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High-temperature phase of the 2D Coulomb gas model near the Kosterlitz–Thouless phase transition

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We show that the high-temperature phase of the 2D Coulomb scalar gas model near the Kosterlitz–Thouless phase transition at a large length scale is equivalent to the free fermion model. We also estimate more precisely the correlation length.

1. Kosterlitz and Thouless [1,2] have proposed a theory for a phase transition in two-dimensional planar spin systems and helium films, and a two-dimensional theory of melting. These systems have singularities, vortices for planar spins and helium, dislocation in the case of melting, which interact by a logarithmic potential, just like charged particles in 2D. The 2D Coulomb scalar gas model (2DCSGM) is defined by the partition function

$$Z_{\mathrm{CG}} = \sum_{s(\boldsymbol{r})=\pm 1} \exp\left(\frac{\epsilon}{T}\sum_{\boldsymbol{r}'} s(\boldsymbol{r}) \ln\left|\frac{\boldsymbol{r}-\boldsymbol{r}'}{a}\right| s(\boldsymbol{r}') - \frac{\mu}{T}\sum_{\boldsymbol{r}} s^2(\boldsymbol{r})\right), \tag{1}$$

where ϵ is the coupling constant, μ is the fugacity, T the temperature and a is the elementary scale in the lattice of the model. The harmonic approximation of the Kosterlitz–Thouless theory is accurate enough at all temperatures below the transition point, provided one only investigates fluctuations at a sufficiently large length scale. It also allows for a renormalization of the constants $K \equiv \epsilon/T$, $y \equiv \mu/T$ of the theory due to bound pairs. Above the transition temperature T_{KT} correlations are expected to decay exponentially with a correlation length of the order of the mean spacing between free vortices, but there is no full information about the high-temperature phase.

In the high-temperature phase the Kosterlitz–Thouless renormalization group (RG) rescaling from the initial elementary scale a to a renormalized elementary scale R_{D} and, correspondingly, from initial constants K, y to renormalized constants K_{D}, y_{D} is correct to leading order in $z = \exp(-y)$, $z_{\mathrm{D}} \ll 1$. We show that it is possible to choose a renormalized scale R_{D} in such a way that the 2DCSGM with renormalized constants K_{D}, y_{D} is equivalent to the free fermion model. The latter allows one to improve the Kosterlitz estimate for the correlation length in the high-temperature phase [2].

This Letter is organized as follows. In section 2 we briefly introduce the 2DCG self-consistent RG and present a crude estimate of the correlation length in the high-temperature phase [1–4]. In section 3, we recapitulate the equivalence between the massive Thirring, the quantum sine-Gordon and the 2DCSG models [5–7]. Using the RG analysis from section 2 and results from section 3, in section 4 the exponentially decaying connected correlation function and the correlation length in the high-temperature phase are obtained.

2. The central result of the Kosterlitz–Thouless theory is a self-consistent integral equation for $\epsilon(r)$, the scale-dependent constant, to leading order in $\exp(-y) \equiv z$:

$$\epsilon(r) = \epsilon(a) + 2\pi \int_a^r n(r')\alpha(r')\frac{r'\,\mathrm{d}r'}{a^2}, \tag{2}$$

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where $n(r)$ is the density of charged pairs separated by r,

$$n(r)=\exp(-2y)(r/a)^{K(r)}\,, \tag{3}$$

and $\alpha(r)$ is the polarizability of a single dipole with separation r. Eq. (2) is invariant under scale transformations of the constants (K, z),

$$a\to a(1+\xi)\,,\quad K^{-1}\to K^{-1}+4\pi z^2\,,$$
$$z\to(1+\xi)^{2-K/2}z\,. \tag{4}$$

