Passive advection in nonlinear medium

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Forced advection of passive tracer in nonlinear medium by a smooth flow is considered. Effective theory for small scale scalar fluctuations is shown to be linear (asymptotic free) and universal. Structure functions demonstrate an extremely anomalous, intermittent behavior while the dissipative anomaly is absent in the problem. © *1999 American Institute of Physics.*
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I. INTRODUCTION

Turbulence is a very nonequilibrium state of nature, which becomes stationary if energy is supplied permanently at large scales. To construct a theory of turbulence means to describe temporal and spatial distributions of velocity and variety of different thermodynamic characteristics of the fluid, i.e., density if turbulence is compressible, temperature if thermo-advection is applied, relative concentration of components in the case of multicomponent (color) flow, magnetic field distribution in a conducting fluid, etc. Dynamics of different fields describing a real turbulent flow is both nonlocal and nonlinear. We call the general situation *active* to emphasize the reciprocal character of interaction between velocity field and thermodynamic characteristic(s). However, sometimes the effect of a thermodynamic field on the velocity distribution is suppressed. It takes place, for example, if scales are separated: a typical spatio-temporal scale of velocity is much larger than one of a thermodynamic quantity. The case when it is theoretically justified to neglect the effect of back reaction of thermodynamic field on velocity, in comparison with those of advection and nonlinearity, is called *passive*. The passiveness does not necessarily mean linearity. Moreover, our objective is to study the passive yet nonlinear situation.

We consider a thermodynamic quantity θ governed by

$$\frac{d}{dt}\theta = -\frac{\delta H\{\theta\}}{\delta\theta} + \phi(t;\mathbf{r}), \tag{1}$$

$$H\{\theta\} \equiv \int d\mathbf{r}\left[\frac{\kappa}{2}(\nabla\theta)^2 + U(\theta)\right], \tag{2}$$

where $H\{\theta\}$ is a positive thermodynamic functional of the system, $U(\theta)$ is a confined ($U\to+\infty$ at $\theta\to\pm\infty$) potential, κ is the diffusion coefficient, and $\phi(t;\mathbf{r})$ stands for statistically steady forcing to provide constant supply of (otherwise relaxational) θ dynamics at large scales. We will discuss here the simplest case possible, when the thermodynamic field is a scalar (notice, however, that generalization of the theory discussed for a vector or generally tensorial object is possible). θ is imbedded in a turbulent flow, i.e., the temporal derivative is extended by the sweeping term

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u}\nabla_r, \tag{3}$$

where the incompressible velocity field $\mathbf{u}(t;r)$ is prescribed to be known statistically.

We aim at finding the statistics of the passive scalar θ fixed by (1)–(3) in the inertial interval of scales, i.e., for scales that are less than both the velocity correlation scale L_u and the scale of the scalar supply L, and larger than the diffusion scale. Incompressible velocity field at those scales is modeled by the first term of its local expansion in the separation relating the reference point to the current one,

$$\mathbf{u}(t;\mathbf{r}) = \hat{\sigma}(t)\mathbf{r}. \tag{4}$$

Here $\hat{\sigma}(t)$ is a $d\times d$ traceless random matrix of the velocity's derivatives.

Problems (1)–(4) describe forced advection of a scalar pollutant in the viscous–convective range absorbed or generated, depending on the sign of the nonlinear rate $\partial_\theta^2 U(\theta)$, for example via a chemical reaction with other species presented abundant in the flow. The problem is of a fundamental importance for geophysical atmospheric turbulence (see Ref. 1 for review). Other relevant phenomenon is turbulent thermo-advection in a cell attached to thermal bath (see Ref. 2 and reference therein). Then, $\partial_\theta^2 U(\theta)$ is the nonlinear heat transfer coefficient and $\theta(t;\mathbf{r})$ measures local deviation from the bath temperature. Many regimes of premixed turbulent combustion are also governed by (1)–(4).[3] The last (but not the least) problem to be mentioned is the phase ordering in a system described by a nonconserved scalar order parameter (a very well known object of the phase transition theory, see Refs. 4–7 for reviews) advected by large scale turbulent flow (some interesting problem combining advection and critical dynamics was studied in Ref. 8).

