A statistical mechanics analysis of coded CDMA with regular LDPC codes

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Why CDMA?

Combining LDPC and CDMA

Transition points

Decoding schemes

Summary and future research

Outline
Overview: Mobile communication system
Multiple access

Simultaneous communication of (uncoordinated) users to the same base station

Frequency-division (FDMA)

Time-division (TDMA)

Code-division (CDMA)

Simultaneous communication
Multiple access (2)

How a channel is divided...?
CDMA: Principle

Alice

Signature Seq. $s_1'$

Binary Data $b_1$
CDMA: Principle

Alice

Binary Data \( b_1 \)

Signature Seq. \( f_s \)

Bob

Binary Data \( b_2 \)

Signature Seq. \( f_s \)

Alice
CDMA Principle

Alice

Bob

Base Station

Binary Data

Signature Seq.

Rcvd. Signal

Binary Data

Signature Seq.

Noise

CDMA Principle
CDMA channel ($K$ users)

Rcvd. Signal $y_{\mu} = \frac{1}{\sqrt{G}} \sum_{k=1}^{K} s^\mu_k b_k + n_{\mu}$

($\mu = 1, \ldots, G$)

Channel

Noise $\{n_{\mu}\}$

Binary Data $b_1, b_2, \ldots, b_K$

Signature Seq. $\{s^\mu_1\}, \{s^\mu_2\}, \ldots, \{s^\mu_K\}$
To estimate $q_1, \ldots, q_K$ from $y_1, \ldots, y_G$
Remark 1: Relationship with Perceptron

Detection Problem

To estimate $b = (b_1; \ldots; b_K)^T$ from $(s_1; y_1); \ldots; (s_G; y_G)$, where $s = (s_1; \ldots; s_K)^T$ and $y = 1_{\mathbb{P}(G)} b^T s + \epsilon$.

Problem equivalent to Learning of Binary-weight Linear Perceptron with Additive Output Noise

\[
(\mathcal{G}, \ldots, 1 = n) \quad \eta u + \eta s \cdot q^\mathcal{G} = \eta y
\]

and

\[
\left(\sum_{n=1}^{K} s_n, \ldots, \sum_{n=1}^{K} \right) \equiv \eta s \text{ where } \mathcal{G} \left(\sum_{n=1}^{K} y_n, \ldots, \right) = q^\mathcal{G} \left(\sum_{n=1}^{K} s_n, \ldots, q, \right) = q
\]

Detection Problem: To estimate $b$ with Perceptron

Remark 1: Relationship with Perceptron
Analysis: Bayesian framework

\[ (q)d(q|\mathbf{h})d(\mathbf{h}) = (\mathbf{h})d \quad \frac{(\mathbf{h})d}{(q)d(q|\mathbf{h})d} = (\mathbf{h}|q)d \]

Posterior

\[ n^S \cdot \frac{q_{z/1-n^h}}{1 - \rho^u} - n^h = n^u \]

Channel char: (q|h)d

Prior: (q)d

as defined by pdf of noise
Bayes decision theory

Loss function:

\[ L_k(b) = 1 - b_k \]

\( b_k \): True Data; \( \hat{b}_k \): Estimate

Optimum decision rule (in the sense of minimizing expected loss)

\[ \hat{b}_k = \arg\max_{b_k \in \{1, -1\}} \left( \frac{P_{b_k}(y)}{P_{\hat{b}_k}(y)} \right) \]

Expected loss | Bit error rate (BER)

\[ \langle \hat{h} \| q \rangle_d = \langle h \| \hat{y} \rangle_d = \max_{\{1, -1\}} \arg = \hat{y} \]

(except for all users due to symmetry)

Bayes decision theory
Statistical-mechanical analysis

- Large-system limit: \( K, G \to \infty \) with load \( \beta = K/G = O(1) \)
- Random spreading: \( s_k \): i.i.d.; mean=0, variance=1
Objective: To evaluate Shannon entropy of per user.

Replica method:

\[
\lim_{K \to 1} \lim_{n \to 0} \frac{\int \mathcal{H}_u \left[ (\hat{h}) d \right] (\hat{h})^0 d \int \frac{\mathcal{H}}{1} \frac{u}{e}^0 u \right]}{\lim_{n \to 0}} = \mathcal{I}
\]

Replica analysis:
Binary uniform prior, AWGN

**Assumptions:**

- Binary uniform prior
  
  \[ p(b) = \text{const. over } \{-1, 1\}^K \]

- Additive White Gaussian Noise Channel

  \[ n_i \sim N(0, \sigma^2_0), \text{ i.i.d.} \]
Binary uniform prior, AWGN

Prior: \( p(b) = \text{const. over } f \) 

Conditional: \( p(y_j | b) = \text{G}_1 \) 

Posterior: \( p(b | y) = \text{const. over } f \)

Bayes' formula: \( \frac{p(y | b) p(b)}{p(y)} = p(b | y) \)
\[ p(b|y) = Z \exp \left( \frac{1}{2} b^T W b \right) \]

\[ w_{ij} = 1 \]

\[ \text{Correlation Mtx. of Signature Seq. } h_k = \frac{1}{p(G \times s_k y)} \]

\[ \text{Matched-Filter Output Vector} \]

\[ \text{Matched-Filter Output Vector} : \sum_{n=1}^{n} \left( \frac{G}{H} \right) = \gamma \quad (\gamma) = \eta \]

\[ \text{Correlation Mtx.} : \sum_{n=1}^{n} \left( \frac{G}{H} \right) = \gamma_m \quad (\gamma_m) = M \]

\[ q_L \eta - q M q_L = (q) H \]

\[ \exp \left[ -Z \frac{0.0}{\eta} \right] = (f|q) \]