Due to this invariance, eq. (2) is equivalent to the pair of coupled differential equations

$$\frac{\mathrm{d}K^{-1}}{\mathrm{d}\xi}=4\pi z^2\,,\quad \frac{\mathrm{d}z}{\mathrm{d}\xi}=(2-\tfrac{1}{2}K)z\,, \tag{5}$$

where $\xi=\ln(r/a)$. The trajectories which follow from eqs. (5),

$$4/K+\ln(\tfrac{1}{4}K)-1=2\pi^2z^2-\tfrac{1}{32}c\,, \tag{6}$$

are shown schematically in fig. 1. The trajectories with $c=0$ are drawn as long dashed lines, which separate the plane in three regions. Region 1 is an infinite correlation length phase, i.e. the low-temperature phase of the model. Regions 2 and 3 both are finite correlation length phases with free charges. The separatrix with $K>4$ is the phase transition line. Here we discuss only the high-temperature phase in region 2 near the phase transition trajectory AB, where

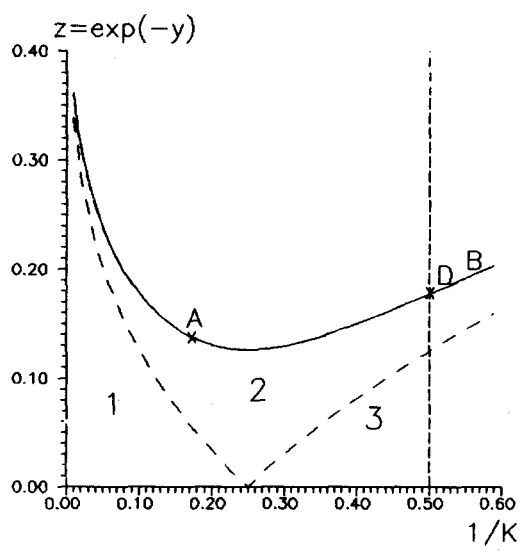

Fig. 1. The RG flow induced by the scaling equation (5). Long dashed lines are separatrices, separating the plane in three regions. The solid line AB is a typical RG trajectory in the high-temperature phase. The dashed vertical line is the bench mark line ($K=2$). Model (1) with constants K, y from the bench mark line is equivalent to a free fermion model.

$$c\sim\tau=\frac{T-T_{\mathrm{KT}}}{T_{\mathrm{KT}}}\ll 1\,,\quad c>0\,. \tag{7}$$

We cannot use (5) for $\xi>\xi_{\mathrm{B}}$, where $z_{\mathrm{B}}\sim 1$, because the used approximation $z\ll 1$ breaks down. However, it is possible to estimate the correlation length above T_{KT} [2]. Let us assume that higher order terms in (5) stop the growth of z near $z_{\mathrm{B}}\sim 1$. This hypothesis looks reasonable, since the dissociation of the pairs starts from the largest scale. The small scale pairs "know nothing" about the behavior of large scale pairs. From (5) and (6) it is straightforward to obtain for small c,

$$\ln(R_{\mathrm{B}}/a)=\xi_{\mathrm{B}}=\int_{4-K_{\mathrm{A}}}^{4-K_{\mathrm{B}}}\frac{\mathrm{d}t}{t^2+c}\simeq\frac{\pi}{\sqrt{c}}\,. \tag{8}$$

This is Kosterlitz's crude estimate for the correlation length.

In section 4 we will show that the trajectory AB intersects the bench mark line (the dashed line in fig. 1) at the scale R_{r}. The 2DCGM with the constants of the model (K, y) from the bench mark line and the smallest scale R_{p} is equivalent to the free fermion theory. The latter permits one to obtain the large scale behavior of the model and to improve Kosterlitz's estimate for the correlation length in the high-temperature phase.

3. We can represent the partition (1) as a functional integral with the action of the sine-Gordon model [5]:

$$Z_{\mathrm{sG}}=\int \mathrm{D}\psi\exp\left(-\int \mathrm{d}^2r\,\mathscr{L}_{\mathrm{sG}}\right), \tag{9a}$$

$$\mathscr{L}_{\mathrm{sG}}=\tfrac{1}{2}(\nabla\psi)^2-\frac{2\exp(-y)}{R^2}\cos(\sqrt{2\pi K}\,\psi)\,. \tag{9b}$$

