An Optimization Approach to Design of Electric Power Distribution Networks

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February 4, 2010

Outline

- Resistive Network Model (DC approximation to AC power flow)
- Optimal line-sizing within a given graph
- Non-convex extension to select the graph (heuristic)
- Adding Robustness to dropping lines/generators

Resistive Network Model

Suppose that we are given:

- A graph G with n nodes and m edges.
- Currents b_i into each node ($\sum_i b_i = 0$). $b_i > 0$ at sources, $b_i < 0$ at sinks and $b_i = 0$ and transmission nodes.
- Conductances $\theta_{ij} > 0$ for all lines $\{i,j\} \in G$.

Node potentials u determined by linear system of equations:

$$K(\theta)u = b$$

Here, $K(\theta)$ is the $n \times n$ *conductance matrix*:

$$K(\theta) = \sum_{i,j} \theta_{ij}(e_i - e_j)^T(e_i - e_j) = A^T \mathrm{Diag}(\theta) A$$

where $\{e_i\}$ are the standard basis vectors and A is $n \times m$ with columns $(e_i - e_j)$ corresponding to lines of G.

Power Loss

For connected G and $\theta > 0$,

$$u = \hat{K}^{-1} b \triangleq (K + 11^T)^{-1} b.$$

Power loss due to resistive heating of the lines:

$$L(\theta) = \sum_{ij} \theta_{ij} (u_i - u_j)^2 = u^T K(\theta) u = b^T \hat{K}(\theta)^{-1} b$$

For random b, we obtain the expected power loss:

$$L(\theta) = \langle b^T \hat{K}(\theta)^{-1} b \rangle = \operatorname{tr}(\hat{K}(\theta)^{-1} \cdot \langle bb^T \rangle) \triangleq \operatorname{tr}(\hat{K}(\theta)^{-1} B)$$

It follows from convexity of $f(X) = \operatorname{tr}(X^{-1})$ for $X \succeq 0$ that $L(\theta)$ is convex.

Convex Network Optimization

This leads to the convex optimization problem of sizing lines (controlling conductances) to minimize power loss subject budget constraint:

$$\begin{aligned} \text{minimize} \quad & L(\theta) \\ \text{subject to} \quad & \theta \geq 0 \\ & a^T\theta \leq C \end{aligned}$$

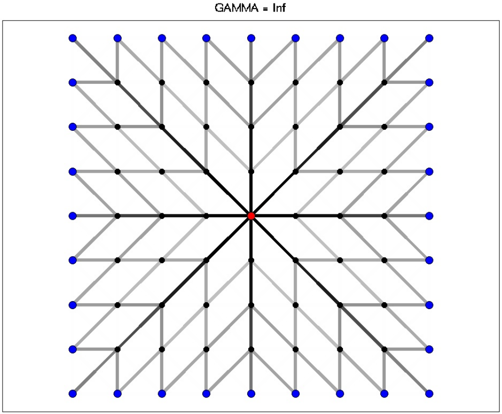
where a_ℓ is cost of conductance on line ℓ (proportional to length of line). Boyd, Ghosh and Saberi formulated this problem for *total resistance* ($B = I$).

Alternatively, one may find the most cost-effective network:

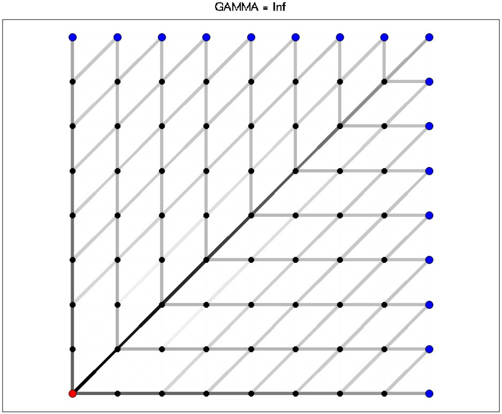
$$\min_{\theta \geq 0}\{L(\theta) + \lambda a^T\theta\}$$

where λ^{-1} represents the cost of power loss (accrued over lifetime of network).

