Phenomenology of Rayleigh-Taylor Turbulence

Michael Chertkov
Theoretical Division, LANL, Los Alamos, New Mexico 87545, USA
(Received 18 June 2003; published 9 September 2003)

I analyze the advanced mixing regime of the Rayleigh-Taylor incompressible turbulence in the small Atwood number Boussinesq approximation. The prime focus of my phenomenological approach is to resolve the temporal behavior and the small-scale spatial correlations of velocity and temperature fields inside the mixing zone, which grows as $\propto t^2$. I show that the "5/3"-Kolmogorov scenario for velocity and temperature spectra is realized in three spatial dimensions with the viscous and dissipative scales decreasing in time, $\propto t^{-1/4}$. The Bolgiano-Obukhov scenario is shown to be valid in two dimensions with the viscous and dissipative scales growing, $\propto t^{1/8}$.

DOI: 10.1103/PhysRevLett.91.115001 PACS numbers: 52.35.Py, 47.27.Te, 47.27.Qb

The hydrodynamic system of equations for velocity and temperature fields in the Boussinesq approximation [1–4] is

$$\partial_t \mathbf{u} + (\mathbf{u}\nabla)\mathbf{u} + \nabla p - \nu \triangle \mathbf{u} = -\beta \boldsymbol{g} T, \qquad (1)$$

$$\nabla \mathbf{u} = 0, \qquad (2)$$

$$\partial_t T + (\mathbf{u}\nabla) T = \kappa \triangle T, \qquad (3)$$

where Eq. (1) is the Navier-Stokes (NS) equation coupled to the advection-diffusion equation (3) and satisfying the incompressibility condition (2). The coupling term on the right-hand side (rhs) of Eq. (1) stands for the buoyancy effect with $\boldsymbol{g}$ being the gravitational acceleration vector directed downwards, i.e., towards smaller z, and β being the thermoexpansion coefficient of the fluid. One assumes that initially (at $t = 0$) the temperature field (counted from the average value) is the step function

$$T(0; \mathbf{r}) = -\mathrm{sgn}(z)\Theta/2, \qquad (4)$$

with Θ being the initial temperature jump (with the colder fluid placed above the hotter one) and $\mathbf{v}(t = 0) = 0$. This initial configuration is unstable.

I do not discuss here the initial stage of the instability assuming that a mixing zone of the width $L(t)$ is already established. The large-scale relation, fixing temporal behavior of L, follows directly from the energy balance between the buoyancy term on the rhs of Eq. (1) and the temporal derivative term on the left-hand side (lhs) of Eq. (1):

$$\frac{u_L(t)}{t} \sim \frac{L(t)}{t^2} \sim \beta g \Theta, \qquad (5)$$

where u_L is the typical velocity on the scale L and the temperature jump Θ between the upper and lower fronts of the mixing zone is constant. The similarity sign, used in Eq. (5), means that a dimensionless factor, e.g., in the $L \sim \beta g \Theta t^2$ relation, remains undetermined. This qualitative, rather than quantitative, style is kept throughout this Letter. [Readers seeking discussions of various quantitative issues of Rayleigh-Raylor (RT) instability and RT turbulence should see [5], which provides a comprehensive review of the subject, and references therein.] Note that the relation (5) is general, i.e., it is valid in any spatial dimension.

The buoyancy term drives kinetic energy in the NS equation (1). This RT pumping, contrasted against the "standard" one leading to the stationary NS turbulence [6,7] (see, e.g., comprehensive pedagogical review of the entire subject of turbulence phenomenology in [8]), is characterized by the following two special features. First, the mixing zone width, $L(t)$, playing the role of the pumping scale, grows quadratically with time. Second, according to Eq. (5), the typical velocity fluctuations at the pumping scale, δu_L, grow linearly with time, $\delta u_L \sim u_L \propto t$. These two major modifications of otherwise fundamental analysis of turbulence are the central assumptions of my analysis.

