On how a joint interaction of two innocent partners (smooth advection and linear damping) produces a strong intermittency

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Forced advection of passive scalar by a smooth d-dimensional incompressible velocity in the presence of linear damping is studied. Acting separately advection and damping do not lead to an essential intermittency of the steady scalar statistics, while being mixed together produce a very strong non-Gaussianity in the convective range: $2n$-th moment of scalar difference, $\langle[\theta(t;\mathbf{r})-\theta(t;0)]^{2n}\rangle$ is proportional to $r^{\xi_{2n}}$, $\xi_{2n}=\min\{2n,\sqrt{d^2/4+2\alpha dn/[(d-1)D]}-d/2\}$, where α/D measures the rate of the damping in the units of the stretching rate. The probability density function (PDF) of the scalar difference is also found. © *1998 American Institute of Physics.*
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Advection of passive scalar $\theta(t;\mathbf{r})$ by an incompressible velocity field is a classical problem in turbulence theory. Kraichnan pioneered analytical study of the problem inventing the temporal short-correlated but spatially nonsmooth model of velocity for which the simultaneous pair correlation function of the scalar was found.[1] However, the question of possible anomalous behavior of higher order $(n>1)$ structure functions $S_{2n}(r)=\langle(\theta(t;r)-\theta(t;0))^{2n}\rangle\sim r^{\xi_{2n}}$ was posed only 25 years later.[2] Next, the anomalous scaling $\Delta_{2n}\equiv n\xi_2-\xi_{2n}$, describing the law of the algebraic growth with L/r (where L is the scale of the scalar pumping) of the dimensionless ratio $S_{2n}(r)/[S_2(r)]^n$, was shown to exist generically.[3–5] The anomalous exponent was calculated perturbatively in expansions of about three nonanomalous $(\Delta_{2n}=0)$ limits, of large space dimensionality d,[3,6] of extremely nonsmooth[4,7] and almost smooth[5] velocities, respectively. A strong anomalous scaling (saturation of ξ_{2n} to a constant) was found for the Kraichnan model at the largest n by a steepest descent formalism.[8] Although the restricted asymptotic information about an anomalous exponent in the model is available a future possibility to establish rigorously the complete dependence of ξ_{2n} on n, d and degree of velocity nonsmoothness seems very unlikely (in a sense, recent Lagrangian numerics[9] compensates the lack of rigorous information).

In the present Brief Communication I discuss yet another passive scalar model with nontrivial anomalous behavior, $\zeta_{2n}<n\xi_2$, which is possible to resolve explicitly for all the values of the governing parameters. The model describes generalization of the smooth (Batchelor) limit of the Kraichnan model on the case of a linear damping of the scalar. The pure Batchelor model (no damping), studied in detail in Refs. 10–15, shows nonanomalous behavior. The advection free limit is also nonanomalous. However [see (5), (6)], a strong intermittency does exist generically: the scaling exponent appears to be a nontrivial function of n, d and a parameter standing for the damping-to-convection ratio. Notice that scaling of convective and damping contributions coincide and there is no Kolmogorov like argument to predict the scaling of all the orders structure functions in the problem. Besides, the problem gives an example of a turbulent situation when the dissipative anomaly is absent (zero diffusivity allows a stationary solution) while a strong intermittency $(\Delta_{2n}\neq 0)$ takes place.

Consider advection of the passive scalar $\theta(t;\mathbf{r})$ by a smooth incompressible velocity field, $\mathbf{u}(t;\mathbf{r})=\hat{\sigma}(t)\mathbf{r}$, in the presence of linear damping and diffusion,

$$\partial_t\theta+\sigma^{\mu\nu}(t)r^{\mu}\nabla_r^{\nu}\theta=\kappa\Delta_r\theta-\alpha\theta+\phi. \quad (1)$$