Single-Generator Examples



Single-Generator Examples



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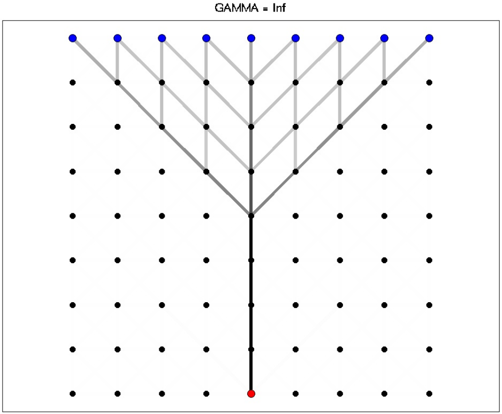

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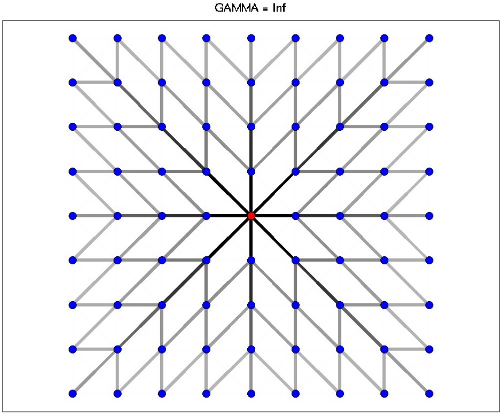

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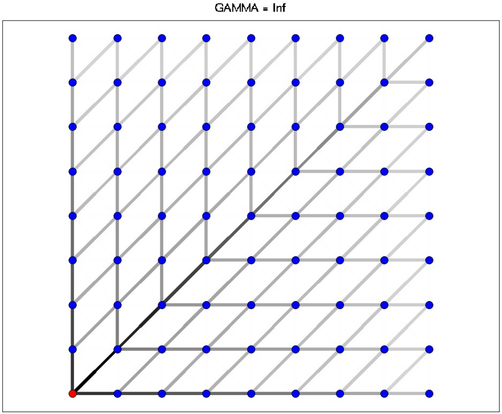
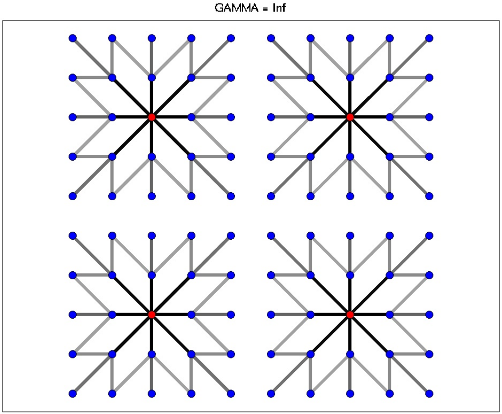


Multi-Generator Example



Imposing Sparsity (Non-convex continuation)

In practice, we may also require that the network should by sparse. We formulate this by adding zero-conductance cost on lines:

$$\min_{\theta \geq 0} \{ L(\theta) + a^T \theta + b^T \phi(\theta) \}$$

where $\phi(t) = 0$ if $\theta = 0$ and $\phi(t) = 1$ if $t > 0$. This is a difficult combinatorial problem.
We smooth this to a continuous optimization, replacing ϕ by:

$$\phi_\gamma(t) = \frac{t}{t + \gamma}$$

The convex optimization is recovered as $\gamma \to \infty$ and the combinatorial one as $\gamma \to 0$. We start with the convex global minimum, and the "track" the solution as $\gamma \to 0$.

Majorization-Minimization Algorithm

To solve each non-convex subproblem (for fixed γ), we iteratively linearize the concave penalty function to recover the convex problem with modified conductance costs.

$$\theta^{(t+1)} = \arg\min_{\theta \geq 0} \{ L(\theta) + [a + b \circ \nabla \phi_\gamma(\theta^{(t)})]^T \theta \}$$

It monotonically decreases the non-convex objective and (almost always) converges to a local minimum. Thus our entire algorithm consists of solving a sequence of convex network optimization problems.

Similar to method of Candes and Boyd for enhancing sparsity in compressed sensing.