Our consideration will be based essentially on understanding, results and general terminology emerged from studies of the pure problem of passive scalar advection (no medium effect at all, $U=0$) having almost 5 decades of history (see Obukhov and Corrsin papers[9,10] for the earliest contributions). Batchelor[11] has pioneered the study of the smooth velocity field limit (4), which nowadays has grown to be (through the important contributions of many people[12–23]) one of the most advanced theories in the field.

A temporal short-correlated but spatially nonsmooth model of velocity, one which was given more than 2 decades later (the first ever analytical evidence of intermittency in turbulence) was invented by Kraichnan.[12,24] Structure functions of scalar difference in the convective range

$$S_q(r)=\langle|\theta(t;r)-\theta(t;0)|^q\rangle\sim r^{\xi_q}, \tag{5}$$

became the key object in the intermittency study. The anomalous scaling, $\Delta_{2n}\equiv n\xi_2-\xi_{2n}$, describing the law of the algebraic growth with L/r of the dimensionless ratio, $S_{2n}(r)/[S_2(r)]^n$, was shown to take place generically.[25–27] The anomalous exponents were calculated perturbatively in expansions about three nonanomalous ($\Delta_{2n}=0$) limits, of large space dimensionality d,[25,28] extremely nonsmooth[26,29] and almost smooth[27] velocities, respectively. A strong anomalous scaling (saturation of ξ_{2n} to a constant) was found for the Kraichnan model at the largest n by a steepest descent formalism.[30,31] Although the restricted asymptotic information about anomalous exponents in the model is available, a future possibility to establish the rigorously complete dependence of ξ_{2n} on n, d and the degree of velocity nonsmoothness seems very unlikely (in a sense, recent Lagrangian numerics[32] compensates for the lack of rigorous information).

Problems (1)–(4) also show anomalous scaling, $\xi_{2n}<n\xi_2$. Here, the intermittency is resolved analytically for an arbitrary asymptotically (at $\theta\to\pm\infty$) convex potential $U(\theta)$: $\delta\theta_r$ obeys the same statistics as one would expect from an auxiliary (linear!) problem with quadratic potential, $U^*(\theta)=\alpha\theta^2$, where α is given by the average of $\partial_\theta^2U(\theta)$ with respect to single point scalar distribution, $\mathcal{P}_1\sim\exp[-U(\theta)/\chi_0]$, with $\chi_0\equiv\int_0^\infty dt\langle\phi(t;0)\phi(0;0)\rangle$. α is always positive, i.e., at the smallest scales the effect of nonlinearity, generally alternating between damping and acceleration, is reduced to a pure linear damping. Finally, for the short-correlated velocity statistics we have found anomalous exponents

$$\xi_q=\min\left\{q,\sqrt{\left[\frac{\bar\lambda}{\Delta}\right]^2+\frac{2\alpha q}{\Delta}}-\frac{\bar\lambda}{\Delta}\right\}, \tag{6}$$

where $\bar\lambda$ and Δ are, respectively, the average and dispersion of the exponential rate of line stretching, $\lambda(t)=t^{-1}\ln[R(t)/R(0)]$, with $\mathbf{R}(t)$, satisfied to $\dot{\mathbf{R}}(t)=\hat\sigma(t)\mathbf{R}(t)$. Generalization of (6) for the case of arbitrary correlated in time velocity is given by (34).

The anomalous behavior (6) differs from that perceived in the Kraichnan model. First of all, ξ_{2n} as a function of n does not saturate to a constant at the largest n but keeps growing with n as $\sqrt{n}$ (that is in the δ-correlated case, in general one gets $n^{1-1/\beta}$, $\beta>1$). The second (and major) difference is associated with the concept of dissipative anomaly. It is generally accepted to talk about dissipative anomaly if some stationary object calculated at zero dissipation ($\kappa=0$) does not coincide with its $\kappa\to0$ counterpart. In the Kraichnan model the anomalous scaling coexists with the dissipative anomaly.[24] However, the nonlinear problems (1)–(4), as well as their linear descendant, show no dissipative anomaly while the anomalous scaling is present. We base the important conclusion on the following *no anomaly* criterion (which, we believe, is general): if zero dissipation analysis produces a normalizable and everywhere positive solution for the probability density functional (PDF) of fluctuated field (θ in our case) then the dissipative anomaly is absent.