It is known that the small scale features of scalar are universal with respect to variation of the pumping $\phi(t;\mathbf{r})$ (see, for example, Ref. 3); therefore for the sake of simplicity the pumping will be considered to be Gaussian thus fixed unambiguously by $\langle\phi(t_1;\mathbf{r}_1)\phi(t_2;\mathbf{r}_2)\rangle=\chi(|\mathbf{r}_1-\mathbf{r}_2|)\delta(t_1-t_2)$, where $\chi(r)$ decays fast enough if r exceeds the integral scale L. The velocity is smooth downscale from L_u, which is supposed to be the largest scale in the problem $(L_u\gg L)$. Aiming for simplicity and compactness of the derivation I consider in this Brief Communication only the Gaussian statistics of $\hat{\sigma}$ fixed by

$$\frac{\langle\sigma^{\eta\mu}(t)\sigma^{\beta\nu}(t')\rangle}{D(d+1)}=\left[\delta^{\mu\nu}\delta^{\eta\beta}-\frac{\delta^{\mu\eta}\delta^{\nu\beta}+\delta^{\mu\beta}\delta^{\nu\eta}}{d+1}\right]\delta(t-t').$$

Generalization of the theory for the case of a finite temporal correlations of velocity will be published elsewhere [it does not change the general structure of all the answers, i.e., (3), (5), (6) derived below, and shows itself only in renormalization of the dimensionless coefficient α/D].

The model describes forced advection of a scalar pollutant in the viscous-convective range $(L_u\sqrt{D/\kappa}\gg 1)$ absorbed instantly and homogeneously, for example, via a chemical reaction with other species presented in abundance in the flow. Linear damping of the pollutant concentration θ is fixed here by the reaction rate α. Another physical situation governed by (1) is turbulent thermoadvection in a cell attached to a thermal bath. Then α is the heat transfer coefficient and $\theta(t;\mathbf{r})$ measures local deviation from the bath temperature.

We start studying the pair correlation function of the scalar field: $F(r_{12})=\langle\theta(t;\mathbf{r}_1)\theta(t;\mathbf{r}_2)\rangle$. Averaging two replicas of (1) one gets

$$\left[-r^{1-d}\partial_r r^d\left(D(d-1)r+\frac{2\kappa}{r}\right)\partial_r+2\alpha\right]F=\chi. \tag{2}$$

Consider the case of a step-like pumping function, when $\chi(r)=P=\text{const}$ at $r<L$, and zero otherwise. Introduce a forced solution of this equation, $F_f(r)=P\vartheta(L-r)/[2\alpha]$. Two different zero modes of the operator from the lhs of (2) should be added to $F_f(r)$, at the upper ($r>L$) and lower ($r<L$) intervals, respectively, to guarantee the continuity of $F(r)$ and its derivative at $r=L$. If the dissipative scale, $r_d=\sqrt{\kappa/D}$, is small enough, one gets

$$F(r)=\frac{P}{2\alpha}\begin{cases}1-\dfrac{\xi_-}{\xi_--\xi_+}\left(\dfrac{r}{L}\right)^{\xi_+}, & r_d\ll r<L,\\[2ex] \dfrac{\xi_+}{\xi_+-\xi_-}\left(\dfrac{r}{L}\right)^{\xi_-}, & r>L\gg r_d;\end{cases} \tag{3}$$

where $\xi_\pm\equiv\pm\sqrt{d^2/4+2\alpha d/[(d-1)D]}-d/2$. Dominant, at $r\ll L$, the contribution into the second order structure function stems from a zero mode of the operator on the rhs of (2), $S_2(r)\sim r^{\xi_-}$.

Come to the study of the scalar difference stationary PDF, $\mathcal{P}\equiv\langle\delta(x-\delta\theta_r)\rangle$. Generally, at zero diffusion ($\kappa\to0$) the stationary limit is perfectly achieved via the direct balance between the pumping ϕ and the α-damping. The scalar pumped at a large scale L and advected downscale is ''eaten'' by damping much before it reaches the dissipative range. Therefore, there is no dissipative anomaly in the case (see below for the proof) and one may derive the Fokker–Planck equation out of (1) neglecting diffusion,

$$D(d-1)r^{1-d}\partial_r r^{d+1}\partial_r\mathcal{P}+\alpha\partial_x(x\mathcal{P})+\tilde{\chi}(r)\partial_x^2\mathcal{P}=0, \tag{4}$$

where $\tilde{\chi}(r)\equiv\chi(0)-\chi(r)$. Even without solving the equation one may get closed equations for the structure functions by means of the integration of (4) against the respective moments of x. The equations supplied by the zero condition at $r\to0$ give the following small scale asymptotics for the even moments (odd moments are constrained to be zero due to isotropy and Gaussianity of the pumping), $S_{2n}/[\theta_L^{2n}]\sim[r/L]^{\xi_{2n}}$,