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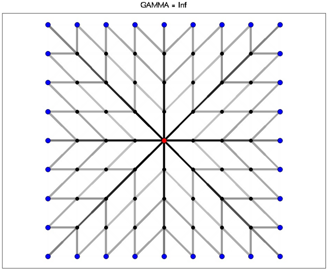

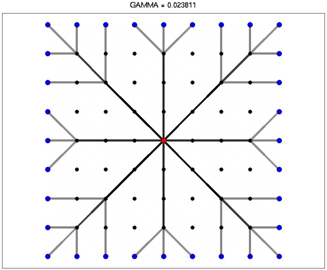

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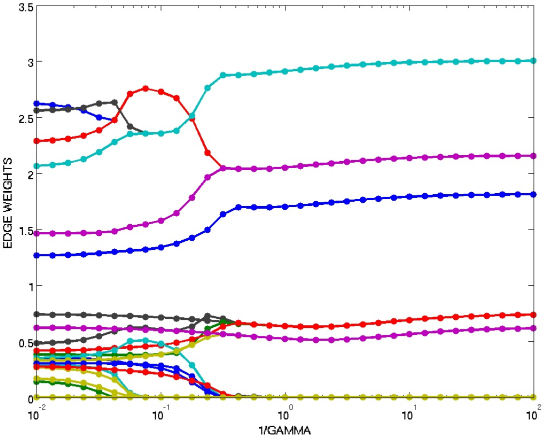

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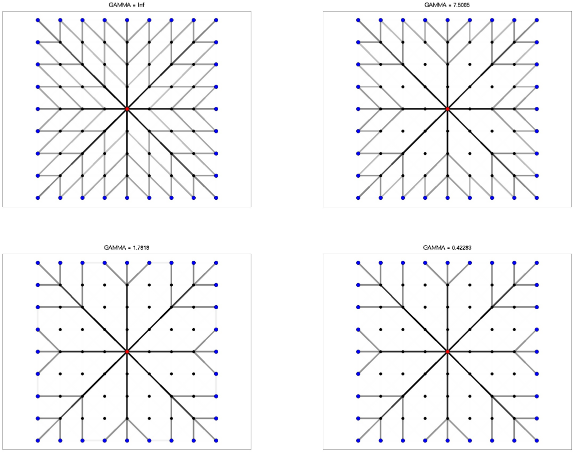
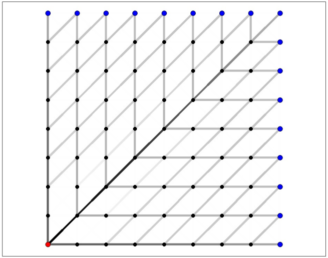
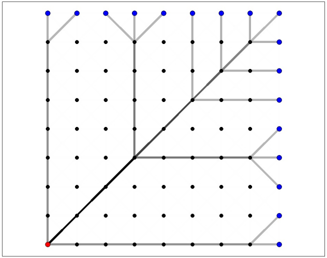


Single-Generator Examples

GAMMA = Inf



GAMMA = 0.023811



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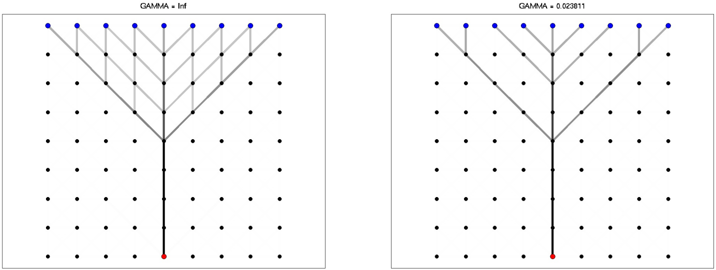

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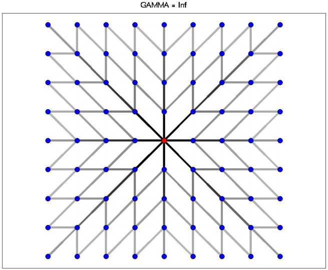

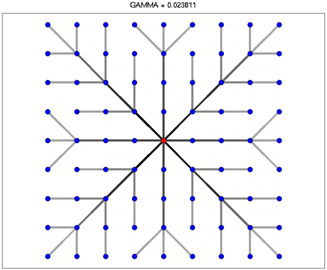

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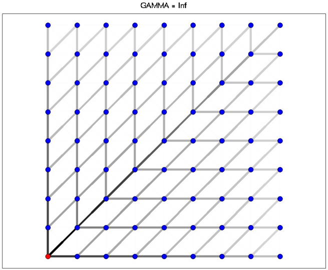

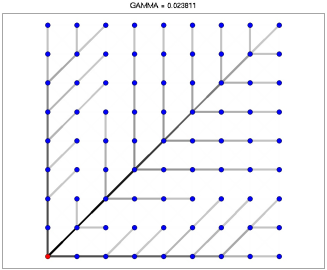

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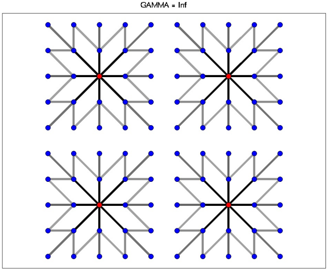