Physics of the no anomaly phenomenon is simple, scalar pumped at the integral scale and transferred downscales by advection is relaxationally destroyed at all the scales. Practically no scalar reaches the dissipative scale and therefore the effect of diffusion is really negligible. The emerging steady state is a result of an interplay between pumping, advection and nonlinearity but the diffusion, if small, is simply not involved.

The problem is formulated in Sec. II. To describe the convective range scalar fluctuations we show how to integrate out the large scale contribution in Sec. III. The scale separation results in suppression of nonlinearity. The effective small scale theory appears to be a linear one with uniform damping. All the final answers emerging from the study of the linear problem are presented in Sec. IV. Section V is reserved for conclusions.

II. FORMULATION OF THE PROBLEM

(1)–(3) describe advection of a passive scalar $\theta(t;\mathbf{r})$ by the smooth incompressible velocity field (4). The scalar is forced by random field $\phi(t;\mathbf{r})$, which for the sake of simplicity is considered to be Gaussian, and therefore fixed unambiguously by

$$\langle\phi(t_1;\mathbf{r}_1)\phi(t_2;\mathbf{r}_2)\rangle=\chi(|\mathbf{r}_1-\mathbf{r}_2|)\delta(t_1-t_2). \tag{7}$$

Here the function $\chi(r)$ decays sufficiently fast at large scales, $r>L$ and $\chi_0=\chi(0)$ is the flux of θ^2 pumped into the system. $\hat\sigma$ is a random in time matrix process described by its PDF, $\Phi\{\hat\sigma(t)\}$, which is supposed to be known. Diffusion is small, i.e., the range of scales in between $r_d=\sqrt{\kappa}/[S/\tau]^{1/4}$ (S and τ are typical values of the strain and velocity correlation time, respectively) and L (usually called the convective interval) is sufficiently large, $L/r_d\gg1$.

Our main purpose is to find the two-point scalar PDF,

$$\mathcal{P}_2(x_+,x_-|r)\equiv\langle\delta[x_--\theta(t;0)+\theta(t;\mathbf{r})]\delta\times[x_+-\theta(t;0)-\theta(t;\mathbf{r})]\rangle, \tag{8}$$

and the scalar structure functions

$$\mathcal{S}_{2n}(r)\equiv\langle[\theta(t;\mathbf{r})-\theta(t;0)]^{2n}\rangle, \tag{9}$$

where averaging with respect to both $\hat\sigma(t)$ and $\phi(t;\mathbf{r})$ is assumed.

Other important objects used in the course of the forthcoming calculations will be the two-point scalar PDF conditioned by $\hat\sigma(t)$

$$\mathcal{G}_2(x_1,x_2|\mathbf{r}_{1,2};t;\{\hat\sigma(t');-\infty\leqslant t'\leqslant t\})\equiv\langle\delta[x_1-\theta(t;\mathbf{r}_1)]\delta[x_2-\theta(t;\mathbf{r}_2)]\rangle_\phi, \tag{10}$$

and the single point scalar PDF

$$\mathcal{P}_1(x)\equiv\langle\delta[x-\theta(t;\mathbf{r})]\rangle. \tag{11}$$

Deep inside the convective range (at $L \gg r_{12}$), $|\theta_1-\theta_2| \ll |\theta_1+\theta_2|$, and (10) can be decomposed into the product

$$\mathcal{G}_2(\theta_1,\theta_2|\mathbf{r}_{1,2};t;\{\hat\sigma(t');-\infty\leq t'\leq t\}) = \mathcal{P}_1(\theta_1) * \mathcal{G}_-(\theta_1-\theta_2|\mathbf{r}_1-\mathbf{r}_2;t;\{\hat\sigma(t');-\infty\leq t'\leq t\}), \tag{12}$$

where spatial homogeneity of the Batchelor case was taken into account. The average of (12) over $\hat\sigma$ reads as

$$\mathcal{P}_2(\theta_1,\theta_2|r_{12}) = \mathcal{P}_1(\theta_1) * \mathcal{P}_-(\theta_1-\theta_2|r_{12}). \tag{13}$$

The assumption on *the absence of the dissipative anomaly* in the case of a very small diffusion lies in the core of our consideration. The formal consequence of the statement is the possibility to omit the dissipative κ-dependent term from (2) already on the dynamical (yet unaveraged) level. The no-anomaly assumption will be justified by the positivity and normalizability of the derived answers for PDFs.

