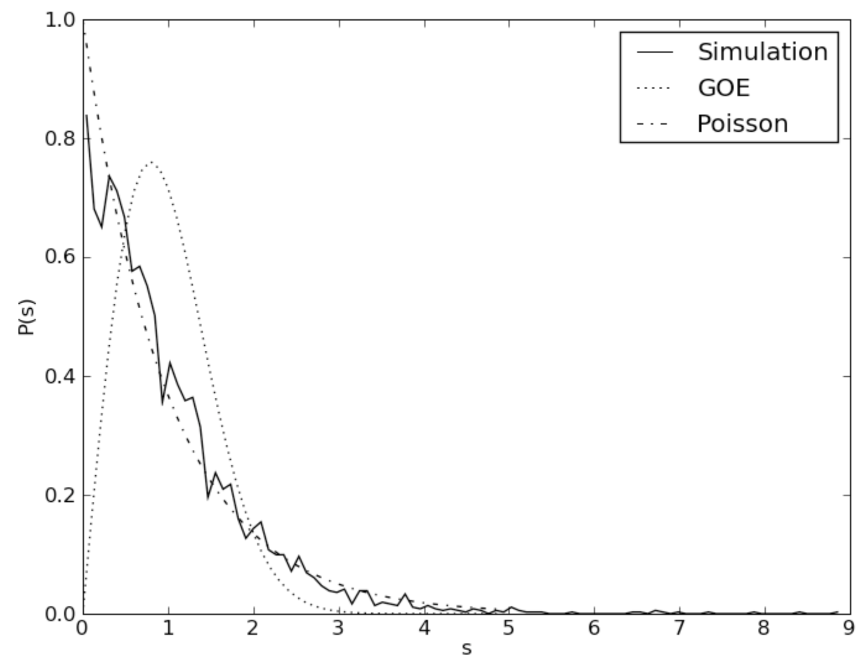
Distributed Networks

Small clusters linked together

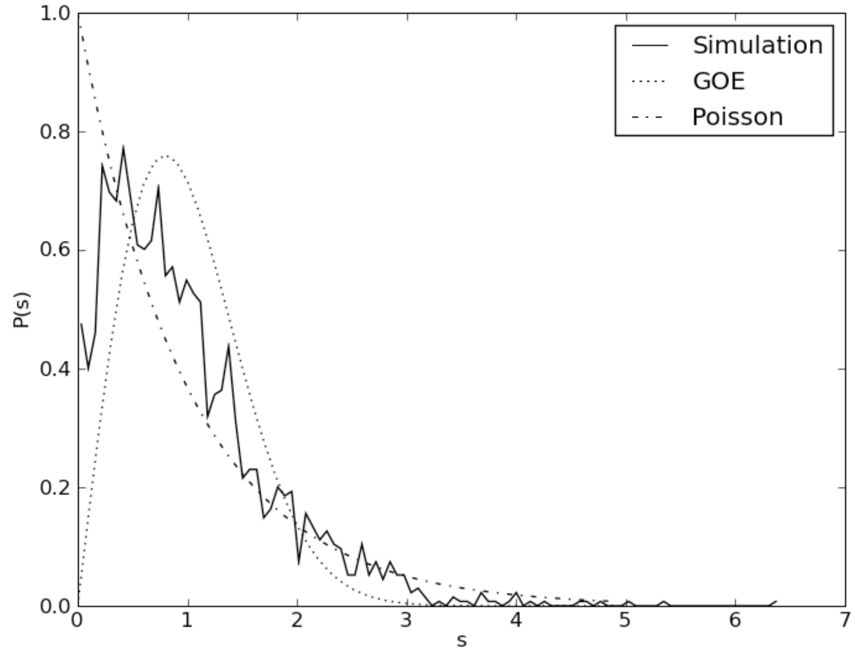
Distributed networks

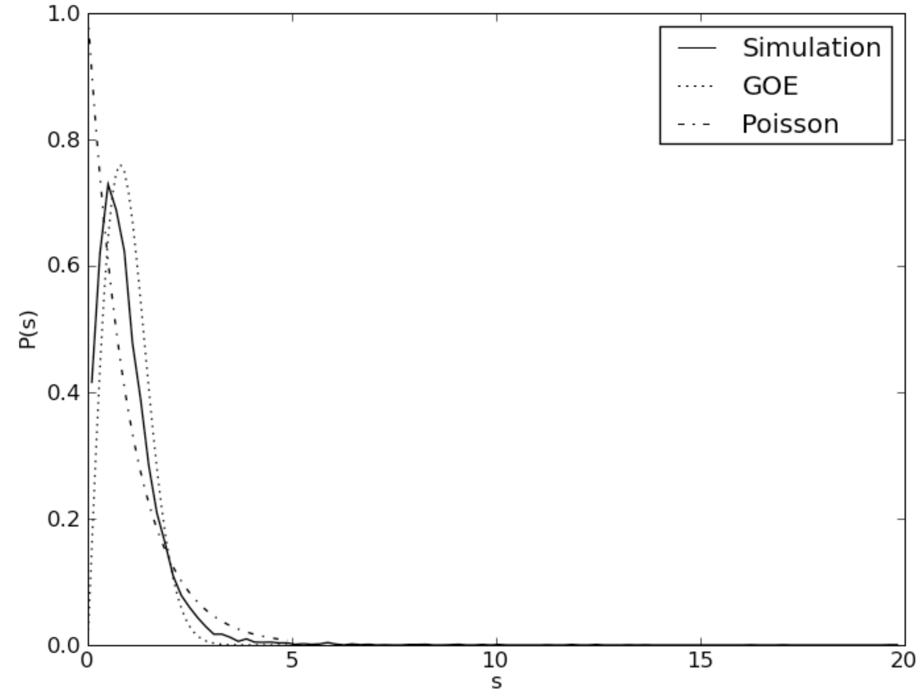
- Model by linking m identical regions of equal size N.
- I=number of interconnectors. For I/N <<<1, get Poisson statistics for m of order N/100.
- However, for sufficiently large I/N, retain GOE distribution even for m of order N.
- Within each region, eigenvalues highly correlated, so any fault or fluctuation propagates through entire region.





Single interconnector: get Poisson distribution for m=10





I=2: do not get Poisson distribution even for m=50.

Conclusions

- Grids matter. They provide context and help us to ask the right questions.
- Surprising tools from nuclear physics can help us understand the connectivity and correlation properties of large complex systems.
- The NNSD helps to illustrate correlations between nodes in a system and provides insight into how failures and fluctuations propagate.
- More work is needed to incorporate network topology information with existing load flow models.
- Controlling chaos through strategic load management?