Propagation of a Huygens Front Through Turbulent Medium

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The dynamics of a thin Huygens front propagating through a turbulent medium is considered. A rigorous asymptotic expression for the effective velocity v_F proportional to the front area is derived. The small-scale fluctuations of the front position are shown to be strongly intermittent. This intermittency plays a crucial role in establishing a steady state magnitude of the front velocity. The results are compared with experimental data. [S0031-9007(98)05738-X]

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The problem of propagation of a thin passive front (of flame, phase transition, etc.) through turbulent flow has attracted much attention since the early 1940s [1,2] when it was realized that velocity fluctuations tend to generate a strongly convoluted front, thus dramatically increasing their area. In premixed combustion processes the flame front area is related directly to the speed of the front propagation. The first expression for the flame front velocity $u_F \propto v_{\rm rms}$, valid in the limit $v_{\rm rms} \gg u_0$, where $u_{\rm rms}$ is the root-mean-square velocity of the integral eddy and u_0 is the laminar flame speed depending on the details of chemical kinetics, was proposed by Schelkin [2] (see Refs. [3–5] for modern reviews on the theory of turbulent combustion). The problem is also important for describing light propagation in a medium with fluctuating dielectric constant, shock wave fronts, etc. [6]. The renormalization group approach [7], which included some theoretically unjustified steps, yielded an expression, differing from the Schelkin result by a logarithmic factor, which agreed with experimental data [8–10] over a wide range of parameter variation. Still, despite substantial activity, no rigorous derivation of the front speed appeared, and the question of the dependence of u_F on both u_0 and the Reynolds number Re remained open. It is shown in this paper that recent advances in the theory of a passive scalar, advected by turbulence, enable one to accurately account for the small-scale intermittency of a scalar field, which is crucial for the description of the front fluctuations and derivation of the effective velocity u_F.

We consider a problem of propagation of a passive front through turbulent flow. The front can be described by the equation for a passive scalar [3,11] (so-called "G" equation)

$$\partial_t G + (\mathbf{v}\partial_{\mathbf{r}})G = u_0|\partial_{\mathbf{r}} G|, \tag{1}$$

where $G(t,\mathbf{r})$ is a scalar field whose level surface, say $G=0$, represents the thin front position. The statistics of turbulent velocity $\mathbf{v}$ is presumed to be known. Equation (1) with $\mathbf{v}=0$ describes a front propagating with the constant speed u_0 (laminar front speed) normally to the local orientation of the front. For example, if the front at time $t=0$ is defined at the x-y plane, it will propagate with a constant speed u_0 and constant area $S_0 =$ const in the z direction. The role of the random field $\mathbf{v}$ is in the generation of a strongly convoluted ("wrinkled") front with a substantially increased area $S_T > S_0$. When $\mathbf{v}=0$ the mass of the reactants consumed per unit time is $dm/dt = u_0 S_0 =$ const and does not change in time. If u_0 is constant (the laminar flamelet model), then $dm/dt = u_0 S_T \equiv u_F S_0$, where S_T is the area of the wrinkled front. This gives a definition of a turbulent or effective front velocity, $u_F = u_0 S_T/S_0$. Assuming that a steady (both u_F and S_T do not depend on time after a long evolution) regime is realized, we introduce a new variable, $G(t,\mathbf{r}) \equiv u_F t - z + h(t,\mathbf{r})$. Then, (1) reads

$$\partial_t h + (\mathbf{v}\partial_{\mathbf{r}})h = v_z + u_0\sqrt{(\partial_{\mathbf{r}} h)^2 + 1 - 2\partial_z h} - u_F\,. \tag{2}$$

In the moving frame the fluctuations of the front position h are assumed to be in a statistically steady state that fixes the value of the front speed u_F which, in the turbulent regime discussed ($u_F \gg u_0$, when $|\partial_r h|^2 \gg |\partial_z h| \gg 1$), is given by

$$u_F = u_0\langle|\partial_{\mathbf{r}} h|\rangle \sim u_0\langle|\delta h(r_0)|\rangle/r_0\,. \tag{3}$$

Here, $|\delta h(r_0)|$ is the magnitude of the velocity difference at the scale r_0 where the "chemical" (u_0 dependent) and advective contributions to (2) balance each other. Averaging over the turbulent velocity is assumed in (3). Since $u_{\rm rms} \gg u_0$, $r_0 \ll L$, where L is the scale of the turbulence source. This means that the scalar, injected at the scale L, is dissipated at the propagating front as a result of the generation of very sharp cusps of the radius $r_0 \ll L$. Formation of such cusps has been observed in numerical simulations [12,13]. Derivation of the characteristic width r_0, magnitude $|\delta h(r_0)|$ of these cusps, and, as a result, u_F is the goal of the theory.