III. REDUCTION OF THE NONLINEAR PROBLEM TO A LINEAR ONE

In the absence of diffusion (1)–(4) can be integrated along the Lagrangian trajectories (characteristics)

$$\frac{d}{dt'}\theta[t';\rho(t')] = -\left.\frac{dU}{d\theta}\right|_{\theta[t';\rho(t')]} + \phi[t';\rho(t')], \tag{14}$$

$$\frac{d}{dt'}\rho(t') = \hat\sigma(t')\rho(t'), \quad \rho(t)=\mathbf{r}, \quad -\infty<t'<t. \tag{15}$$

The nonlinearity leads to dumping of the scalar in regions of convex ($\partial_\theta U>0$) potential while it generates the scalar fluctuations anywhere else. Fokker–Planck equations (see Ref. 33 for similar calculations) derived out of (14), (15) by means of direct averaging over the Gaussian noise ϕ are

$$\left[\partial_\theta \frac{dU(\theta)}{d\theta} - \chi(0)\partial_\theta^2\right]\mathcal{P}_1 = 0, \tag{16}$$

$$\left[\partial_t + \sum_{i=1,2}\left(\sigma^{\mu\nu}(t) r_i^\mu \partial_{r_i}^\nu - \partial_{\theta_i}\frac{dU(\theta_i)}{d\theta_i}\right) - \sum_{i,j=1,2}\chi(r_i-r_j)\partial_{\theta_i}\partial_{\theta j}\right]\mathcal{G}_2 = 0, \tag{17}$$

where $\mathcal{G}_2$ is not stationary, since it does depend on time explicitly through $\hat\sigma(t)$. Integrating (17) with respect to $\theta_+ = \theta_1+\theta_2$ and assuming that the integral is formed at $|\theta_1-\theta_2| \ll |\theta_1+\theta_2|$, where (12) is valid, we arrive at the close equation for the scalar difference PDF

$$\{\partial_t + (\sigma^{\mu\nu}(t) r^\mu \partial_r^\nu - \alpha\partial_x x) - 2[\chi(0)-\chi(r)]\partial_x^2\} \times \mathcal{G}_-(x|\mathbf{r};t;\{\hat\sigma(t');-\infty\leq t'\leq t\}) = 0. \tag{18}$$

Here α is defined as the following average over the large scale θ statistics

$$\alpha \equiv \left\langle \frac{d^2U(\theta)}{d\theta^2}\right\rangle_{\mathrm{LS}} \equiv \int_{-\infty}^{\infty} d\theta \frac{d^2U(\theta)}{d\theta^2}\mathcal{P}_1(\theta). \tag{19}$$

The normalized and everywhere positive solution of (16) is

$$\mathcal{P}_1(\theta) = \frac{\exp[-U(\theta)/\chi_0]}{\int_{-\infty}^{\infty} d\theta \exp[-U(\theta)/\chi_0]}. \tag{20}$$

Substitution of (20) into (19) gives

$$\alpha = \frac{\langle [dU(\theta)/d\theta]^2\rangle_{\mathrm{LS}}}{\chi_0} = \frac{\int_{-\infty}^{\infty} d\theta [dU(\theta)/d\theta]^2 \exp[-U(\theta)/\chi_0]}{\chi_0 \int_{-\infty}^{\infty} d\theta \exp[-U(\theta)/\chi_0]}, \tag{21}$$

i.e., α is always positive constant if the potential $U(x)$ is asymptotically ($x\to\pm\infty$) convex. Therefore, we have found that deep inside the convective interval regions of scalar generation are suppressed statistically.

