

Effects of surface tension on immiscible Rayleigh-Taylor turbulence

M. Chertkov^a, I. Kolokolov^{a,b}, V. Lebedev^{a,b}

^a *Theoretical Division, LANL, Los Alamos, NM 87545, USA*

^b *Landau Institute for Theoretical Physics, Moscow, Kosygina 2, 119334, Russia*

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We present phenomenology describing the internal structure of a turbulent zone, produced as the result of the push of a heavy fluid into a light one, for the case of immiscible fluids. One finds that the Kolmogorov cascade is realized within a range that grows with time, viz., scales between the mixing zone width, $L \propto t^2$, and the viscous scale, $\eta \propto t^{-1/4}$. Surface tension effects lead to formation of an emulsion-like state. Density fluctuations on scales larger than the typical drop size, l , are governed by the Obukhov-Corrsin cascade. If $l \gg \eta$, a wave energy cascade, related to capillary waves propagating along the surfaces of drops, is formed at scales below l , $l \propto t^{-2/5}$.

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Introduction. If a heavy fluid lies above a light one, the gravity-driven Rayleigh-Taylor (RT) instability develops [1–4]. At later stages, this unstable flow becomes turbulent. The most striking feature of RT turbulence is the formation of a turbulent mixing zone of width L that grows quadratically with time [5]:

$$L \approx \alpha \mathcal{A} g t^2. \quad (1)$$

Here, \mathcal{A} is the Atwood number, related to the fluid densities $\rho_{1,2}$ by $\mathcal{A} \equiv (\rho_1 - \rho_2)/(\rho_1 + \rho_2)$, and g is the acceleration of gravity. The law (1) was observed in many numerical and laboratory experiments (see Refs. [6, 7] for recent reviews). Numerical and experimental values of the dimensionless coefficient α in Eq. (1) vary from 0.02 to 0.07.

Recently one of us (M.C.) proposed a phenomenological theory explaining the hierarchy of scales and the spectra of velocity and density fluctuations in a specific regime of $3d$ RT turbulence: for low \mathcal{A} (i.e., in the Boussinesq approximation) and for miscible fluids [8]. The theory is based on the law (1) and also on a common feature of multi-scale organization in hydrodynamic turbulence, viz., that small scales adjust adiabatically to changes in large scale characteristics. The phenomenology predicts that, in the wide range of scales between the integral scale, L , and the viscous scale, η , energy cascades down scale (as observed numerically and experimentally [9–12]) and the Kolmogorov estimate for the velocity increment (difference) [13–15],

$$\delta v_r \sim (\epsilon r)^{1/3}, \quad (2)$$

holds. Here ϵ is the energy flux per unit mass, $\epsilon \sim \mathcal{A}^2 g^2 t$, which grows linearly with time. It was shown in Ref. [8] that the Kolmogorov scenario is self-consistent, in the sense that even though the RT turbulence is buoyancy driven at scales $\sim L$, the effect of buoyancy on turbulence becomes irrelevant at smaller scales, $r \ll L$. This self-consistent logic is an adaptation (to the RT turbulence setting) of the Shraiman-Siggia arguments [16], introduced in the context of Boussinesq convection. The

phenomenology also predicts (in agreement with the numerical analysis of Ref. [17]) that the viscous scale η decreases with time as

$$\eta \sim \left(\frac{\nu^3}{\mathcal{A}^2 g^2 t} \right)^{1/4}, \quad (3)$$

where ν is the kinematic viscosity. (We assume that the kinematic viscosities of the fluids are of the same order.) Comparing Eq. (1) and Eq. (3) one finds that the turbulent description is self-consistent, i.e., $L \gg \eta$, for $t \gg \nu^{1/3} \mathcal{A}^{-2/3} g^{-2/3}$.

It is clear that the adiabatic and Kolmogorov-like arguments leading to the estimate (2) are not restricted to the miscible case considered in Ref. [8]. In particular, the general argument suggests that the Kolmogorov picture also holds within some range of scales for the immiscible case. In this case, however, surface tension should play an essential role in the mixing zone. The problem addressed in the present letter is to identify and study phenomena related to surface tension.

