Synchronization and Kron Reduction in Power Networks

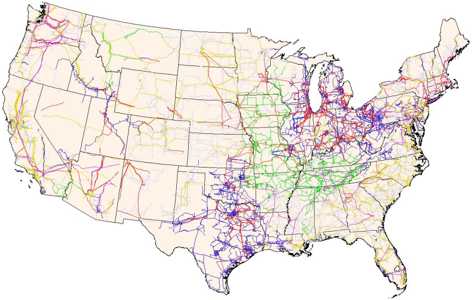
Florian Dörfler and Francesco Bullo

Center for Control,
Dynamical Systems & Computation
University of California at Santa Barbara
http://motion.me.ucsb.edu

Center for Nonlinear Studies
Los Alamos National Labs

Los Alamos, New Mexico, June 8, 2011

Motivation: the current power grid is . . .



“. . . the greatest engineering achievement of the 20th century.”
[National Academy of Engineering ’10]

1. large-scale, complex, & rich nonlinear dynamics
2. 100 years old and operating at its capacity limits $\Rightarrow$ **BLACKOUTS**

The New York Times — **The Blackout of 2003: Failure Reveals Creaky System, Experts Believe** 8/15/2003

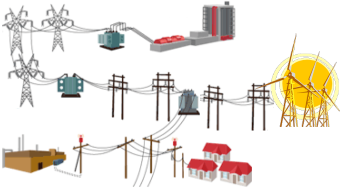
Motivation: the envisioned power grid

Energy is one of the top three national priorities

Expected developments in "**smart grid**":

1. large number of distributed power sources
2. increasing adoption of renewables

$\Rightarrow$ large-scale, complex, & heterogeneous networks with stochastic disturbances



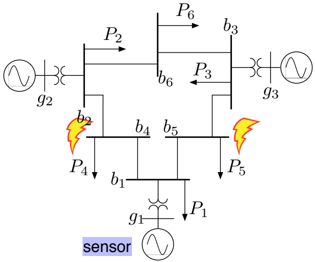
Some **smart grid keywords**:

"control/sensing/optimization" $\oplus$ *"distributed/coordinated/decentralized"*

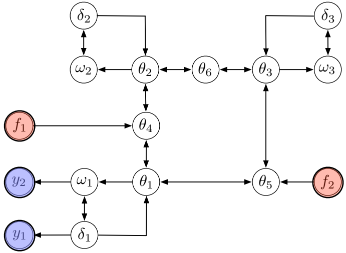
Open problem [D. Hill & G. Chen '06]: power network dynamics $\overset{?}{\leftrightsquigarrow}$ graph

Projects at UCSB: "power systems engineering" $\oplus$ "networked control"

1. **detection and identification** of faults & cyber-physical attacks (together with F. Pasqualetti)



9 bus power grid



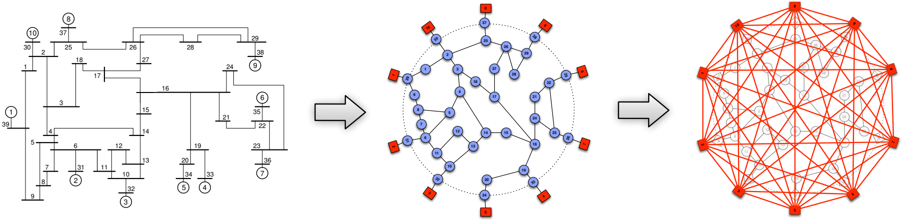
signal flow in dynamics

Questions: Is the attack or fault detectable/identifiable by measurements?
How to design (distributed) filters for detection/identification?

Motivation: the envisioned power grid – our viewpoint

Projects at UCSB: "power systems engineering" $\oplus$ "networked control"

2. **Kron reduction** – model reduction using algebraic graph theory



New England power grid | graph | equivalent reduced grid

Questions: How are the two electrically-equivalent networks related in terms of graph topology, spectrum, resistance, . . . ?

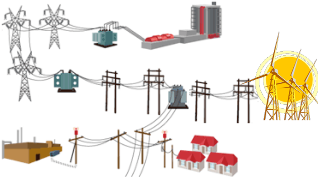
Applications of Kron reduction in smart grid problems?

Motivation: the envisioned power grid – our viewpoint

Projects at UCSB: "power systems engineering" $\oplus$ "networked control"

3. **Synchronization** & transient stability

Generators have to swing synchronously despite severe fluctuations in generation/load or faults in network/system components



Observations from distinct fields:

- power networks are coupled oscillators
- coupled oscillators synchronize for large coupling
- graph theory quantifies coupling, e.g., λ_2

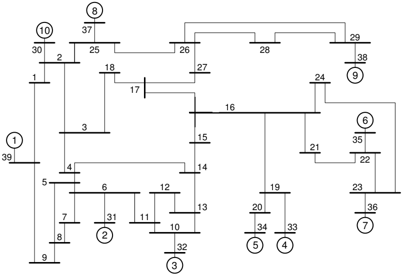
$\Rightarrow$ hence, power networks synchronize for large λ_2

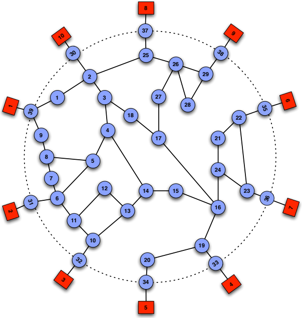
Today's talk: theorems about these observations

Outline

1. Introduction and Motivation
2. **Mathematical Modeling of a Power Network**
3. Synchronization in the Kuramoto Model
4. From the Kuramoto Model to the Power Network Model
5. From Network-Reduced to Structure Preserving Models
6. Conclusions

"New England Power Grid"





Power network topology:

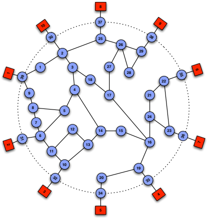
1. n generators ■, each connected to a bus ●
2. m buses ● form a connected graph
3. admittance matrix $\mathbf{Y}_{\text{network}} \in \mathbb{C}^{(n+m)\times(n+m)}$ characterizes the network