The presence of the three characteristic scales, L (integral scale), η (Kolmogorov viscous scale), and r_0, defines three possible flame regimes:

$$A:\quad L \gtrsim \eta \gg r_0, \qquad B:\quad L \gg r_0 \gg \eta, \qquad C:\quad L \gg \eta \gg r_0\,. \tag{4}$$

Generation of the small-scale scalar fluctuations (direct cascade) [14,15] takes place in both inertial-convective, $L > r > \eta$, and dissipative-convective, $\eta > r > r_0$ (which is valid in the A, C cases but not B), intervals. Thus, the problem is naturally divided into two: First, we need to describe scalar (height of the flame brush) correlations in both convective ranges $L > r \gg r_0$. Once the solutions in the convective intervals are found, we will be ready to resolve the second and principal part of the problem: to calculate the value of the dissipative scale r_0 and, matching dissipative and convective intervals, find turbulent speed of the front, u_F. Therefore, naturally, we are starting from the first task, considering all of the regimes (A–C) one at a time.

Regime A.—Batchelor (a pure viscous-convective) regime. Case A corresponds to a well-studied situation, first discussed by Batchelor [16] and developed further in Refs. [17–19]. Without loss of generality and following [19], we will consider the velocity difference to be Gaussian in the case

$$\langle \delta v_r^\alpha(t) \delta v_r^\beta(t) \rangle = \frac{D}{\tau} [2\delta^{\alpha\beta} r^2 - r^\alpha r^\beta] \exp\left[-\frac{t}{\tau}\right], \tag{5}$$

where τ is the turnover time of the integral (L-size) eddy. The pair correlation function of the scalar obeys the famous logarithmic law in the convective interval of scales, $L \gg r \gg r_0$ [17–19],

$$\langle h_1 h_2 \rangle = DL^2 \ln[L/r_{12}]/\lambda, \tag{6}$$

where DL^2 describes the fluctuation of the "source" function v_z while λ stands for the Lyapunov exponent corresponding to the rate of Lagrangian stretching. Correlation between the source and convective terms in (2) does not contribute to (6). Accounting for these correlations slightly modifies the higher-order moments generating subleading contributions and a mere renormalization of bare coefficients. The Lyapunov exponent λ as a function of τ, D was found for the two-dimensional version of the model (5) in [19]. Asymptotics of the large and small τ were described explicitly in [19]. Generally, the problem of finding λ was reduced in [19] to a well-defined auxiliary quantum mechanics problem which was easy to solve numerically. An interpolation formula for λ, fitting well all of the known asymptotical formulas is

$$\lambda = \sqrt{D/\tau}\, \tanh[\sqrt{D\tau}]. \tag{7}$$

This formula, valid in the space of arbitrary dimensionality $d > 2$, holds up to $\mathcal{O}(d)$ corrections. The statistics of the scalar fluctuations in the convective interval are shown to be Gaussian [19]. Therefore, the typical fluctuation of the height h at r_0, estimated by the second moment (6), is

$$|\delta h_{r_0}| \sim L\sqrt{D/\lambda}. \tag{8}$$

The expression (8) is derived without any spatial averaging over the large-scale ($\sim L$) structures, which are always present in a real flow where all macroscopic characteristics, including the Lyapunov exponents λ, are slightly modulated on the integral scale. Accounting for this spatial variation gives an estimate following directly from (6):

$$|\delta h_{r_0}| \sim L\sqrt{D/\lambda}\, \sqrt{\ln[L/r_0]}. \tag{9}$$

The modified regime A, accounting for the large-scale averaging, will be denoted hereafter by A'.