Surface Tension Effects. We examine the dynamics of two immiscible fluids when the heavier fluid is placed initially above the lighter one. This configuration leads to RT instability, which eventually develops into RT turbulence. The size of the turbulent mixing zone (and thus the amount of fluid entrained in the turbulent motion) grows according to Eq. (1). Hydrodynamic motion at scales $\sim L$ is driven by buoyancy. At smaller scales the direct (i.e., directed towards smaller scales) cascade of (kinetic) energy is realized, leading to the estimate (2). The cascade is accompanied by mutual penetration of the fluids, which is initiated by the injection of pure fluid jets into the mixing zone. The collision of jets of different fluids produces complex (fractal) interfacial structures. Drops of both types are permanently shed from the interface; the result is the creation of an emulsion-like state. A schematic view of a snapshot taken inside the mixing zone, illustrating the density distribution, is shown in Fig. 1. Notice that the exact shapes of the drops are by no means fixed, as fluctuations in the lo-

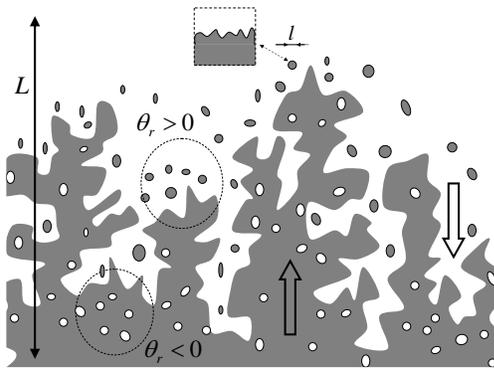


FIG. 1: Schematic view of the mass distribution snapshot. Different mass densities are marked as grey/light and white/heavy, respectively. Circular domains bounded by dashed lines correspond to regions dominated by heavy ($\theta_r > 0$) and light ($\theta_r < 0$) fluids. Density distribution within any of the domains is not homogeneous: drop-rich (emulsion) regions alternate with drop-free regions. Arrows indicate mean direction of the flow in the corresponding regions. Drops are shed from single-phase tongues possessing fractal shape. The inset on the top of the Figure illustrates that surfaces of the drops are populated by capillary waves.

cal radius of curvature of the interface are of the order l , i.e., the typical drop size. Surface tension does not allow drops to have size much smaller than l .

Let us consider the case when the typical drop size is larger than the viscous scale, $l \gg \eta$. Then the size l can be estimated to be the scale where the kinetic energy density of the fluids, $\rho(\delta v_l)^2$, and the interfacial energy density, σ/l , are of the same order:

$$l \sim \left(\frac{\sigma^3}{\mathcal{A}^4 \rho^3 g^4 t^2} \right)^{1/5}, \quad (4)$$

where ρ is the mean mass density, $\rho = (\rho_1 + \rho_2)/2$, and σ is the surface tension coefficient. According to Eq. (4), the characteristic drop size l decreases with time t , generating an emulsion that is progressively more dispersed. Dynamically, the permanent decrease in the typical drop size is realized through creation (shedding) of new drops as well as through breakup of already existing drops into smaller ones.

The estimate (4) is correct provided that the scale l is much smaller than L ; this requirement corresponds to the condition

$$t \gg \left(\frac{\sigma}{\mathcal{A}^3 \rho g^3} \right)^{1/4}. \quad (5)$$

This inequality emphasizes that at large scales, $\sim L$, gravity overcomes surface tension (which tends to stabilize the RT instability). Another condition, $l \gg \eta$,

results in

$$t \ll t_0, \quad t_0 = \frac{\sigma^4}{\mathcal{A}^2 \rho^4 g^2 \nu^5}. \quad (6)$$

This inequality means that the Kolmogorov cascade is insensitive to viscosity at scales $\sim l$. We assume that the inequalities (5) and (6) are compatible, thus leading to the condition

$$\sigma^3 \gg \mathcal{A} \rho^3 g \nu^4. \quad (7)$$

Below we discuss separately scales larger and smaller than l .

Density Fluctuations. As the mixing zone grows, new portions of both heavy and light fluids are entrained in the turbulent region. Jets of the fresh portions, of the typical size L , move from the mixing zone periphery towards the mixing zone center through the counter-propagating emulsion containing drops of other fluid (see Fig. 1). The interfacial contact of the counter-propagating jets generates increasingly complex (fractal) structures evolving passively at scales larger than l , where surface tension is not relevant and the interface dynamics does not exert any back reaction on the flow (so that it cannot lead to any interface rupture). Surface tension becomes relevant at the scale l , leading to interface breakdown, i.e., to formation of drops of sizes $\sim l$. Notice that as time advances, old drops (i.e., drops of larger size formed earlier, because at the time of their formation l was larger) are broken, so that a majority of drops inside the mixing zone at any given time have size $\sim l$. The concentration of drops is also inhomogeneous, implying scale-dependent density variations.