Mathematical model of a power network

Structure-preserving power network model – **frequency-dependent loads:**

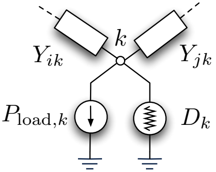
1. n generators ■ :
   - modeled by dynamic swing equations:

$$M_i \ddot{\theta}_i + D_i \dot{\theta}_i = P_{\text{mech.in},i} - P_{\text{electr.out},i}$$



2. transmission network: lossless & real power

3. m buses ● :
   - load model: constant power & linear frequency dependence



$$D_i \dot{\theta}_i + P_{\text{load},i} = -\sum_{j=1}^{n+m} |V_i| \cdot |V_j| \cdot |\mathbf{Y}_{ij}| \cdot \sin\left(\theta_i - \theta_j\right)$$

Mathematical model of a power network

Classic **ODE structure-preserving model** [A.R. Bergen & D. Hill '81]:

$$M_i\ddot{\theta}_i + D_i\dot{\theta}_i = P_i - \sum\nolimits_{j=1}^{n+m} P_{ij}\sin(\theta_i - \theta_j)\,, \qquad i \in \{1,\dots,n\}$$

$$D_i\dot{\theta}_i = P_i - \sum\nolimits_{j=1}^{n+m} P_{ij}\sin(\theta_i - \theta_j)\,, \qquad i \in \{n+1,\dots m\}$$

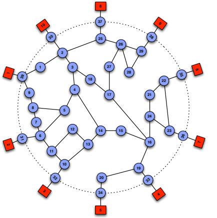
$P_{ij} = |V_i| \cdot |V_j| \cdot |\mathbf{Y}_{ij}| \ \geq 0$ — max. power transferred $i \leftrightarrow j$

$$P_i = \begin{cases} P_{\text{mech.in},i} & \text{for } i \in \{1,\dots,n\} \\ -P_{\text{load},i} & \text{for } i \in \{n+1,\dots m\} \end{cases}$$

real power injection at i

Mathematical model of a power network

Structure-preserving power network model – **linear loads:**



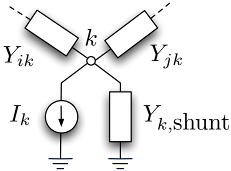
1. n generators ■ :
   - modeled by dynamic swing equations:

$$M_i\ddot{\theta}_i + D_i\dot{\theta}_i = P_{\text{mech.in},i} - P_{\text{electr.out},i}$$

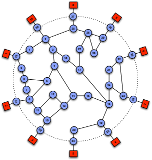
2. transmission network: linear

3. m buses ● :
   - load model: constant impedance & current
   - algebraic & linear Kirchhoff equations:

$$I = \mathbf{Y}_{\text{network}} V$$



Kron reduction to an ODE power network model:

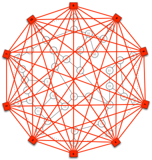


- Kirchhoff equations partitioned via boundary nodes & interior nodes :

$$\begin{bmatrix} I_{\text{boundary}} \\ \hline I_{\text{interior}} \end{bmatrix} = \left[\begin{array}{c|c} Y_{\text{boundary}} & Y_{\text{bound-int}} \\ \hline Y_{\text{bound-int}}^T & Y_{\text{interior}} \end{array}\right] \begin{bmatrix} V_{\text{boundary}} \\ \hline V_{\text{interior}} \end{bmatrix}$$

Schur complement / Kron reduction

$$Y_{\text{reduced}} = \mathbf{Y}_{\text{network}} / Y_{\text{interior}} \quad \Longrightarrow \quad I_{\text{boundary}} + I_{\text{red}} = Y_{\text{reduced}} V_{\text{boundary}}$$



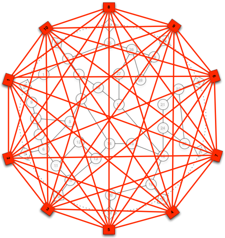
- network reduced to active nodes (generators)
- Y_{reduced} induces complete "all-to-all" coupling graph
- local loads at ■ : $Y_{\text{red, } i, i} \cdot |V_i|^2 + I_{\text{boundary}} + I_{\text{red}}$

Network-Reduced ODE power network model:

classic **interconnected swing equations**

[P.M. Anderson et al. '77, M. Pai '89, P. Kundur '94] :

$$M_i\ddot{\theta}_i + D_i\dot{\theta}_i = P_i - \sum\nolimits_{j=1,\, j\neq i}^{n} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij})$$



"all-to-all" reduced network Y_{reduced} with

$P_{ij} = |V_i| \cdot |V_j| \cdot |Y_{\text{red},i,j}| \; > 0$ max. power transferred $i \leftrightarrow j$

$\varphi_{ij} = \arctan(\Re(Y_{\text{red},i,j})/\Im(Y_{\text{red},i,j})) \; \in [0, \pi/2[$ reflect losses $i \leftrightarrow j$

$P_i = P_{\text{mech.in},i} - |V_i|^2 \Re(Y_{\text{red},i,i}) + \Re(V_i \cdot I_{\text{red},i})$ effective power input at i

Transient stability in power networks: problem statement

$$M_i\ddot{\theta}_i + D_i\dot{\theta}_i = P_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

1. **General synchronization problem:**
   $|\theta_i - \theta_j|$ bounded & $\dot{\theta}_i = \dot{\theta}_j$ in presence of transient disturbances

2. **Classic analysis methods**: Assumptions & Hamiltonian arguments

$$M_i\ddot{\theta}_i + D_i\dot{\theta}_i = -\nabla_i U(\theta)^T$$

Energy function analysis or analysis of reduced gradient flow:
[N. Kakimoto et al. '78, H.-D. Chiang et al. '94, ...]

$$\dot{\theta}_i = -\nabla_i U(\theta)^T$$

Key objective: compute domain of attraction via numerical methods

Transient stability in power networks: problem statement

$$M_i\ddot{\theta}_i + D_i\dot{\theta}_i = P_i - \sum\nolimits_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij})$$

1. **General synchronization problem:**
   $|\theta_i - \theta_j|$ bounded & $\dot{\theta}_i = \dot{\theta}_j$ in presence of transient disturbances

2. **Classic analysis methods**: Assumptions & Hamiltonian arguments

$$M_i\ddot{\theta}_i + D_i\dot{\theta}_i = -\nabla_i U(\theta)^T$$

Open Problem "power sys dynamics + complex nets" [D. Hill & G. Chen '06]