$$\xi_{2n}=\min\left\{2n,\sqrt{\frac{d^2}{4}+\frac{2\alpha dn}{(d-1)D}}-\frac{d}{2}\right\}, \tag{5}$$

where $\theta_L\sim P/\max\{\alpha,D\}$ stands for the amplitude of a scalar typical fluctuation at the integral scale L and $\tilde{\chi}(r)$ is assumed to have a regular expansion in r^2 about the origin. Here (5) holds for any d, n, and α. According to (5) all the moments are anomalous if α is small enough, otherwise the lowest moments are normal. Notice that the continuous dependence of the anomalous exponents on the damping rate originates from the coincidence of the scaling dimensions (zero in the Batchelor case of smooth velocity) of the bare eddy diffusivity operator and the damping-dependent correction to it.

Although the calculation of anomalous exponents was our main goal, $\mathcal{P}$ solving (4) may also be found. Consider such a moderate α, $\alpha<\alpha_c=(d-1)D$, that the pumping term in (4) can be neglected at $r\ll L$. Then, all the information about the pumping enters into consideration only through the boundary condition at the integral scale. For example, the solution of (4) giving the Gaussian PDF at the integral scale is

$$\begin{aligned}\mathcal{P}(x;r)&=\frac{\ln[L/r]}{\pi x\sqrt{a}}\left(\frac{L}{r}\right)^{d/2}Q\left(\frac{x}{\theta_L};\ln[L/r]\right),\\ Q(y;z)&\equiv\int_0^{\ln[1/y]}\exp\left[-\frac{z^2}{4at}-\frac{d^2a}{4}t-y^2\exp[2t]\right]\frac{dt}{t^{3/2}},\end{aligned} \tag{6}$$

where $a\equiv D(d-1)/\alpha$. At $L\gg r$, $\mathcal{P}(x;r)$ shows a change in behavior at $x_c\equiv\theta_L[r/L]^{1/(da)}$: $Q(y;z)$ is finite at the origin, $Q(0;z)\sim(r/L)^{d/2}/\ln[L/r]$; further, the algebraic in x decay at $x\ll x_c$, $Q(0;z)-Q(y;z)\sim y^{d^2a/4}$, turns into $Q(y;z)\sim y^{d^2a/4}\exp[-z^2/(a\ln[1/y])]$ at $x\gg x_c$. One gets particularly that at $\alpha>\alpha_c$, the anomalous result (5) is applicable for all the positive moments of $|\delta\theta_r|$. All the negative moments are divergent.

The possibility of the two-point consideration explained above is based on the absence of the dissipative anomaly. To prove this and also to clarify the dynamical origin of anomalous behavior I consider Lagrangian multi-point representation and show how does it lead to (5). Here (1) is equivalent to

$$\theta(t;\mathbf{r})=\int_0^\infty dt'\exp[-\alpha t']\phi(t';\rho(t-t')), \tag{7}$$

$$\frac{d}{dt}\rho(t)=\hat{\sigma}(t)\rho(t)+\xi(t),\quad \rho(0)=\mathbf{r}, \tag{8}$$

where $\xi(t)$ is the Langevin noise, $\langle\xi^\alpha(t)\xi^\beta(t')\rangle=2\kappa\delta(t-t')$. Averaging the simultaneous product of $2n$ different replicas of (7), following trajectories of $2n$ particles ρ_i fixed by (8), one gets $F_{2n}\equiv\langle\theta_1\cdots\theta_{2n}\rangle$. The $\kappa\to0$ limit is well defined for the general object. Indeed, very small but finite diffusion tends to separate otherwise coinciding particles, however it does not affect the evolution of particles separated initially. Another point, for the purpose of the $2n$-th structure function calculation, it is utmost enough to consider F_{2n} at the colinear configuration, $\mathbf{r}_i=\mathbf{n}r_i$. The observations allow, first, to integrate (7) at $\kappa\to0$ and, second, to reduce the $2n\times(d-1)$ parametric average to the following single-parametric one:

$$F_{2n}=\sum_{\{i_1,\ldots,i_{2n}\}}^{\{1,\ldots,2n\}}\left\langle\prod_{k=1}^{n}\int_0^\infty dt_k e^{-\alpha t_k}\chi[e^{\eta(t_k)}r_{i_k;i_{k+1}}]\right\rangle, \tag{9}$$

where $r_{ij}\equiv|\mathbf{r}_i-\mathbf{r}_j|$. The longitudinal stretching rate, $\eta(t)\equiv\ln|\hat{W}(t)\mathbf{n}|$, is the only fluctuating quantity left; $\hat{W}(t)$ satisfies $d\hat{W}(t)/dt=\hat{\sigma}(t)\hat{W}(t)$. The $\alpha=0$ version of (9) was calculated in Ref. 14 for the $d=2$ case and generalized for any $d\ge2$ in Ref. 15 via a change of variables and a further straightforward transformation of the path integral standing for the average over $\hat{\sigma}(t)$. The η measure, known from Ref. 15 and applied to (9), produces

$$\frac{F_{2n}}{n!}=\int\left[\prod_{i=1}^{n} dt_i d\eta_i\right]\exp\left[\frac{d}{2}\eta_1-\frac{D(d-1)d}{4}t_1\right]$$
$$\times\sum_{\{k_i,\ldots,k_{2n}\}}^{\{1,\ldots,2n\}}\prod_{i=1}^{n}\left[e^{2\alpha t_i}\chi(e^{\eta_i}r_{k_{2i},k_{2i+1}})\right.$$
$$\left.\times G(t_{i-1,i};\eta_{i-1,i})\right], \tag{10}$$

where η_i integrations are not restricted, $0\leq t_n\leq\cdots\leq t_1\leq\infty$, $t_{n+1}=\eta_{n+1}=0$, $t_{i,k}=t_i-t_k$, with equivalent notations for η, and $G(t;\eta)\equiv[\sqrt{\pi d/4(d-1)tD}]\exp[-d\eta^2/4(d-1)tD]$. The integrand of (10) decays exponentially in time with the major contribution into the integral formed at $t_i\sim 1/\alpha$. Therefore, it does not depend on any r_{ij} and gives no contribution into the structure function. The first actual r-dependent contribution stems from $n-1$ temporal integrals formed at $\tau\sim 1/\alpha$, and one at $t_i\sim\tau_r\sim\ln[L/r]/\max\{\alpha,D\}$. This special integration brings a spatial dependence into the object, therefore, on a single distance. Generally, there exists a variety of terms with all the possible combinations, like term with k integration formed at τ, while $n-k$ ones at τ_r, and therefore dependent explicitly on $2(n-k)$ points. However, we are looking exclusively for a term dependent on all the $2n$ points since only such a term contributes $S_{2n}(r)$. It is really simple to calculate the scaling of this term making use of the temporal separation, $\tau_r\gg\tau$. Indeed, the large time contribution may be extracted out of (10) in a saddle-point calculation. A variation of all the exponential terms in (10) with respect to t_i gives a chain of saddle equations. The χ functions in the integrand of (10) limits the η integrations from above by $\ln[L/r]$. Therefore, the desirable $2n$-points contribution forms at $t_i=\sqrt{d/[4(d-1)D(2\alpha n+D(d-1)d/4)]}\ln[L/r]$, and $\eta_i=\ln[L/r]$. Substituting the saddle-point values of t_i and η_i into (10) we arrive at the anomalous part of (5).

The basic physics of nonzero ξ_{2n} (means deviating from the naive balance of pumping and advection) and generally anomalous ($\Delta_{2n}\neq 0$) scaling at $\alpha>0$ can be stated quite clearly. According to (9) the advection changes scales but not the amplitude, while the amplitude of the injected scalar field decays exponentially from the time of injection at the constant rate α. The temporal integrals in (10) forms at the mean time to reach a scale which is proportional to the negative log of the scale. However, the effective spread in the factor by which the amplitude has decayed, upon reaching a given scale, increases as the scale decreases. It is why $\xi_{2n}>0$. Also there is more room for fluctuations about the mean time due to the interference between the exponential decay of the scalar amplitude and fluctuations of the stretching rate η. Thus intermittency increases with a scale size decrease.

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