On the basis of (16) and (18) we conclude that from the point of view of the small scale statistics of scalar difference our problem is equivalent to the linear one, with $dU(\theta)/d\theta$ being replaced just by $\alpha\theta$. In other terms, we may proceed averaging the linear dynamical equation

$$\partial_t\theta + \sigma^{\mu\nu}(t) r^\mu \nabla_r^\nu \theta = -\alpha\theta + \phi(t;\mathbf{r}), \tag{22}$$

instead of the original nonlinear one. The steady distribution of the scalar difference enforced by (22) was the subject of a recent paper,[34] the method and results of which will be briefed and generalized in Sec. IV.

IV. VELOCITY AVERAGING: ANOMALOUS SCALING

The linear analog of (14) is

$$\theta(t;\mathbf{r}) = \int_0^\infty dt' \exp[-\alpha t']\phi[t';\rho(t-t')], \tag{23}$$

where ρ is the Lagrangian trajectory fixed by (15).

Consider the case of δ correlated in time velocity

$$\langle \sigma_{\alpha\beta}(t_1)\sigma_{\mu\nu}(t_2)\rangle = D[(d+1)\delta^{\alpha\mu}\delta^{\beta\nu} - \delta^{\alpha\nu}\delta^{\beta\mu} - \delta^{\alpha\beta}\delta^{\mu\nu}]\delta(t_1-t_2), \tag{24}$$

where d is space dimensionality. For the purpose of the $2n$th structure function calculation it is enough to discuss the simultaneous product, $F_{1\cdots 2n} \equiv \langle \theta_1\cdots\theta_{2n}\rangle$, which according to (23), (15) is

$$F_{1\cdots 2n} = \sum_{\{i_1,\dots,i_{2n}\}}^{\{1,\cdots,2n\}} \left\langle \prod_{k=1}^{n} \int_0^\infty dt_k\, e^{-\alpha t_k}\chi[\hat W(t_k) r_{i_k;i_{k+1}}]\right\rangle_{\hat\sigma}, \tag{25}$$

$$\hat W(t) \equiv T\exp\left[\int_0^t dt' \hat\sigma(t')\right], \quad \frac{d\hat W(t)}{dt} = \hat\sigma(t)\hat W(t). \tag{26}$$

Calculation of $F_{1\cdots 2n}$ is essentially simplified for the collinear configuration $\mathbf{r}_i = \mathbf{n} r_i$ (we will argue below why the scaling results derived in the collinear way hold generally). Then the $2n\times(d-1)$parametric average (25) is reduced to the following single-parametric one:

$$F_{1\cdots 2n}=\sum_{\{i_1,\ldots,i_{2n}\}}^{\{1,\cdots,2n\}}\left\langle\prod_{k=1}^{n}\int_0^\infty dt_k\, e^{-\alpha t_k}\chi[e^{\eta(t_k)}r_{i_k;i_{k+1}}]\right\rangle, \tag{27}$$

with $r_{ij}\equiv|\mathbf{r}_i-\mathbf{r}_j|$. Here, the longitudinal stretching rate, $\eta(t)\equiv\ln|\hat{W}(t)\mathbf{n}|$, is the only fluctuating quantity left. The $\alpha=0$ version of (27) was calculated in Ref. 17 for the $d=2$ case and generalized for any $d\geqslant 2$ in Ref. 18 via a change of variables and further straightforward transformation of the path integral standing for the average over $\hat{\sigma}(t)$. It is shown in Refs. 17 and 18 that the η measure (of the exponential stretching rate of a line element) is a shifted Gaussian one

$$\mathcal{D}\eta(t)\exp\left[-\int_0^\infty dt\frac{(\dot{\eta}-\bar{\lambda})^2}{2\Delta}\right], \tag{28}$$

characterized by mean Lyapunov exponent $\bar{\lambda}=Dd(d-1)/2$, and dispersion $\Delta=D(d-1)$. (28) applied to (27) produces

$$\frac{F_{1\cdots 2n}}{n!}=\int\left(\prod_{i=1}^{n}dt_i\,d\eta_i\right)\exp\left(\frac{\bar{\lambda}}{\Delta}\eta_1-\frac{\bar{\lambda}^2}{2\Delta}t_1\right)\times\sum_{\{k_i,\cdots,k_{2n}\}}^{\{1,\cdots,2n\}}\prod_{i=1}^{n}[e^{-2\alpha t_i}\chi(e^{\eta_i}r_{k_{2i},k_{2i+1}})\times G(t_{i-1,i};\eta_{i-1,i})], \tag{29}$$

where η_i $(i\leqslant n)$ integrations are not restricted, $0\leqslant t_n\leqslant\cdots\leqslant t_1\leqslant\infty$, $t_{n+1}=\eta_{n+1}=0$, $t_{i,k}\equiv t_i-t_k$ (with equivalent notations for η) and

$$G(t;\eta)\equiv\frac{\exp[-(\eta^2/2\Delta t)]}{\sqrt{2\pi\Delta t}}. \tag{30}$$