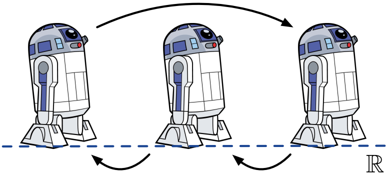
transient stability, performance, and robustness of a power network
$\overset{?}{\leftrightsquigarrow}$ underlying graph properties (topological, algebraic, spectral, etc)

Detour: consensus protocols & Kuramoto oscillators

Consensus protocol in $\mathbb{R}^n$:

$$\dot{x}_i = -\sum_{j \neq i} a_{ij}(x_i - x_j)$$

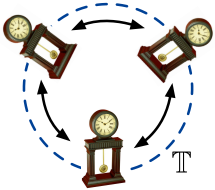
- n identical **agents** with state variable $x_i \in \mathbb{R}$
- **application:** agreement and coordination algorithms, ...
- **references:** [M. DeGroot '74, J. Tsitsiklis '84, L. Moreau '04, ...]



Kuramoto model in $\mathbb{T}^n$:

$$\dot{\theta}_i = \omega_i - \frac{K}{n}\sum_{j \neq i} \sin(\theta_i - \theta_j)$$

- n non-identical **oscillators** with phase $\theta_i \in \mathbb{T}$ & frequency $\omega_i \in \mathbb{R}$
- **application:** sync phenomena in nature, Josephson junctions, ...
- **references:** [C. Huygens XVII, Y. Kuramoto '75, A. Winfree '80, ...]

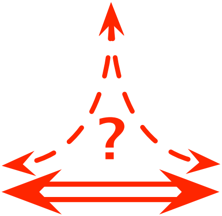


“The big picture”

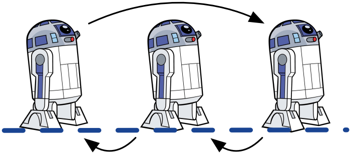
Open problem: sync in power networks

state, parameters, and topology of graph

$$M_i\ddot{\theta}_i + D_i\dot{\theta}_i = P_i - \sum\nolimits_{j\neq i} P_{ij}\sin(\theta_i - \theta_j + \varphi_{ij})$$

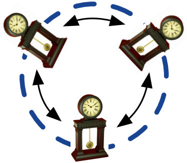


Consensus protocols:

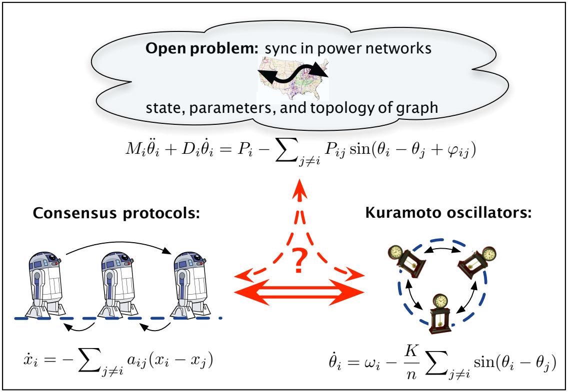


$$\dot{x}_i = -\sum\nolimits_{j\neq i} a_{ij}(x_i - x_j)$$

Kuramoto oscillators:



$$\dot{\theta}_i = \omega_i - \frac{K}{n}\sum\nolimits_{j\neq i}\sin(\theta_i - \theta_j)$$



Previous "observations" about this connection:

Power systems: [D. Subbarao et al., '01, G. Filatrella et al., '08, V. Fioriti et al., '09]
Networked control: [D. Hill et al., '06, M. Arcak, '07]
Dynamical systems: [H. Tanaka et al., '97, A. Arenas '08]

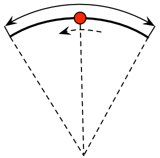
Outline

1. Introduction and Motivation
2. Mathematical Modeling of a Power Network
3. **Synchronization in the Kuramoto Model**
4. From the Kuramoto Model to the Power Network Model
5. From Network-Reduced to Structure Preserving Models
6. Conclusions

Synchronization in the Kuramoto Model

Classic Homogeneous Kuramoto model

$$\dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j=1}^{n} \sin(\theta_i - \theta_j)$$



Notions of synchronization:

- phase synchronization: $\theta_i(t) = \theta_j(t)$

Classic intuition:

- Kuramoto model is a forced gradient system $\dot{\theta}_i = \omega_i - \nabla_i U(\theta)$
- critical points $\{\nabla U(\theta) = \mathbf{0}\} \; = \; \{\text{phase sync}\} \cup \{\text{saddle points}\}$

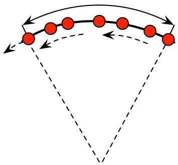
$\Rightarrow$ almost global phase sync $\Leftrightarrow$ $\omega_i = \omega_j$

Classic Homogeneous Kuramoto model

$$\dot{\theta}_i = \omega_i - \frac{K}{n} \sum\nolimits_{j=1}^{n} \sin(\theta_i - \theta_j)$$

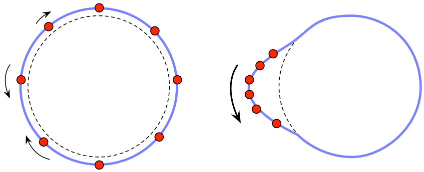
Notions of synchronization:

1. phase cohesiveness: $|\theta_i(t) - \theta_j(t)| < \gamma$ for small $\gamma < \pi/2$ ... arc invariance
2. frequency synchronization: $\dot{\theta}_i(t) = \dot{\theta}_j(t)$



Classic intuition:

- K small & $|\omega_i - \omega_j|$ large
  $\Rightarrow$ incoherence & no sync
- K large & $|\omega_i - \omega_j|$ small
  $\Rightarrow$ cohesiveness & frequency sync



Synchronization in the Kuramoto Model

Classic homogeneous Kuramoto model: $\dot{\theta}_i = \omega_i - \frac{K}{n}\sum_{j=1}^{n} \sin(\theta_i - \theta_j)$

Necessary and sufficient condition [F. Dörfler & F. Bullo '10]

The following two statements are equivalent:

1. Coupling dominates non-uniformity, i.e., $\boxed{K > K_{\text{critical}} \triangleq \omega_{\max} - \omega_{\min}}$.
2. The Kuramoto model achieves phase cohesiveness & exponential frequency synchronization for all $\omega_i \in [\omega_{\min}, \omega_{\max}]$.