Regime B.—Pure inertial-convective range. An inertial-convective range is realized at the scales $r \gg \eta \gg r_0$ where one cannot neglect small-scale advection contribution. In the case of a general nonsmooth velocity, one finds strongly intermittent behavior, manifested in the anomalous scaling of the scalar structure functions,

$$S_{2n}(r) = \langle [h(\mathbf{r}_1) - h(\mathbf{r}_2)]^{2n} \rangle \sim L^{2n-\zeta_{2n}} r_{12}^{\zeta_{2n}}, \tag{10}$$

valid at the separations $r_{12} \ll L$. The fundamental origin of the anomalous scaling, $\zeta_{2n} < n\zeta_2$, was discovered recently [20–22]. It was understood that the anomalous exponents originate from zero modes of the eddy-diffusivity operator: ζ_{2n} are universal numbers, solely defined by the velocity statistics and independent on the properties of the pumping term. The exponents were analytically calculated for the case of the velocity field, rapidly varying in time, introduced by Kraichnan [23] in $1/d$, $2 - \zeta_2$, and ζ_2 expansions [20–22], respectively. An instanton approach [24] yields yet another large n ($n \gg d$) asymptotic for $\zeta_{2n} \to \zeta_\infty(\zeta_2, d)$ when $n \to \infty$. The constant $\zeta_\infty(\zeta_2, d)$ was explicitly calculated. The saturation of exponents ζ_n was predicted also in [25]. The instanton consideration, applied to a general passive scalar problem, always results in the collapse of exponents $\zeta_{2n} \to \zeta_\infty < \infty$, with the asymptotic value ζ_∞ to be a complicated functional of the velocity field statistics. It can be easily understood: The $2n$th moment of the scalar can be represented as a path integral over $2n$ fluid particles. The dominant contribution into the high ($2n$th) order structure function originates from the most probable $2n$-particle trajectory. In the incompressible world it is impossible to avoid a divergence of particles in at least one of the directions and it has been shown in [24] that a contribution to the path integral from a single diverging trajectory is sufficient to cause saturation of the exponents ζ_n. This effect has a very strong and important influence on the parametric dependence of an effective front speed u_F calculated below.

Regime C.—Consecutive inertial-convective and dissipative-convective ranges. The description of the scalar correlations in the inertial-convective regime (ICR) does not deviate from the one considered above for case B. However, in the dissipative-convective interval (DCR), $L \gg \eta \gg r \gg r_0$, the behavior of the scalar is very different from that observed in the low-Reynolds-number Batchelor regime when $L \approx \eta$. The crucial difference stems from the essential non-Gaussianity and intermittency of the scalar field on the velocity dissipation scale η, where the solutions in both intervals have to match. In this case, the scale η is an integral scale for the DCR $r_0 \ll r \ll \eta$, where a powerful injection of all higher-order integrals of motion ($\int d\mathbf{r} h^n$) takes place. The point is that a $2n - 2$ moment defines a pumping for the next $2n$th order moment. Thus, considering a generalization of the Kraichnan model for the case C (and neglecting the correlations between the source and convective terms, which can slightly renormalize some constants), one obtains the exact equations for the multipoint correlation functions $F_{2n}(\mathbf{r}_1, \dots, \mathbf{r}_{2n}) \equiv \langle h(\mathbf{r}_1) h(\mathbf{r}_2) \cdots h(\mathbf{r}_{2n}) \rangle$,

$$\hat{L} F_{2n} = V^{11}(r_{12}) F_{2n-2}(\mathbf{r}_3, \dots, \mathbf{r}_{2n}) + \text{ permutations}, \tag{11}$$

where $\hat{L} \equiv -\mathcal{K}^{\alpha\beta}(\mathbf{r}_{ij}) \nabla_i^\alpha \nabla_j^\beta$, $V^{\alpha\beta}(r) = K^{\alpha\beta}(L) - K^{\alpha\beta}(r)$, and $K(r) \sim r^{2-\gamma} D$ at $L > r > \eta$, while $K(r) \sim r^2 \eta^{-\gamma} D$ at $\eta > r$. The operator $\hat{L}$ has the $-\gamma$ dimensionality (it scales as $r^{-\gamma}$) in the inertial-convective range while it is $O(r^0)$ in the dissipative-convective range. As a result, the $2n$th moment F_{2n}, explicitly depending on all $2n$th points, is dominated by the forced "logarithmic" solution of (11) in the DCR. In the ICR, however, it is a zero mode of the operator $\hat{L}$ which dominates the solution. Therefore, the $2n$th moment of the scalar difference, $S_{2n}(r) = \langle \delta h_r^{2n} \rangle$ is estimated by $S_{2n}(r) = a_n L^2 [r/r_0] S_{2n-2}$, where a_n are dimensionless n-dependent constants. On the another hand, at the scale η (13), this expression should match the anomalous structure functions (10) from the upper (inertial-convective) interval giving

$$S_{2n}(r) \sim \eta^{\zeta_{2n}} L^{2n-\zeta_{2n}} \frac{\ln^n[r/r_0]}{\ln^n[\eta/r_0]}. \tag{12}$$