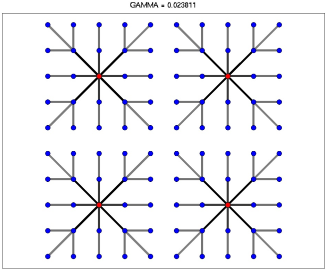

Adding Robustness

To impose the requirement that the network design should be robust to failures of lines or generators, we use the worst-case power dissipation:

$$\hat{L}^{\setminus k}(\theta) = \max_{z \in \{0,1\}^m | 1^T z = m-k} L(z \circ \theta)$$

It is tractable to compute only for small values of k.

Note, the point-wise maximum over a collection of convex function is convex.

So the linearized problem is again a convex optimization problem at every step continuation/MM procedure.

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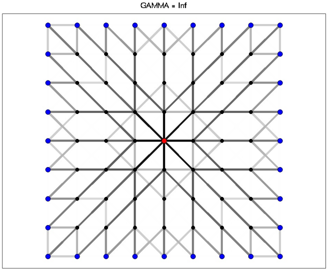

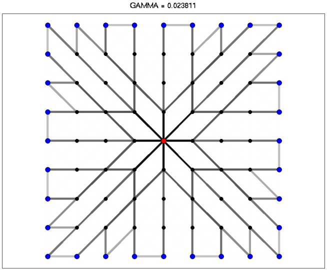

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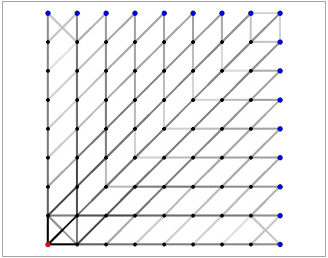

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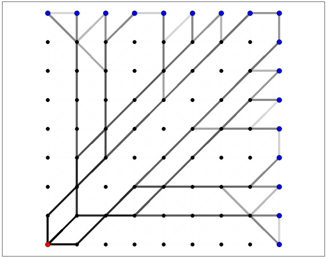

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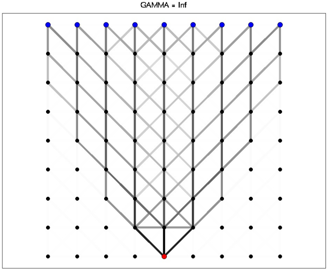

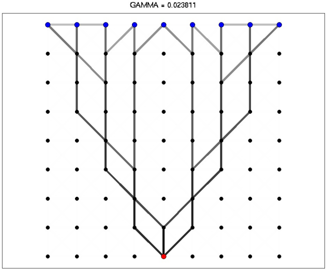

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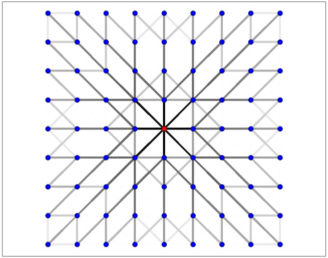

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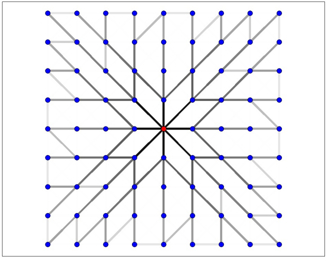

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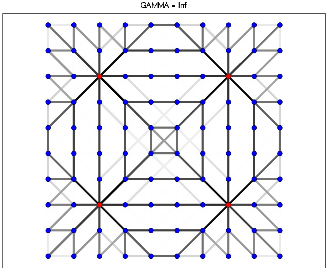

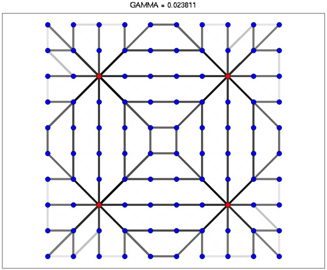

Conclusion

A promising heuristic approach to design of power distribution networks. However, cannot gaurantee global optimum.

Future Work:

- Bounding optimality gap?
- Use non-convex continuation approach to place generators
- possibly useful for graph partitioning problems
- adding further constraints (e.g. don't overload lines)
- extension to (exact) AC power flow?

Thanks!