Define $\gamma_{\min}$ & $\gamma_{\max}$ by $K_{\text{critical}}/K = \sin(\gamma_{\min}) = \sin(\gamma_{\max})$, then

1) **phase cohesiveness** for all arc-lengths $\gamma \in [\gamma_{\min}, \gamma_{\max}]$;

2) **practical phase synchronization:** from $\gamma_{\max}$ arc $\rightarrow$ $\gamma_{\min}$ arc; &

3) exponential **frequency synchronization** in the interior of $\gamma_{\max}$ arc.

Synchronization in the Kuramoto Model

Classic homogeneous Kuramoto model: $\dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j=1}^{n} \sin(\theta_i - \theta_j)$

Necessary and sufficient condition [F. Dörfler & F. Bullo '10]

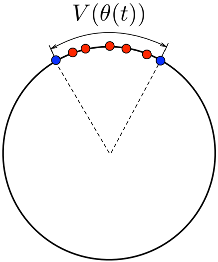
The following two statements are equivalent:

1. Coupling dominates non-uniformity, i.e., $\boxed{K > K_{\text{critical}} \triangleq \omega_{\max} - \omega_{\min}}$.
2. The Kuramoto model achieves phase cohesiveness & exponential frequency synchronization for all $\omega_i \in [\omega_{\min}, \omega_{\max}]$.

- improves existing sufficient bounds [F. de Smet et al. '07, N. Chopra et al. '09, G. Schmidt et al. '09, A. Jadbabaie et al. '04, S.J. Chung et al. '10, J.L. van Hemmen et al. '93, A. Franci et al. '10, S.Y. Ha et al. '10]
- tight w.r.t. continuum-limit [G.B. Ermentrout '85, A. Acebron et al. '00]
- tight w.r.t. implicit conditions for particular configurations [R.E. Mirollo et al. '05, D. Aeyels et al. '04, M. Verwoerd et al. '08]

Main proof ideas

1. **Cohesiveness** $\theta(t)$ in γ arc $\Leftrightarrow$ arc-length $V(\theta(t))$ is non-increasing



$$\Leftrightarrow \begin{cases} V(\theta(t)) = \max\{|\theta_i(t) - \theta_j(t)| \mid i,j \in \{1,\dots,n\}\} \\ D^+ V(\theta(t)) \overset{!}{\leq} 0 \end{cases}$$

$\sim$ **contraction property** [D. Bertsekas et al. '94, L. Moreau '04 & '05, Z. Lin et al. '08, ...]

$$\Leftrightarrow \quad K \sin(\gamma) \overset{!}{\geq} K_{\text{critical}}$$

2. **Frequency synchronization** in γ arc $\Leftrightarrow$ consensus protocol in $\mathbb{R}^n$

$$\frac{d}{dt}\dot{\theta}_i = -\sum\nolimits_{j \neq i} a_{ij}(t)(\dot{\theta}_i - \dot{\theta}_j)\,,$$

where $a_{ij}(t) = \frac{K}{n}\cos(\theta_i(t) - \theta_j(t)) > 0$ for all $t \geq 0$

Synchronization in the Kuramoto Model

Topological and heterogeneous Kuramoto model

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

1. **Assumption**: coupling graph is undirected and connected
2. Locally stable **phase sync** iff $\omega_i = \omega_j$ [P. Monzon '06, R. Sepulchre et al. '07]
3. **Necessary conditions**: (no sync if)

   $\dot{\theta}_i = \dot{\theta}_j \;\Rightarrow\; \sum_{k=1}^{n} (a_{ik} + a_{jk}) \leq |\omega_i - \omega_j|$

   $\dot{\theta}_i = \omega_{\text{sync}} \;\Rightarrow\; \sum_{j=1}^{n} a_{ij} \leq \left|\omega_i - \frac{\sum_{k=1}^{n} \omega_k}{n}\right|$

4. Various **sufficient conditions** in the literature: [S.J. Chung et al. '10, A. Franci et al. '10, A. Jadbabaie et al. '04, F. Dörfler et al. '09, L. Buzna et al. '09]

$\Rightarrow$ same implication: "coupling dominates non-uniformity" $\Rightarrow$ sync

Topological and heterogeneous Kuramoto model

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

1. **Assumption I**: coupling graph is undirected and complete

$\Rightarrow$ **Sufficient condition I**: $\sum_{\ell=1}^{n} \min_{k \in \{i,j\} \setminus \{\ell\}} \{a_{k\ell}\} > |\omega_i - \omega_j|$

reduces to exact condition in homogeneous case

2. **Assumption II**: coupling graph is undirected and connected

$\Rightarrow$ **Sufficient condition II**: $\lambda_2(L(a_{ij})) > \|(\omega_1 - \omega_2, \dots)\|_2$

$\Rightarrow$ suff. & tight **conjecture**: $\lambda_2(L(a_{ij})) > \left\| \omega - \frac{\sum_{j=1}^{n} \omega_j}{n} \mathbf{1} \right\|_2 \cdot \frac{2}{\sqrt{n}}$

Outline

From the Kuramoto model to the power network model

1) Structure-preserving power network model on $\mathbb{T}^{n+m} \times \mathbb{R}^n$:

$$M_i \ddot{\theta}_i + D_i \dot{\theta}_i = P_i - \sum_{j=1}^{n+m} P_{ij} \sin(\theta_i - \theta_j)\,, \qquad i \in \{1, \dots, n\}$$
$$D_i \dot{\theta}_i = P_i - \sum_{j=1}^{n+m} P_{ij} \sin(\theta_i - \theta_j)\,, \qquad i \in \{n+1, \dots m\}$$

2) Non-uniform variation of Kuramoto model on $\mathbb{T}^{n+m}$:

$$\dot{\theta}_i = P_i - \sum_{j=1}^{n+m} P_{ij} \sin(\theta_i - \theta_j)\,, \qquad i \in \{1, \dots, n+m\}$$

Synchronization & stability conditions can be related via

1. singular perturbation analysis [F. Dörfler & F. Bullo '09]
2. extension of 1^{st}-order Lyapunov funcs [D. Koditschek '88, Y.P. Choi et al. '10]
3. local topological conjugacy arguments [F. Dörfler & F. Bullo '11]