Let us proceed now with the second task and estimate the value of the dissipative scale, r_0, required for the calculation of the front velocity u_T. The point of crucial importance is a necessity to distinguish between the different moments of the front fluctuations at the dissipative scale, δh_{r_0}. Usually one estimates the typical fluctuations as a root-mean-square value of the corresponding variable. This is fine in the case of "normal scaling" or if the small-scale intermittency is not too strong (cases A and A'). However, in the cases B and C, intermittency in the convective intervals is extremely strong: Estimates for a typical front fluctuation based on various moments, $\delta h_r^{(n)} \approx \langle \delta h_r^n \rangle^{1/n}$, are very different. As a result, we get from (8)–(10) and (12) the following n-dependent estimate at the dissipative scale r_0:

$$|\delta h_{r_0}^{(n)}| \sim L \begin{cases} \sqrt{D/\lambda}, & A, \\ \sqrt{D/\lambda}\sqrt{\ln[L/r_0]}, & A', \\ [r_0/L]^{\zeta_n/n}, & B, \\ [\eta/L]^{\zeta_n/n}/\sqrt{\ln[\eta/r_0]}, & C, \end{cases} \tag{13}$$

for the cases A, A', B, and C, respectively. On the other hand, the value of $|\delta h_{r_0}|$ is defined by the equilibration of the convective and "dissipative" terms in (2) giving

$$|\delta h_{r_0}^{(n)}| \sim u_F T, T \sim \begin{cases} \lambda^{-1} \ln[L/r_0], & A, A', \\ L/v_{\text{rms}}, & B, C. \end{cases} \tag{14}$$

It shows that the linear dimension of the flame brush, calculated on the dissipative scale, is defined by the turbulent flame speed and the overall time of Lagrangian evolution which is the typical time for Lagrangian separation, initially equal to r_0, to reach the integral scale. In the B, C regimes the overall time is r_0 independent (note that the time of evolution from r_0 to η is neglected in comparison with one describing the inertial-convective stage of evolution, from η to L). Which number n is essential for defining u_F via (14) is yet to be discussed. One should add (3) to (13) and (14). All of the relations are collected in Table I. As was cited before, $\zeta_{2n}/[2n] \to 0$ and respectively, $\beta_n \to 1/[1 - \zeta_1] > 0$, at $n \to \infty$. Since, intermittency of the scalar is growing downscale from the scale of the pumping, the higher the ratio of the integral scale to the dissipative one (of the turbulent velocity to the chemical one), the higher the moment-number n entering the actual stationary velocity of the flame (which is defining the actual width of the front). The highest value of $\delta h_{r_0}^{(n)}$ is reached at $n \to \infty$. In other words, the stationary speed of the front is defined by such optimal and rare (instanton) configurations which give the dominant contribution into the highest moments of the scalar field h. The optimal configurations define the maximal spreading of the flame brush and control (mainly) the dissipation and, finally, the stationary velocity of the brush. The necessity for accounting for the highest moments of the δh_{r_d} has a simple Lagrangian explanation. The point of crucial importance is a failure of a $2n$-particle description of the $2n$-order correlation function of the scalar at scales smaller than the dissipative ones. This fact can be easily illustrated by a numerical procedure attempting to describe the initially plane front in terms of the collection of n Lagrangian particles [26], each moving with velocity $u_0 \mathbf{n} + \mathbf{v}(\mathbf{x}, \mathbf{t})$, where $\mathbf{n}$ is a unit vector in the z direction. The initial ($t = 0$) distance between these particles is $l_0 \approx 1/n$. Very soon after beginning the simulation the sharp cusps started to form, and one has to add more and more particles to preserve continuity of the front. The development of these very sharp gradients drives the necessary number of particles $n \to \infty$ (any regularization of the cusps, for example, through a weak dependence of u_0 on the local curvature of the front would replace ∞ by a finite but still very large number). We would like to reiterate a very important and profound peculiarity of the situation: Table I, taken at the largest n, corresponds to a very rare optimal (instanton) configuration, responsible for the highest moments of the scalar difference. This means that to describe the small-scale ($r \ll r_0$) dynamics of the front height $h(t; \mathbf{r})$, one has to evaluate contributions from these configurations. We postpone the more accurate investigation (which requires some further development of