In the immiscible case, the “microscopic” density ρ is a two-valued quantity, $\rho = \rho_{1,2}$. Therefore, a spatial distribution of the mass density at the scales larger than l has to be described in the framework of a coarse-graining procedure, i.e., in terms of averaged quantities. For this purpose, we introduce the quantity θ_r , which is the deviation of the mass density from its mean value ρ coarse-grained at a scale r . Values of θ_r can be positive or negative, signaling which of the two fluids dominates the r -vicinity of a given point \mathbf{R} . (The situation is illustrated in Fig. 1, where regions of a size r are enclosed by dashed circles.) Accounting for mass advection and neglecting surface tension effects, one arrives at the following continuity equation for θ_r (see, e.g., Ref. [18]):

$$\partial_t \theta_r(\mathbf{R}) = -\mathbf{v}_r(\mathbf{R}) \nabla \theta_r(\mathbf{R}), \quad (8)$$

where $\mathbf{v}_r(\mathbf{R})$ is defined as the velocity field, coarse-grained at the same scale r . In complete analogy with the miscible Boussinesq description of Ref. [8], one finds that the field θ_r is a passive tracer advected in the inertial-convective range of scales, $L \gg r \gg l$. Therefore, in accordance with the Obukhov-Corrsin law [23, 24], the

scalar increment (difference) at the scale r is estimated as

$$\delta\theta_r \sim \mathcal{A}\varrho(r/L)^{1/3}. \quad (9)$$

From Eqs. (2,9) one derives that the power generated by the gravity force (per unit mass) at the scale r , $\sim \varrho^{-1}\delta\theta_r g \delta v_r$, is much less than the Kolmogorov energy flux (per unit mass), ϵ . As a result, the direct cascade of passive density fluctuations is established in the range of scales between L and l , thus confirming self-consistency of the Obukhov-Corrsin picture.

Capillary Wave Range. If $t \ll t_0$, then the drop size, l , lies in between the integral scale, L , and the viscous scale, η . We have argued above that in the range of scales bounded from above by L and from below by l , the Kolmogorov in-volume cascade is realized. Moreover, one finds that at smaller scales, $r < l$, turbulence inside and outside drops is also of the Kolmogorov type. As far as dynamics on the interface (surfaces of the drops) is concerned, we claim that a turbulent cascade of capillary waves takes place. The capillary wave dynamics opens an additional channel for energy transfer to small scales. The energy flux, coming from the integral scale L , splits in two parts at the scale l : a part of the energy cascades further (towards η) in the bulk (the mechanism being equivalent to that for single-phase turbulence) while the remainder (which is roughly of the same order as the volume part) feeds capillary fluctuations, giving rise to the capillary wave energy cascade at the surfaces of drops.

Capillary waves are excited at the scale l by the inertial motion; then capillary wave interactions lead to the formation of a cascade in which waves with smaller and smaller wave lengths r , $r \ll l$, are produced. The cascade is of a weak turbulence kind, i.e., the roughness (degree of non-flatness) of the interface decreases with scale. Therefore, zoomed at the scale $r \ll l$, the interface can be viewed as an almost flat one populated by capillary waves. Such zoomed portion of the interface is shown schematically as an inset in Figure 1. The fluctuation spectra for the capillary wave cascade were derived by Zakharov and co-workers [19–21]. Using their results, one finds that the pair correlation function of the wave-generated velocity field, measured at two points on the interface lying distance r apart from each other, is

$$\langle \mathbf{v}(\mathbf{R})\mathbf{v}(\mathbf{R} + \mathbf{r}) \rangle \sim (\epsilon l)^{2/3} (l/r)^{1/4}. \quad (10)$$

The typical surface elevation between the two points is estimated as $h_r \sim r(r/l)^{3/8}$. Therefore, the typical slope, h_r/r , characterizing an effective nonlinearity of the problem, decreases with the scale. This estimate confirms that the wave turbulence at the interface is weak. It is also straightforward to check that the nonlinear interaction time at the scale r within the wave turbulence range decreases with scale, $\propto r^{3/4}$, thus making our adiabatic description well justified.

We also find that velocity fluctuations induced by the capillary waves (10) are stronger than respective fluctuations in the bulk, described by Eq. (2). Therefore, the interface turbulence is insensitive to fluctuations in the bulk. On the other hand, velocity fluctuations at a scale r generated by surface waves become negligible beyond distance r from the interface. This explains why turbulent fluctuations in the bulk are insensitive to fluctuations at the interface.

Comparing the capillary waves dispersion law, $\omega_k = \sqrt{\sigma/(2\varrho)} k^{3/2}$ (where ω_k is the frequency of a wave characterized by the wave vector k), with the viscosity enforced dissipation rate, $\sim \nu k^2$, one finds that the capillary waves are dissipated at the scale

$$r_0 = \varrho\nu^2/\sigma. \quad (11)$$

Combining Eqs. (4,3,11) one concludes that the capillary wave interval, bounded by l from above, by r_0 from below and containing η scale in between, shrinks with time, so that the three scales become comparable at t_0 .