From the Kuramoto model to the power network model

1. **singular perturbation analysis** [F. Dörfler & F. Bullo '09]

- time-scale separation if $\boxed{\epsilon = \min_i M_i/D_i \text{ is sufficiently small}}$

$$M_i\ddot{\theta}_i + D_i\dot{\theta}_i = P_i - \sum_{j=1}^{n+m} P_{ij}\sin(\theta_i - \theta_j)\,, \qquad i \in \{1, \dots, n\}$$

$$D_i\dot{\theta}_i = P_i - \sum_{j=1}^{n+m} P_{ij}\sin(\theta_i - \theta_j)\,, \qquad i \in \{n+1, \dots m\}$$

- reduced slow dynamics $\Rightarrow$ **non-uniform Kuramoto model:**

$$D_i\dot{\theta}_i = P_i - \sum_{j=1}^{n+m} P_{ij}\sin(\theta_i - \theta_j) \qquad (\star)$$

Tikhonov's Theorem: Assume cohesiveness & frequency sync for $(\star)$. Then $\forall\,(\theta(0), \dot{\theta}(0))$ there exists $\epsilon^* > 0$ such that $\forall\,\epsilon < \epsilon^*$ and $\forall\, t \geq 0$

$$\theta_i(t)_{\text{power network}} - \theta_i(t)_{\text{non-uniform Kuramoto model}} = \mathcal{O}(\epsilon)\,.$$

2 extension of 1st-order Lyapunov functions

1) strict Lyapunov functions for mechanical systems

- potential energy $U(\theta)$ is strict Lyapunov function for the **kinematics**:

$$\dot{\theta}_i = P_i - \nabla_i U(\theta)$$

$\Rightarrow$ can construct strict a Lyapunov function for the equilibria of the **dissipative dynamics** [Lord Kelvin 1886, D. Koditschek 1988]:

$$M_i \ddot{\theta}_i + D_i \dot{\theta}_i = P_i - \nabla_i U(\theta)$$

2) Extension of contraction argument $V(\theta(t)) = \max_{i,j}\{|\theta_i(t) - \theta_j(t)|\}$ to 2nd-order system [Y.P. Choi et al. '10]

$\Rightarrow$ the application to power network dynamics is currently under development [F. Dörfler & F. Bullo '11]

From the Kuramoto model to the power network model

3. **local topological conjugacy arguments** [F. Dörfler & F. Bullo '11]

1) Structure-preserving power network model on $\mathbb{T}^{n+m} \times \mathbb{R}^n$:

$$M_i \ddot{\theta}_i + D_i \dot{\theta}_i = P_i - \nabla_i U(\theta)\,, \qquad i \in \{1, \dots, n\}$$
$$D_i \dot{\theta}_i = P_i - \nabla_i U(\theta)\,, \qquad i \in \{n+1, \dots m\}$$

2) Non-uniform Kuramoto model & frequency dynamics on $\mathbb{T}^{n+m} \times \mathbb{R}^n$:

$$\frac{d}{dt}\,\dot{\theta}_i = -M_i^{-1} D_i \cdot \dot{\theta}_i\,, \qquad i \in \{1, \dots, n\}$$
$$\dot{\theta}_i = P_i - \nabla_i U(\theta)\,, \qquad i \in \{1, \dots, n+m\}$$

The following **facts** are true $\forall\ M_i > 0$ & $D_i > 0$:

- 1) & 2) have the **same equilibrium manifolds**
- equilibria 1) & 2) have the **same local stability properties**

From the Kuramoto model to the power network model

3. **local topological conjugacy arguments** [F. Dörfler & F. Bullo '11]

1) Structure-preserving power network model on $\mathbb{T}^{n+m} \times \mathbb{R}^n$:

$$M_i\ddot{\theta}_i + D_i\dot{\theta}_i = P_i - \nabla_i U(\theta)\,, \qquad i \in \{1,\dots,n\}$$
$$D_i\dot{\theta}_i = P_i - \nabla_i U(\theta)\,, \qquad i \in \{n+1,\dots m\}$$

2) Non-uniform Kuramoto model & frequency dynamics on $\mathbb{T}^{n+m} \times \mathbb{R}^n$:

$$\frac{d}{dt}\,\dot{\theta}_i = -M_i^{-1}D_i \cdot \dot{\theta}_i\,, \qquad i \in \{1,\dots,n\}$$
$$\dot{\theta}_i = P_i - \nabla_i U(\theta)\,, \qquad i \in \{1,\dots,n+m\}$$

The following **facts** are true $\forall\;\; M_i > 0$ & $D_i > 0$:

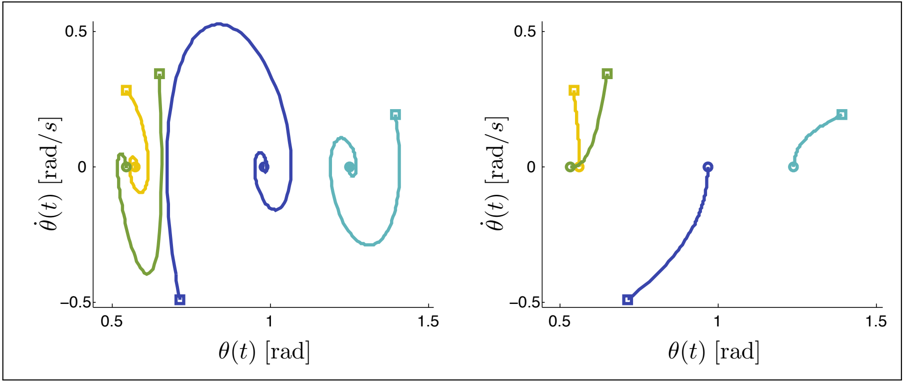
$\Rightarrow$ 1) synchronizes $\Leftrightarrow$ 2) synchronizes

$\Rightarrow$ near the equilibrium manifolds 1) & 2) are **topologically conjugate**

3. **local topological conjugacy arguments** [F. Dörfler & F. Bullo '11]

$\Rightarrow$ 1) synchronizes $\Leftrightarrow$ 2) synchronizes

$\Rightarrow$ near the equilibrium manifolds 1) & 2) are **topologically conjugate**