Three-dimensional case.—The Kolmogorov-Obukhov picture of steady NS turbulence [6,7] is based on the assumption that the kinetic energy flux is scale independent. The major technical point of this Letter is to show that the stationary picture and the flux approach allow a quasistationary, adiabatic generalization. Adiabaticity means that the flux, as a large-scale object, changes slowly so that smaller scale fluctuations have enough time to adjust themselves to the new, current value of the flux. Then, estimation of the kinetic energy flux, ε, comes from simple dimensional construction [the only independent and large-scale dimensional quantities at my disposal are $L(t)$ and $u_L(t)$]: $\varepsilon(t) \sim u_L^3/L$. Thus, velocity fluctuations, δu_r, in the range of scales naturally called the inertial interval, $\eta(t) \ll r \ll L(t)$, are described by

$$\delta u_r(t) \sim \delta u_L(t)\left(\frac{r}{L(t)}\right)^{1/3} \sim (\beta g \Theta)^{2/3} r^{1/3} t^{1/3}. \qquad (6)$$

This picture follows directly from Eq. (1) under the assumption that the buoyancy term drives kinetic energy at the adiabatically changing scale $L(t)$. Then, the nonlinear self-advection term in Eq. (1) establishes a constant

 0031-9007/03/91(11)/115001(4)$20.00

(for a given moment of time) flux of energy smaller scales than $L(t)$. The most important assumption in this picture is that temperature fluctuations, produced by advection-diffusion dynamics (3), can be neglected at small scales, $r \ll L(t)$. In other words, I claim (and this is to be checked later) that the small-scale, $r \ll L$, component of the temperature field, δT_r, is passively advected by the velocity field, and the distortion of the kinetic energy balance due to the buoyancy term is negligible in comparison with the kinetic self-advection flux term:

$$\delta u_r^3/r \gg \beta g \delta T_r \delta u_r. \quad (7)$$

[Note that a similar Kolmogorov-Obukhov–type scenario was suggested by Shraiman and Siggia [9] to explain the Rayleigh-Benard (RB) turbulent convection problem. The RB problem in the Boussinesq approximation is described by the same set of dynamical Eqs. (1)–(3), with constant temperature boundary conditions with a colder top plate and warmer bottom plate, instead of the initial condition Eq. (4) describing the RT setting. A comprehensive discussion of the RB problem is given in the reviews [10,11].]

As was already stated, the small-scale part of the temperature field, governed by Eq. (3), is passive. Thus, similar to what was introduced above as a generalization of the Kolmogorov-Obukhov theory, one can also construct an adiabatic RT generalization of the Obukhov-Corrsin theory of passive scalar advection [12,13], assuming that the temporal change of the temperature fluctuations flux, $\varepsilon_T(t) \sim \delta T_r^2 \delta u_r/r \sim \Theta^2 u_L/L$, is adiabatic, i.e., the small-scale fluctuations of the temperature field, δT_r, rapidly adjust themselves to the large-scale (and thus slow) change in the temperature flux. One finds that

$$\text{for } \eta(t) < r < L(t), \qquad \delta T_r \sim \Theta\left(\frac{r}{L(t)}\right)^{1/3}. \quad (8)$$

It is straightforward to check, combining the results from Eqs. (6) and (8), that the relation (7) is indeed satisfied, as the rhs in Eq. (7) is suppressed in comparison with the lhs by the asymptotically small factor, $\sim(r/L)^{2/3}$.

I consider the case of large Prandtl number turbulence, $\nu \gg \kappa$. Then, the scaling picture described by Eq. (6) obviously breaks down for $r \lesssim \eta(t)$, where η is the RT generalization of the Kolmogorov scale. The Kolmogorov, or simply viscous, scale can be found by matching the inertial self-advection and viscous terms in Eq. (1), $\delta u_\eta \eta \sim \nu$. Accounting additionally for Eqs. (5) and (6), with the latter relation extended down to the viscous scale, one derives

$$\eta \sim \left(\frac{\nu}{\delta u_L}\right)^{3/4} L^{1/4} \sim \frac{\nu^{3/4}}{t^{1/4}\sqrt{\beta g \Theta}}, \quad (9)$$

$$\delta u_\eta \sim \frac{\nu}{\eta} \sim (\nu t)^{1/4}\sqrt{\beta g \Theta}. \quad (10)$$

Note that the $t^{-1/4}$ viscous scale decrease was originally found in numerical simulations of [14], where also a self-similar closure theory, resolving z (layer) dependence of the large-scale velocity moments and resulting in the $t^{-1/4}$ law, was proposed. Velocity is obviously smooth at scales smaller than the Kolmogorov (viscous) scale, $\delta u_r \sim \delta u_\eta r/\eta$.