Remark 2: Relationship with Hopfield models

Ising spin systems - Hopfield models
The problem: \( z = A \mathbf{r} \pmod{2} \)

The received vector \( \overrightarrow{\mathbf{r}} + q \overrightarrow{\mathbf{c}} = \mathbf{u} \) \((\mathbf{u} \in \mathbb{F}_2^N)\)

Generator matrix \( \overrightarrow{\mathbf{A}}^{-1} \mathbf{B} | \mathbf{I} = \mathbf{G} \) \((\mathbf{A}, \mathbf{B} \in \mathbb{F}_2^{M \times N})\)

Encoding: \( q \overrightarrow{\mathbf{c}} = \overrightarrow{\mathbf{t}} \)

Decoding: \( \overrightarrow{\mathbf{u}} = \overrightarrow{\mathbf{A}} \mathbf{r} = \overrightarrow{\mathbf{A}} G \mathbf{t} \pmod{2} \)
Motivation: LDPC-coded CDMA

Need for coding: Performance of uncoded CDMA

LDPC codes: High rate / low decoding complexity

Related work: LDPC-coded CDMA on the basis of stripping (Caire et al. 2003)

System > minimal QoS
Serial concatenation of LDPC code and CDMA channel

Assume: \( N/M \to 1 \) but \( R = N/M = O(1) \)

Assume: \( K/G \to 1 \) but \( R = K/G = O(1) \)

Average taken w.r.t both \( s \) and \( A \)

Performance of optimum joint detection/decoding

Regular Gallager codes used (\( C, I \): Left/Right degree)

LDPC-coded CDMA
What we want to calculate is the free energy (mutual information per symbol per user between received and sent symbols)

\[
\lim_{M,K \to \infty} [\mathbb{E}_{S} \mathbb{E}_{A} (A,S)f] \rightarrow f
\]

We average over \( A \) and \( S \)

\[
\{(A,J) \} \equiv J
\]

\[
\left[ \mathbb{E} \left\langle \left( \mathcal{L} \right| \tilde{h}\right\rangle \mathcal{P} \left( \mathcal{L} \right| \tilde{h}\right\rangle \langle \mathcal{L} \rangle \tilde{h} \mathcal{P} \langle \mathcal{L} \rangle \rangle - \mathcal{P} \mathcal{P} \mathcal{P} \right] \frac{M}{N} = f
\]
The Hamiltonian has components (for each user)

$$0 = \left[ \left( \sum_{k=1}^{\infty} \frac{\mathbf{G}^\top}{\mathbf{I}} - \frac{\mathbf{h}}{2} \right) \frac{\mathbf{2}\Omega}{\mathbf{I}} - \frac{\mathbf{1}}{\mathbf{I}} \right]$$

For each codeword bit, spreading chip and user

- The parity checks are obeyed and 0 otherwise
- The parity checks are $\infty = (\cdots) \chi$ if parity checks are $\left( \sum_{k=1}^{\infty} \frac{\mathbf{1}}{\mathbf{I}} \right) \chi$

Use Nishimori’s condition = correct prior, RS representing the channel noise.

LDPC-coded CDMA - Hamiltonian
\[
\left[ x^p (x^q) \right] \prod_{i=1}^{\infty} \left( x^p \prod_{i=1}^{\infty} - x \right) \psi \int = (x)^q \\
\left[ x^p (x^q) \right] \prod_{i=1}^{\infty} z \mathcal{A} \left[ \left( x^p \prod_{i=1}^{\infty} \tanh \sum_{i=1}^{\infty} A + z A^\wedge \right) \tanh - x \right] \psi \int = (x)^q \\
\frac{z[(b-1)g + z^q]}{(b + wz - 1)g + z^q} = A \quad \frac{(b-1)g + z^q}{z} = A \\
\left[ x^p (x^q) \right] \prod_{i=1}^{\infty} z \mathcal{A} \left( x^p \prod_{i=1}^{\infty} \tanh \sum_{i=1}^{\infty} A + z A^\wedge \right) x \psi \int = b \\
\left[ x^p (x^q) \right] \prod_{i=1}^{\infty} z \mathcal{A} \left( x^p \prod_{i=1}^{\infty} \tanh \sum_{i=1}^{\infty} A + z A^\wedge \right) x \psi \int = m \\
\]
Bit error rate:

LDPC-coded CDMA
LDPC-coded CDMA

Typical structure of $P_b$-σ diagram

Spinodal; Thermo. trans
As $\beta$ increases $\sigma_0 \rightarrow 0$, irregular codes?
As the load increases, theoretical thresholds approach single-user channel capacity.
Individual optimum decoding

Received signal

CDMA Detection

Decoder

S_1, S_2, \ldots, S_L
Detection and decoding

Minimum MSE multi-user detection (H/S) per user

Individual optimum detection (H/S) and decoding

Joint detection and decoding
Individual optimum decoding
Individual optimum decoding II
Summary and future directions

Introduction of CDMA multiuser detection problem

Statistical-mechanics analysis using replica method

Coding prior to modulation has great potential

Current problem - dynamical transition point

Future directions - irregular constructions, joint modulation and coding

Future directions

Supported by FP5 RTN - STIPCO

http://www.ncrg.aston.ac.uk

Introduction of CDMA multiuser detection problem