Instructive note: topological conjugacy confirms validity of ϵ-assumption

Main Synchronization Results

Structure-preserving power network model with frequency-dependent loads:

$$M_i\ddot{\theta}_i + D_i\dot{\theta}_i = P_i - \sum\nolimits_{j=1}^{n+m} P_{ij}\sin(\theta_i - \theta_j)\,, \qquad i \in \{1,\dots,n\}$$

$$D_i\dot{\theta}_i = P_i - \sum\nolimits_{j=1}^{n+m} P_{ij}\sin(\theta_i - \theta_j)\,, \qquad i \in \{n+1,\dots m\}$$

Results:

1. locally stable **phase sync** $\Leftrightarrow$ $P_i/D_i = const. \quad \forall\, i$

2. **necessary condition**: Let $\omega_{\text{sync}} = \sum_{k=1}^{n+m} P_k / \sum_{k=1}^{n+m} D_k$, then

   $\sum_{j=1}^{n+m} P_{ij} \leq |P_i - D_i \cdot \omega_{\text{sync}}| \quad \Rightarrow \quad$ no freq. sync

3. **exact condition for homogeneous case**, i.e., $P_{ij} = K/(n+m)$:

   $K > \max_{\{i,j\}} |\tilde{P}_i - \tilde{P}_j| \quad \Leftrightarrow \quad$ freq. sync & cohesiveness

   where $\tilde{P}_i = P_i - D_i \cdot \omega_{\text{sync}}$.

Main Synchronization Results

Structure-preserving power network model with frequency-dependent loads:

$$M_i\ddot{\theta}_i + D_i\dot{\theta}_i = P_i - \sum_{j=1}^{n+m} P_{ij} \sin(\theta_i - \theta_j)\,, \qquad i \in \{1, \dots, n\}$$

$$D_i\dot{\theta}_i = P_i - \sum_{j=1}^{n+m} P_{ij} \sin(\theta_i - \theta_j)\,, \qquad i \in \{n+1, \dots m\}$$

Results:

4. **sufficient condition**: Let $\tilde{P}_i = P_i - D_i \cdot \omega_{\text{sync}}$, then

$$\lambda_2(L(P_{ij})) > \left\| \left(\tilde{P}_1 - \tilde{P}_2\,, \dots \right) \right\|_2 \quad \Rightarrow \quad \text{freq. sync \& cohesiveness}$$

5. suff. & tight **conjecture**: $\lambda_2(L(P_{ij})) > \left\| \tilde{P}_i \right\|_2 \cdot 2/\sqrt{n+m}$

bottom line: *"coupling dominates imbalance in active power/damping"*

Main Synchronization Results

Network-reduced power system model without transfer conductances:

$$M_i\ddot{\theta}_i + D_i\dot{\theta}_i = P_i - \sum_{j=1}^{n} P_{ij}\sin(\theta_i - \theta_j)$$

Remark: all results can also be stated with transfer conductances

Results:

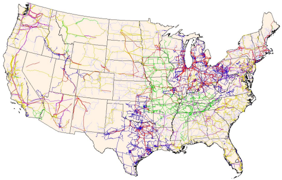
... same conditions as before ... and

6. **sufficient conditions**: Let $\tilde{P}_i = P_i - D_i \cdot \omega_{\text{sync}}$, then

$$\left.\begin{array}{l} \lambda_2(L(P_{ij})) > \left\|\left(\tilde{P}_1 - \tilde{P}_2\,, \dots\right)\right\|_2 \\ \text{or} \\ \sum_{\ell=1}^{n} \min\limits_{k\in\{i,j\}\setminus\{\ell\}} \{P_{k\ell}\} > \left|\tilde{P}_i - \tilde{P}_j\right| \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \text{freq. sync} \\ \;\& \\ \text{phase cohesiveness} \end{array}\right.$$

bottom line: *"coupling dominates imbalance in active power/damping"*

Conclusions so far . . .

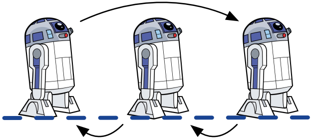


Open problem in synchronization and transient stability in **power networks:**

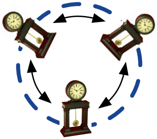
relation to underlying network state, parameters, and topology

singular perturbations, strict Lyapunov functions, & topological conjugacy

Time–varying Consensus Protocols



Non–uniform Kuramoto Oscillators



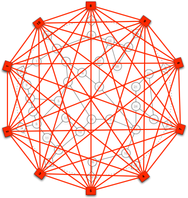
Kuramoto, consensus, and nonlinear control tools

Outline

1. Introduction and Motivation
2. Mathematical Modeling of a Power Network
3. Synchronization in the Kuramoto Model
4. From the Kuramoto Model to the Power Network Model
5. **Kron Reduction:**
   from network-reduced to structure preserving models
6. Conclusions

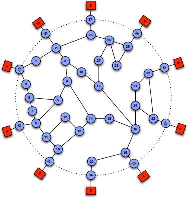
Kron Reduction

Network-reduced power system model:



- synchronization conditions on $\lambda_2(L(P_{ij}))$ and P_{ij}
- all-to-all reduced admittance matrix for lossless network uniform generator voltage levels $|V_i| = V$:
  $L(P_{ij}) = \mathrm{i} \cdot V^2 \cdot Y_{\text{reduced}}$

Structure-preserving power system model:



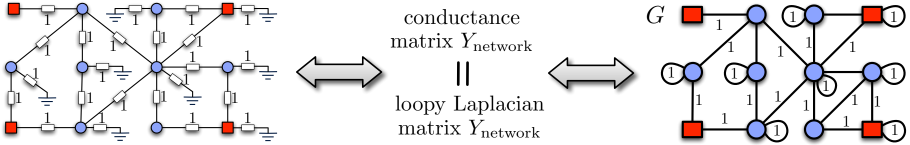
- topological bus admittance matrix $\mathbf{Y}_{\textbf{network}}$
- **Kron reduction** via Schur complement:

$$Y_{\text{reduced}} = \mathbf{Y}_{\textbf{network}} / Y_{\text{interior}}$$

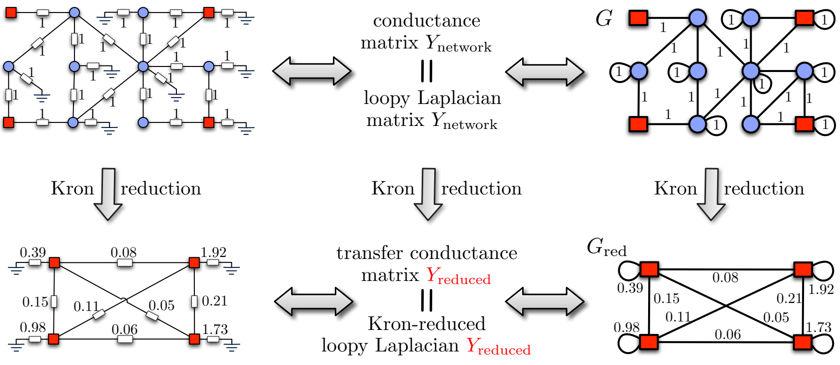
Kron reduction of graphs

Consider either of the following three equivalent setups:

1. a connected electrical network with conductance matrix $\mathbf{Y}_{\text{network}}$, terminals ■ , interior nodes ● , & possibly shunt conductances
2. a symmetric and irreducible loopy Laplacian matrix $\mathbf{Y}_{\text{network}}$ with partition (■ ,●), & possibly diagonally dominance
3. an undirected, connected, & weighted graph with boundary nodes ■ , interior nodes ● , & possibly self-loops



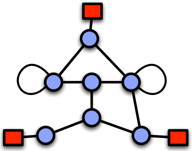
Kron reduction via Schur complement: $Y_{\text{reduced}} = \mathbf{Y}_{\text{network}} / Y_{\text{interior}}$



- **Engineering applications:** smart grid monitoring, circuit theory (star-Δ transformation), power network model reduction, power electronics, large-scale integration chips, electrical impedance tomography, data-mining, ...
- **Mathematics applications:** sparse matrix algorithms, finite-element methods, sparse multi-grid solvers, Markov chain reduction, stochastic complementation, applied linear algebra & matrix analysis, Dirichlet-to-Neumann map, ...
- **Physics applications:** knot theory, Yang-Baxter equations and applications, high-energy physics, statistical mechanics, vortices in fluids, entanglement of polymers & DNA, ... [F. Dörfler & F. Bullo '11, J.H.H. Perk & H. Au-Yang '06]

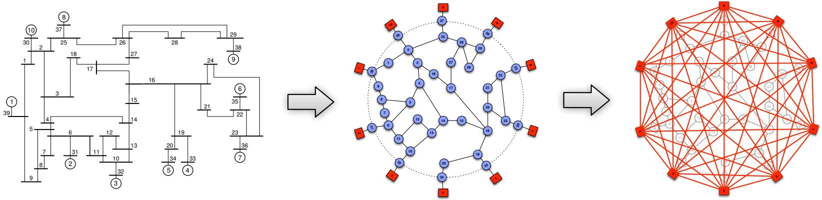
Kron reduction of a graph with

- boundary ■ , interior ● , non-neg self-loops ⟲
- loopy Laplacian matrix $\mathbf{Y}_{\text{network}}$
- Schur complement: $Y_{\text{reduced}} = \mathbf{Y}_{\text{network}} / Y_{\text{interior}}$



Properties of Kron reduction:

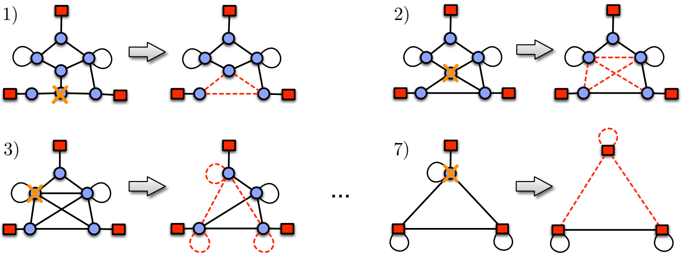
1. **Well-posedness:** set of loopy Laplacian matrices is closed



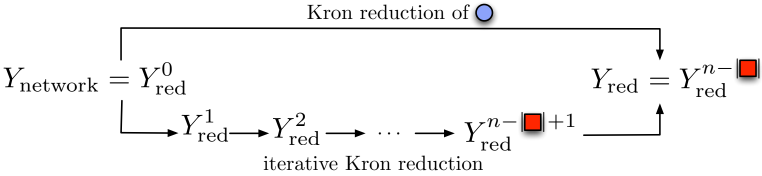
Kron reduction of graphs: properties

2. **Iterative 1-dim Kron reduction**: $\mathbf{Y}_{\text{reduced}}^{k+1} = \mathbf{Y}_{\text{reduced}}^{k} / \bullet$

   $\Rightarrow$ topological evolution of the corresponding graph

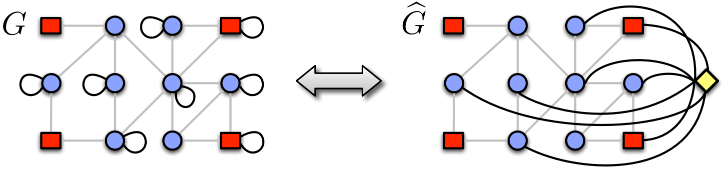


   $\Rightarrow$ **Equivalence**: the following diagram commutes:

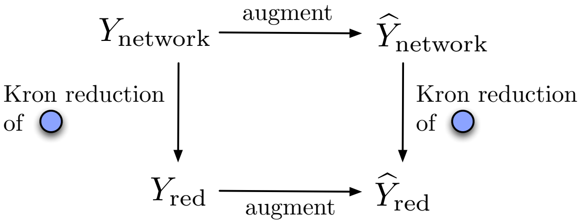


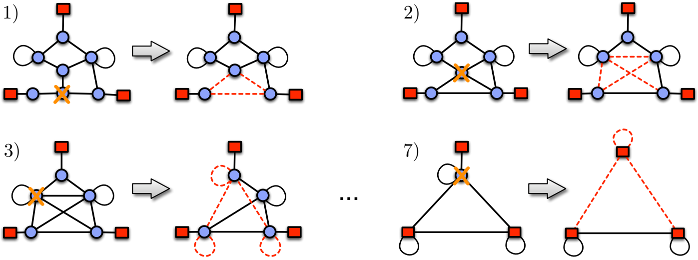
Kron reduction of graphs: properties

3. **Augmentation**: replace self-loops ⟲ by edge to grounded node ◊



⇒ **Equivalence**: the following diagram commutes:





4. **Topological properties:**
   - interior network connected ⇒ reduced network complete
   - at least one node in interior network features a self-loop ⟲
     ⇒ all nodes in reduced network feature self-loops ⟲

5. **Algebraic properties:** self-loops in interior network
   - decrease mutual coupling in reduced network
   - increase self-loops in reduced network

6. **Spectral properties**:
   - interlacing property: $\lambda_i(\mathbf{Y}_{\text{network}}) \leq \lambda_i(Y_{\text{reduced}}) \leq \lambda_{i+n-|\blacksquare|}(\mathbf{Y}_{\text{network}})$

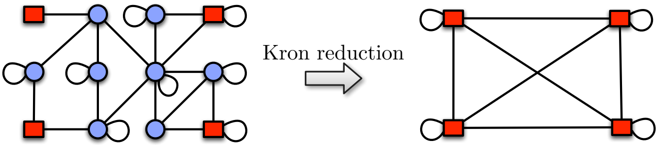
   $\Rightarrow$ algebraic connectivity λ_2 is non-decreasing

   - effect of self-loops $\circlearrowleft$ on loop-less Laplacian matrices:

   $$\lambda_2(L_{\text{reduced}}) + \max\{\circlearrowleft\} \geq \lambda_2(L_{\text{network}}) + \min\{\circlearrowleft\}$$

   $\Rightarrow$ self-loops weaken the algebraic connectivity λ_2

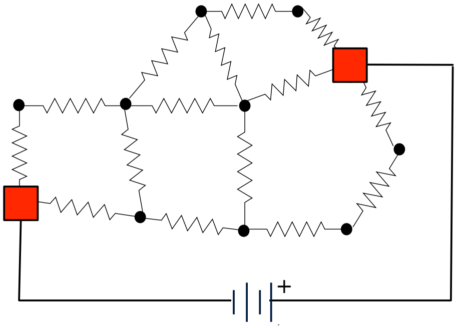
Example: all mutual edges have unit weight



without self-loops: $\lambda_2(L_{\text{network}}) = 0.39 \leq 0.69 = \lambda_2(L_{\text{reduced}})$
with unit self-loops: $\lambda_2(L_{\text{network}}) = 0.39 \geq 0.29 = \lambda_2(L_{\text{reduced}})$

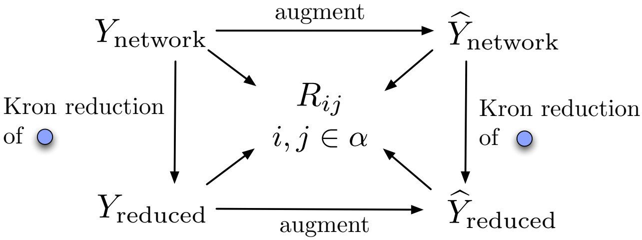
Kron reduction of graphs: properties

7. **Effective resistance** R_{ij} :



7. **Effective resistance** R_{ij} :

- **Equivalence and invariance** of R_{ij} among ■ nodes:



- no self-loops: R_{ij} among ■ uniform $\Leftrightarrow$ $\frac{1}{R_{ij}} = \frac{■}{2} |Y_{\text{red}}(i,j)|$
- self-loops: R_{ij} among ■ & ◆ uniform $\Leftrightarrow$ $\frac{1}{R_{ij}} = \frac{■}{2} |Y_{\text{red}}(i,j)| + \max\{⭘\}$

Synchronization conditions in structure-preserving model

Assumption I: lossless network and uniform voltage levels V at generators

1. **Spectral sync condition:** $\lambda_2(L(P_{ij})) \geq \ldots$ becomes

$$\lambda_2(i \cdot \mathbf{Y}_{\text{network}}) > \left\| \left(\tilde{P}_1 - \tilde{P}_2, \ldots \right) \right\|_2 \cdot \frac{1}{V^2} + \max\{\circlearrowleft\}$$

or

Assumption II: effective resistance R among generator nodes is uniform

2. **Resistance-based sync condition:** $\sum_{\ell=1}^{n} \min_{k \in \{i,j\} \setminus \{\ell\}} P_{k\ell} \geq \ldots$ becomes

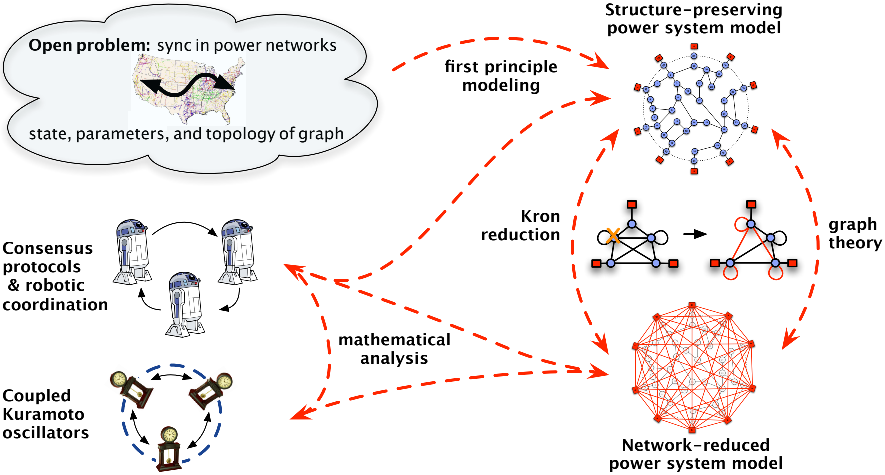
$$\frac{1}{R} > \max_{i,j} \left| \tilde{P}_i - \tilde{P}_j \right| \cdot \frac{1}{2V^2} + \max\{\circlearrowleft\}$$

$\Rightarrow$ structure-preserving power network model with linear loads synchronizes

Outline

1. Introduction and Motivation
2. Mathematical Modeling of a Power Network
3. Synchronization in the Kuramoto Model
4. From the Kuramoto Model to the Power Network Model
5. Kron Reduction
6. **Conclusions**

Conclusions



Ambitious workplan: sharpest conditions for most realistic models