The integrand of (29) decays exponentially in time and is dominated by the contribution into the integral formed at $t_i\sim 1/\alpha$. The leading term does not depend on any r_{ij} and gives no contribution into $2n$th order structure function. The first actual r-dependent contribution stems from $n-1$ temporal integrals formed at $\tau\sim 1/\alpha$, and one at $t_i\sim\tau_r\sim\ln[L/r]/\max\{\alpha,D\}$. This special integration brings a spatial dependence into the object, therefore on a single distance. Generally, there exists a variety of terms with all the possible combinations, like term with k integration formed at τ, with $n-k$ ones at τ_r, and therefore dependent explicitly on $2(n-k)$ spatial points. However, we are looking exclusively for a term dependent on all the $2n$ points since only such a term contributes $S_{2n}(r)$. It is simple to calculate the scaling of this term making use of the temporal separation $\tau_r\gg\tau$. Indeed, the large time contribution may be extracted out of (29) in a saddle-point calculation. Variation of all the exponential terms in (29) with respect to t_i gives a chain of saddle equations. The χ functions in the integrand of (29) limits the η integrations from above by $\ln[L/r]$. Therefore, the desirable $2n$-point contribution forms at $t_i=\sqrt{\bar{\lambda}/[\Delta(2\alpha n\Delta+\bar{\lambda}^2/2)]}\ln[L/r]$, and $\eta_i=\ln[L/r]$, where it is assumed that in the leading logarithmic order there is no need to distinguish between contributions of different separations r_{ij}. Substituting the saddle-point values of t_i and η_i into (29) one arrives at the anomalous part of (6), with $q=2n$. The normal-scaling counterpart of (6) originates from expansion of the integrand of (29) in a regular series in r^2.

The basic physics of nonzero ξ_{2n} (means deviating from the naive balance of pumping and advection) and generally anomalous ($\xi_{2n}<n\xi_2$) scaling at $\alpha>0$ can be stated quite clearly. According to (23) the advection changes scales but not amplitude, while the amplitude of the injected scalar field decays exponentially from the time of injection at the constant rate α. The temporal integrals in (29) form at the mean time to reach a scale which is proportional to the negative log of the scale. However, the effective spread in the factor by which amplitude has decayed, upon reaching a given scale, increases as the scale decreases. This is why $\xi_{2n}>0$. Also there is more room for fluctuations about the mean time due to the interference between the exponential decay of the scalar amplitude and fluctuations of the stretching rate η. Thus intermittency increases when the scale size decreases.

Another way to derive (6) from (23) is to construct $S_{2n}(r)$ directly. It is easy to check that the structure functions of different orders are produced by the PDF satisfied to

$$\bar{\lambda}r^{1-2\bar{\lambda}/\Delta}\partial_r r^{1+2\bar{\lambda}/\Delta}\partial_r\mathcal{P}_-+\alpha\partial_x(x\mathcal{P}_-)+[\chi_0-\chi(r)]r^2\partial_x^2\mathcal{P}_-=0. \tag{31}$$

The solution of (31), in the regime where you can neglect the χ-dependent term is

$$\mathcal{P}_-(x|r)=\frac{1}{2\pi i}\frac{1}{\theta_L}\int_{0^+-i\infty}^{0^++i\infty}ds\left(\frac{\theta_L}{x}\right)^{s+1}\times\left(\frac{r}{L}\right)^{\sqrt{d^2/4+\alpha s/[D(d-1)]}-d/2}a_s. \tag{32}$$

Here, a_s is a function fixed by matching at the integral scale, roughly, $\mathcal{P}_-(x|L)\sim\mathcal{P}_1(x)$, where $\mathcal{P}_1(x)$ is given by (20). The PDF (32) appears to be positive and normalizable, therefore confirming the initial hypothesis on the absence of dissipative anomaly. Also, (32) shows that (6) holds for general (not only even integer) positive q.