Eq. (10) gives an estimate for velocity fluctuations at the interface. Therefore, if the velocity spectrum is calculated as a full volume average, an additional small factor r/l for the capillary wave contribution emerges due to the aforementioned localization of the capillary wave dynamics in some close proximity of any given drop surface. One concludes that the overall (volume-averaged) contribution into the velocity increment at $r \gg \eta$ is dominated by the bulk, i.e., by the $3d$ Kolmogorov cascade term, which masks the wave turbulence contribution. On the other hand, the capillary-wave spectrum extends to scales smaller than η , $\eta \gg r \gg r_0$, where the volume contribution is already damped by viscosity. Therefore, the interfacial contribution should be clearly seen in the velocity fluctuation spectrum within this special range of scales.

Advanced Stage. When time t approaches t_0 , both l and η reach r_0 simultaneously and the capillary interval collapses. Later on, for $t \gg t_0$, the characteristic drop size l becomes smaller than η , which, in turn, becomes smaller than r_0 . Therefore, the capillary cascade is absent at this stage. The scale l emerges now as the result of a balance between the capillary force σl and the viscous force $\mathcal{A}\varrho\nu(\delta v_\eta/\eta)l^2$ at the scale l . Taking into account estimates (2,3) one arrives at

$$l \sim \frac{\sigma}{\mathcal{A}\varrho g} \frac{1}{\sqrt{\nu t}}, \quad (12)$$

which guarantees that l decreases with time faster than η , with the viscous scale being described by Eq. (3).

For $t \gg t_0$ a new range of scales, bounded by η from above and by l from below, emerges. In this range, the velocity fluctuations, e.g., those entering the expression (8) for the dynamics of coarse-grained density field, are spatially smooth; that is, the fluctuations are of the so-called Batchelor kind [25, 26]. Thus, fluctuations of the

coarse-grained density field are described by the following second order structure function [25–27]:

$$\langle(\delta\theta_r)^2\rangle \sim \mathcal{A}^2 g^2 (\eta/L)^{2/3} \ln(\eta/r). \quad (13)$$

Notice that if condition (7) is reversed, i.e., if $\sigma^3 \ll \mathcal{A}g^3 g\nu^4$, then the RT instability develops into turbulence for $t \gg [\nu/(\mathcal{A}^2 g^2)]^{1/3}$, when l is already smaller than η , so that turbulence begins immediately in the regime just discussed.

Conclusions. We examined the effect of surface tension on immiscible RT turbulence. It was shown that surface tension effects lead to the formation of an emulsion-like state, with the typical drop size l decreasing in time. We found that the character of the density fluctuations on the scales larger than l is insensitive to the immiscible nature of the problem. If the size l is larger than the viscous scale, turbulence in capillary waves propagating along the drops’ surfaces is realized in parallel with the Kolmogorov turbulence inside, and also outside, the drops (i.e., in the bulk). Thus, the energy is carried towards small scales by both inertial and surface wave cascades simultaneously. This is the regime realized at moderate time as well as if the effects of surface tension are stronger than those of viscosity. Later in time, the wave turbulence interval collapses, leading to the formation of a finely dispersed emulsion with the typical drop size being much smaller than the viscous scale.

Let us mention, for the sake of completeness, that immiscible RT turbulence in $2d$, which is frequently addressed in numerical simulations, is very different from that in $3d$, which has been the focus of this letter. One expects, in analogy with the Boussinesq case considered in [8], that the Bolgiano-Obukhov regime [28, 29], rather than the Kolmogorov regime, is realized in $2d$. Besides, it is easy to estimate that viscous scale, η , and the capillary scale, l , both increase in $2d$, contrary to what was concluded above for $3d$.

The description of RT turbulence proposed in this letter is phenomenological. The phenomenology ignores effects associated with the spatial inhomogeneity of the mixing zone. It also ignores the effects of intermittency, leading to anomalous scaling of higher order velocity and density increments [15]. These and other issues (e.g. analyzing the case when two viscosities are parametrically different) should be addressed in the future.

Even though our theory is specific to RT turbulence, we can apply it in other situations, for instance, when immiscible fluids are driven into a turbulent regime by a mechanism other than constant gravity. Two interesting examples of this kind are (a) the statistically steady regime realized under permanent forcing (e.g., in a Taylor-Couette apparatus or when two immiscible fluids are pushed through a pipe) and (b) the decaying Rightmayer-Meshkov regime realized after an initially large acceleration is switched off. Although the overall temporal picture of the flow requires serious modifi-

cation, the spatial picture of the immiscible turbulence reported in this letter will still be applicable. In particular, one finds that the multi-drop (emulsion) picture discussed above and the splitting of the energy cascade in two at scales smaller than the scale of the typical drop size should be seen as well in these other immiscible turbulence problems.

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