The small-scale part of the temperature field, corresponding to scales smaller than the Kolmogorov scale, is advected by smooth velocity (one which in the modern turbulent jargon is frequently said to be of a Batchelor kind [15,16]), so that the respective modification of Eq. (8) is

$$\delta T_r \sim \delta T_\eta \sqrt{\ln[\eta(t)/r]}, \quad \text{for } r_d(t) < r < \eta(t). \quad (11)$$

Here r_d, denoting the scale of the smallest structure (ramp) of the temperature field (and it is also the scale where the scalar, temperature flux stops), is estimated to be

$$r_d(t) \sim \sqrt{\frac{\kappa \eta(t)}{\delta u_\eta(t)}} \sim \left(\frac{\nu}{t}\right)^{1/4} \sqrt{\frac{\kappa}{\beta g \Theta}}. \quad (12)$$

Note that the $\eta(t)/r_d(t)$ ratio is invariant, $\sim\sqrt{\nu/\kappa}$, i.e., it does not change with time.

Two dimensions.—One would expect that two-dimensional physics should be different from the three-dimensional physics discussed above. This expectation is based on the following standard phenomenological consideration: if pumping and dissipation are not taken into account, the velocity field dynamics is characterized by two globally conserved quantities, i.e., in addition to kinetic energy, the system also conserves enstrophy. Thus, in the steady case when both energy and enstrophy are permanently injected at the pumping scale, one expects to find two cascades, of energy and enstrophy, describing the transport of these two fundamental kinematic quantities upscale and downscale, respectively [17,18]. However the fundamental RT relation (5) breaks the stationarity and one finds that the inverse cascade (of energy) cannot be realized simply because the "pumping" scale $L(t)$ grows too fast for the larger [than current value of $L(t)$] structures to become established. (The situation is in a sense similar to what is known as a lack of inverse cascade in 2D turbulence decay; see, e.g., [8].) Moreover, an additional line of arguments leads to the conclusion that the direct (enstrophy) cascade is also not realizable in the 2D RT turbulence because the major assumption of the 3D RT turbulence, that the buoyancy term is subleading at scales smaller than L, breaks down in two dimensions. Indeed, if we assume the opposite, i.e., that solutions corresponding to the enstrophy and temperature fluctuations are realized, one derives that $\delta v_r \sim \delta v_L(r/L)$ and $\delta T_r \sim \Theta$ (where the estimates are valid up to logarithmic factors) and, finally, that the ratio of the rhs of Eq. (7) to the expression on the lhs of Eq. (7) is of the order of L/r, i.e., it is not small. Thus, I have just proved by counterexample that the enstrophy flux solution is not realizable in 2D either, and the buoyancy term from the rhs of Eq. (1) cannot be considered small in comparison with the nonlinear self-advection term from the lhs of the same equation.

The resolution of this problem comes through the so-called Bolgiano-Obukhov (BO) scenario introduced in the context of the Rayleigh-Bernard convection [19,20]. (See also [10] and references therein.) The BO scenario assumes equipartition of the buoyancy and nonlinear terms in Eq. (1) at all scales smaller than the energy-containing one, $L(t)$:

$$\frac{\delta u_r^2}{r} \sim \beta g \delta T_r. \tag{13}$$

Even though the cascades of kinetic energy and enstrophy are prohibited, the temperature fluctuations continue to cascade to small scales. Thus, from Eq. (3) one derives

$$\varepsilon_T(t) \sim \frac{\delta u_r \delta T_r^2}{r} \sim \frac{\delta u_L \Theta^2}{L}. \tag{14}$$

Combining Eqs. (13) and (14) one arrives at

$$\delta T_r \sim \left(\frac{r}{L(t)}\right)^{1/5}, \tag{15}$$

$$\delta u_r \sim \left(\frac{r}{L(t)}\right)^{3/5} u_L(t) \sim \frac{r^{3/5}(\beta g \Theta)^{2/5}}{t^{1/5}}. \tag{16}$$

Considering the case of large Pr number, $\nu \gg \kappa$, one defines the viscous scale η, as a scale at which both buoyancy and nonlinear self-advection terms from Eq. (1) become of the order of the viscous term. From Eq. (16) extended down to the viscous scale, and the relation, $\delta u_\eta \eta \sim \nu$, one derives

$$\eta(t) \sim \left(\frac{\nu}{u_L}\right)^{5/8} L^{3/8} \sim \frac{\nu^{5/8} t^{1/8}}{(\beta g \Theta)^{1/4}}. \tag{17}$$

Thus, in contrast with the 3D expression Eq. (9), Eq. (17) shows that the 2D viscous scale grows with time. Note also that the quadratic growth of $L(t)$ with time is still rapid enough to guarantee an asymptotic increase of the Bolgiano-Obukhov range dimensionless width, $L(t)/\eta(t) \propto t^{15/8}$.