TABLE I. Summary of all relations; $\beta_n \equiv [1 - \zeta_1 + \zeta_n/n]^{-1}$

	A	A'	B	C
u_F	$\frac{L\sqrt{D\lambda}}{\ln[L\lambda/v_0]}$	$\frac{L\sqrt{D\lambda}}{\ln[L\lambda/v_0]}$	$v_{\rm rms}\left[\frac{v_0}{v_{\rm rms}}\right]^{\zeta_n\beta_n/n}$	$v_{\rm rms}\left[\frac{v_0}{v_{\rm rms}}\right]^{\zeta_n\beta_n/n}\ln^{-1/2}\left[\frac{n}{L}\left(\frac{v_{\rm rms}}{v_0}\right)^{\beta_n}\right]$
r_0	$\frac{v_0}{\lambda}\ln[L\lambda/v_0]$	$\frac{v_0}{\lambda}\ln[L\lambda/v_0]$	$L\left[\frac{v_0}{v_{\rm rms}}\right]^{\beta_n}$	$L\left[\frac{v_0}{v_{\rm rms}}\right]^{\beta_n}$

the instanton technique of [24] accounting for the entire dynamics of the front surface) for a future publication.

One of the principal results of this work, given in Table I at $n \to \infty$, is that turbulent front speed strongly depends on the ratio $\Gamma = r_0/\eta$, which is a novel dimensionless parameter of the problem [notice a difference between r_0 and the so-called Gibson scale, where the convection and front propagation velocities are comparable [3]; for example, in the regime C and $\zeta_1 = 1/3$, the Gibson scale is $L(u_0/v_{\rm rms})^3$ while r_0 is $L(u_0/v_{\rm rms})^{3/2}$]. Therefore, in order to compare experimental data with theoretical predictions, we must first estimate Γ and determine the front propagation regime. Recent experimental studies of a passive front propagating in turbulent media were performed in the flows generated by vibrating grids, capillary waves, Taylor-Couette flow, and Hele-Shaw cells. All experiments (for the turbulent regime, $v_{\rm rms} \gg u_0$) corresponded to the case C. For example, in the vibrating grid experiments [10] the values of $u_{\rm rms} \approx 1$ cm/sec, $L \approx 1$–10 cm, Re $\approx 10^2$–10^3, and $u_0/v_{\rm rms} \approx 10^{-2}$–$10^{-3}$. (In the case of the Kolmogorov turbulence, one has an estimation $\eta \approx 10L\,{\rm Re}^{-3/4}$, where $\Re$ is the Reynolds number and the factor $c \approx 10$ agrees with available experimental data). For the experimental conditions of [9], one gets $\Gamma \approx 10^{-3}$–10^{-4}. The expression for the front velocity, following from Table I, evaluated at r_0 corresponding to $n \to \infty$, is

$$u_F \sim \frac{u_{\rm rms}}{\sqrt{\frac{2}{3}\ln[\eta/L] + \ln[U]}}\bigg|_{U\to\infty} \to \frac{u_{\rm rms}}{\sqrt{\ln[U]}}, \qquad (15)$$

where $U \equiv u_{\rm rms}/u_0$. The right-hand side of (15) (a similar formula was derived in [7], see also [27], where the dynamic renormalization group has been used for evaluation of the turbulent flame speed u_F) agrees very well with experimental data on u_F in a variety of turbulent flows in a wide range of variation of the dimensionless turbulent intensity $U \le 20$–500. This universality can be readily understood: In all of the experimental situations the $O(\ln[\eta/L])$ contribution is not large and can be neglected in comparison with the $O(\ln[U])$ term. In a typical case, $U \approx 20$–100, the transitional $u_{\rm rms}$-based Reynolds number is ${\rm Re}_c \approx 10^3$–10^4, which is very high. This explains the relatively broad applicability of the large U asymptotic of (15) and, therefore, motivates a higher Re experiment to test the general answer.

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