Expanding (9a) in powers of $h\equiv 2\exp(-y)/R^2$, one can obtain the correspondence between the quantum sine-Gordon and the massive Thirring partitions [6–8]. The Thirring model is defined in terms of Dirac fermions,

$$Z_T = \int D\bar{\varphi}\, D\varphi \exp\left(-\int d^2r\, \mathscr{L}_T\right), \tag{10a}$$

$$\mathscr{L}_T = -i\bar{\varphi}\gamma_\mu \partial_\mu \varphi - \tfrac{1}{2}g^2(\bar{\varphi}\gamma_\mu\varphi)^2 + im\bar{\varphi}\varphi\,, \tag{10b}$$

where γ_μ are γ-matrices in 2D. The correspondence between the constants is

$$1+g/\pi = 2/K\,, \tag{11a}$$

$$m = \frac{\exp(-y)}{\pi K R^2}\left(\frac{R}{R'}\right)^{K/2} R'\,, \tag{11b}$$

where R is the minimal length scale and R' is the arbitrary length scale of the models. The correspondence between the connected correlation functions can be calculated as well:

$$\langle\langle \cos[\sqrt{2\pi K}\,\psi(r)]\cos[\sqrt{2\pi K}\,\psi(r')]\rangle\rangle_{sG} = -\tfrac{1}{4}m^2R^4\exp(2y)\langle\langle\bar{\psi}(r)\psi(r)\bar{\psi}(r')\psi(r')\rangle\rangle_T\,. \tag{12}$$

We observe that the value $K=2$ corresponds to a free fermion field theory. Thus, the line $K=2$ is the bench mark line.

4. The trajectory AB intersects the bench mark line in the point D. $K_D=2$, y_D are the renormalized constants and R_D is the minimal length scale R in the point D. We find z_D and R_D to leading order in c from (5), (6):

$$z_D = \exp(-y_D) = \sqrt{\frac{1-\ln 2}{2\pi^2}}\left(1+\frac{c}{64(1-\ln 2)}\right) \simeq 0.125(1+0.05c)\,, \tag{13}$$

$$\ln(R_D/a) = \int_K^2 \frac{dK}{4K+K\ln(\frac{1}{4}K)-(1-\frac{1}{32}c)K^2}\,. \tag{14}$$

All the following results hold to leading order for $z_D \ll 1$. Using results from section 3, we see that the model (1) is equivalent to the free fermion field theory with mass

$$m = \frac{1}{2\pi R_D}\exp(-y_D)\,. \tag{15}$$

It is natural to identify the inverse mass with the correlation length R_c,

$$R_c/R_D = 2\pi\exp(y_D) \simeq 50.2(1-0.051c) \gg 1\,. \tag{16}$$

Thus, one obtains the dependence R_c on c, τ (eqs. (14), (16)), estimating more precisely the correlation length.

According to (12), the connected correlation function at the point D takes the form

$$\begin{aligned}\langle\langle \cos[\sqrt{2\pi K}\,\psi(r)]\cos[\sqrt{2\pi K}\,\psi(r')]\rangle\rangle_{sG} &= \tfrac{1}{4}m^2R_D^4\exp(2y_D)\int\frac{d^2p\,d^2p'\exp[i(\boldsymbol{p}+\boldsymbol{p}')\cdot\boldsymbol{r}]}{(2\pi)^4(p^2+m^2)(p'^2+m^2)}\\ &\quad\times \mathrm{Tr}[(m-i\hat{p})(m-i\hat{p}')]\\ &= \frac{m^2R_D^4}{4\pi^2}\exp(2y_D)[K_0^2(mr)-\tfrac{1}{2}K_1^2(mr)]\\ &\simeq \frac{m^3R^4}{8\pi}\frac{\exp(-2mr)}{r}\,,\end{aligned} \tag{17}$$

where $\hat{p}=\gamma_\mu p_\mu$ and $K_0(r)$, $K_1(r)$ are modified Bessel functions. Thus, one obtains the exponentially decaying order (17) above T_{KT} at large scales, contrary to the order with a power-law decay in the low-temperature phase [8].

Finally, it should be emphasized that the equivalence of the 2DCSGM above T_{KT} to the free fermion model has been obtained in the expansion in powers of z_D and only at a large length scale. It differs from the exact equivalence of the 2D Ising model to the free fermion one near the phase transition point [10].

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