At even smaller scales, $r < \eta$, the velocity field is smooth, $\delta u_r \sim \delta u_\eta r/\eta$, and the temperature field down-scale transport occurs according to the Batchelor regime flux solution given by Eq. (11). The 2D version of Eq. (12) is

$$r_d(t) \sim \sqrt{\frac{\kappa \eta(t)}{\delta u_\eta(t)}} \sim \frac{\sqrt{\kappa}(t\nu)^{1/8}}{(\beta g \Theta)^{1/4}}, \tag{18}$$

i.e., the $\eta(t)/r_d(t)$ ratio remains invariant, $\sim\sqrt{\nu/\kappa}$, as in the 3D case.

I find it useful to conclude this Letter with some remarks emphasizing the relation between the above simple phenomenological theory and other relevant descriptions and phenomena. I will also mention some related models and extensions that I believe are worth studying in the future.

In my opinion, spectral analysis of RT turbulence in the low Atwood regime is a surprisingly underdeveloped field, especially given the long history of RT turbulence in the Boussinesq approximation [1–4]. There are only a few recent numerical studies [14,21–23] and even fewer experimental studies [21,24] of fundamental issues of energy and density spectra at scales smaller than $L(t)$. Even though some studies, see, e.g., [23], do discuss 2D numerical analysis of RT turbulence, I was unable to find in the literature any discussion of the energy and/or density spectrum in the 2D case for comparison with my 2D predictions (13)–(18). On the other hand, all the 3D studies known to us [14,21,23,24], in which small-scale spectra were analyzed, are consistent with the Obukhov-Corrsin "5/3"-spectrum described in Eq. (8). As far as the Kolmogorov (viscous) scale dependence on time described by Eq. (9) is concerned, this is the original result of the recent numerical analysis and self-similar closure theory by Clark and Ristorcelli [14]. My obvious conclusion here is that more numerical and experimental studies of the small-scale features of RT turbulence are required to obtain a clear picture of where the phenomenological theory is applicable and where it fails.

The breakdown of this phenomenological (i.e., it is certainly limited and approximate) theory is obviously expected at least for higher-order moments (structure functions) of the fluctuating fields. Indeed, the common expectation is that, even though scaling of low moment structure functions of turbulent fields is usually well approximated by phenomenological estimates, the effect of intermittency, corresponding to the breakdown of the self-similar description for higher moment structure functions, remains out of the scope of this phenomenological approach (see, e.g., [8]). Clearly, recent breakthroughs in understanding the Lagrangian source of intermittency in passive scalar turbulence [25–27] establish an important starting point for the analysis of the RT small-scale intermittency (see, e.g., reviews [28,29]). Thus, I think that it may be useful to study in detail the passive problem, i.e., the one characterized by a statistically given velocity field that is decoupled from the scalar field (temperature) and which obeys a relatively simple (self-similar, and maybe even Gaussian) but time-dependent statistics mimicking the results of the 3D phenomenological theory discussed above. This passive model would be a natural time-dependent generalization of the famous Kraichnan model of passive scalar advection [16] which led to the previously mentioned passive scalar results [25–27]. Theoretical and numerical solutions (e.g., through the Lagrangian particles numerical method [30] developed for Kraichnan model analysis) of intermittency phenomena may become a very important next step for a more sophisticated analysis of mixing in Raleigh-Taylor turbulence.

It also may be useful to generalize the Lagrangian phenomenological approach of Navier-Stokes steady turbulence, developed in [31], to the time-dependent case of RT turbulence. A plausible model here would be one completely neglecting the backreaction of the small-scale temperature fluctuations to the velocity field, i.e., replacing the rhs term in Eq. (1) by a constraint imposed on the large-scale velocity field, consistent with Eq. (5). This modeling may help to describe at least some part of the large-scale anisotropy (and also to test if and how it cascades) that is known to play an essential role in RT turbulence (see, e.g., [14,21–23]).

I also find it important to stress once again the relation and the difference between statistics of small-scale fluctuations in RT turbulence and Rayleigh-Benard convection turbulence, with the latter being studied extensively for the last three decades (see, e.g., reviews [10,11]). The relation is obvious. Even if one neglects the very different boundary and initial conditions imposed in the two cases, the dynamical system of microscopic equations (1)–(3), governing both phenomena in the Boussinesq approximation is the same. The major difference between these problems stems from stationarity of the RB turbulence and thus its extreme sensitivity to the boundary conditions (this point was emphasized in recent studies [32–35]), contrasted with the universality expected from the small-scale structure of RT turbulence, e.g., with respect to the type of initial perturbation of the surface separating the two fluids.