The assumption of collinearity made above is not crucially important for evaluation of (25). The calculations become more evolved (but still doable) mainly because of the necessity to follow additionally the dynamics of the $d-1$ subleading Lyapunov exponents (not just the leading one), which all are entering the argument of the χ function in (25). In the δ-correlated case the Lyapunov exponents' statistics is Gaussian and the Lyapunov spectrum is equidistant.[18,20] Therefore, complications comes through the set of $n(d-1)$ integrals to be inserted in the generalization of (29). Having the large parameter $\ln[L/r]$ in hand we can treat all the integrations in a saddle-point manner and the final answer for the correlation function will be consistent with what was derived above in the collinear case. In the simple calculations we therefore check directly the absence of dissipative anomaly in this case. In Lagrangian terms it can be stated as follows: two initially close Lagrangian trajectories, making a dominant contribution into the correlation functions of the dumped scalar, stay close forever (see relevant discussion in Ref. 21). Notice that the dissipative anomaly is present in the pure Batchelor case (no medium, $U=0$),[21] i.e., integration with respect to subleading Lyapunov exponents in the case results in the effective renormalization of (28): the dispersion of a line element exponential stretching rate $\Delta = D(d-1)$ should be replaced by one characterizing the exponential stretching rate of the largest dimension of a fluid blob, $\Delta_b = 3(d-1)D/(d+1)$.[23]

The last observation is about generalization of the results for the case of finite-correlated velocity statistics.[35] In the general case the η measure is not Gaussian and the positive quadratic form in the integrand of (28) is to be replaced by a convex $S(\dot{\eta})$. Then all the above calculations are easy to redo. The saddle t_i is given by $x_n \ln[L/r]$, where x_n is a solution of

$$x_n^2 \partial_{x_n}[S(x_n)/x_n] = 2\alpha n. \tag{33}$$

Finally, the expression for the exponents generalizing the anomalous part of (6) is

$$\xi_{2n} = \partial_{x_n} S(x_n). \tag{34}$$

One concludes, particularly, that at the largest n, $\xi_{2n} \propto n^{1-1/\beta}$, where $S(x \to +\infty) \to x^\beta$ ($\beta > 1$).

V. CONCLUSION

We have shown that the nonlinear problems (1)–(4) are reduced to a linear one at the smallest (still from convective not dissipative range) scales. We note the asymptotic theory about initial nonlinearity through the effective damping coefficient (19). The linear problem was solved for the general case of the large scale velocity field arbitrarily correlated in time.

The most important feature, guessed initially in the course of calculations, appears to be the absence of dissipative anomaly. Self-consistency of the hypothesis was confirmed afterwards by a direct check of positivity and normalizability of the final expression (32) for PDF. Of course, the absence of dissipative anomaly is not a common situation in turbulence. It is, however, suggested that we start analyzing any new turbulent problem from the simple *no anomaly* test.

We discussed only the case of smooth velocity here and the restriction was very important because of the absence of dissipative anomaly, scales separation and subsequently solvability of the problem. The Batchelor case is very special, since the Lagrangian dynamics of n particles, generally described by $n(d-1)$ degrees of freedom, is reduced to the dynamics of $d-1$ eigenvalues of the stretching matrix. This lies in the core of the Batchelor problem's solvability. Also, in the Batchelor case the scaling dimension of the eddy diffusivity operator coincides with one of the α (damping) dependent term. The coincidence of exponents explains the anomalous scaling, particularly the continuous dependence of the exponents on α. Any multiscale velocity field (say taken from the Kraichnan model) leads to the disbalance of the scaling dimensions of the advective and medium-originated contribution into the eddy-diffusivity operator, resulting in complete screening of any medium effect in the convective range. We conclude by this guess, which is rather brave (and, of course, is not rigorous at all). More studies in this direction, first of all on the nature of dissipative anomaly, are required.

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[35]The paragraph was inspired by general remark of M.Vergassola on averaging of exponential objects in the case of finite correlated velocity statistics.