Note also that the spatial part of my 2D consideration based on the BO scaling of Eqs. (13)–(16) is equally applicable to the RT and the RB cases (in the RB case L and u_L are constants). This is consistent with realizability of the BO scenario for low-order structure functions of temperature and velocity in 2D RB turbulence, recently confirmed numerically [36].

I would like to acknowledge illuminating discussions with T. T. Clark, G. D. Doolen, U. Frisch, V. Lebedev, R. Ristorcelli, D. H. Sharp, and V. Steinberg.

[1] Lord Rayleigh, Proc. London Math. Soc. **14**, 170 (1883).

[2] G. I. Taylor, Proc. R. Soc. London, Ser. A **201**, 192 (1950).

[3] S. Chandrasekhar, *Hydrodynamic and Hydrodynamic Instability* (Dover Publications, New York, 1961).

[4] R. E. Duff, F. H. Harlow, and C.W. Hirt, Phys. Fluids **5**, 417 (1962).

[5] D. H. Sharp, Physica (Amsterdam) **12D**, 3 (1984).

[6] A. N. Kolmogorov, Izv. Akad. Nauk SSSR, Ser. Fiz. **VI**, 56 (1941).

[7] A. M. Obukhov, Dokl. Akad. Nauk SSSR **32**, 22 (1941).

[8] U. Frisch, *Turbulence. The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).

[9] B. I. Shraiman and E. D. Siggia, Phys. Rev. A **42**, 3650 (1990).

[10] E. D. Siggia, Annu. Rev. Fluid Mech. **26**, 137 (1994).

[11] L. P. Kadanoff, Phys. Today **54**, No. 8, 34 (2001).

[12] A. M. Obukhov, Izv. Akad. Nauk SSSR, Geogr. Geofiz. **13**, 58 (1949).

[13] S. Corrsin, J. Appl. Phys. **22**, 469 (1951).

[14] T. T. Clark and R. Ristorcelli, J. Fluid Mech. (to be published).

[15] G. K. Batchelor, J. Fluid Mech. **5**, 113 (1959).

[16] R. H. Kraichnan, Phys. Fluids **11**, 945 (1968).

[17] R. H. Kraichnan, Phys. Fluids **10**, 1417 (1967).

[18] G. K. Batchelor, Phys. Fluids Suppl. II **12**, 233 (1969).

[19] R. Bolgiano, J. Geophys. Res. **64**, 2226 (1959).

[20] A. M. Obukhov, Dokl. Akad. Nauk SSSR **125**, 1246 (1959).

[21] S. B. Dalziel, P. F. Linden, and D. L. Youngs, J. Fluid Mech. **399**, 1 (1999).

[22] A.W. Cook and P. E. Dimotakis, J. Fluid Mech. **443**, 69 (2001).

[23] Y. N. Young, H. Tufo, A. Dubey, and R. Rosner, J. Fluid Mech. **447**, 377 (2001).

[24] P. N. Wilson and M. J. Andrews, Phys. Fluids **14**, 938 (2002).

[25] M. Chertkov, G. Falkovich, I. Kolokolov, and V. Lebedev, Phys. Rev. E **52**, 4924 (1995); Phys. Rev. Lett. **76**, 2706 (1996).

[26] K. Gawedzki and A. Kupiainen, Phys. Rev. Lett. **75**, 3834 (1995).

[27] B. I. Shraiman and E. Siggia, C.R. Acad. Sci., Ser. II **321**, 279 (1995).

[28] B. I. Shraiman and E. D. Siggia, Nature (London) **405**, 639 (2000).

[29] G. Falkovich, K. Gawedzki, and M. Vergassola, Rev. Mod. Phys. **73**, 913 (2001).

[30] U. Frisch, A. Mazzino, and M. Vergassola, Phys. Rev. Lett. **80**, 5532 (1998).

[31] M. Chertkov, A. Pumir, and B. I. Shraiman, Phys. Fluids **11**, 2394 (1999).

[32] S. Ashkenazi and V. Steinberg, Phys. Rev. Lett. **83**, 3641 (1999); **83**, 4760 (1999).

[33] Z. A. Daya and R. E. Ecke, Phys. Rev. Lett. **87**, 184501 (2001).

[34] R. M. Kerr, Phys. Rev. Lett. **87**, 244502 (2001).

[35] S. Grossmann and D. Lohse, Phys. Rev. E **66**, 016305 (2002).

[36] A. Celani, T. Matsumoto, A. Mazzino, and M. Vergassola, Phys. Rev. Lett. **88**, 054503 (2002).