AREA THEOREMS FOR ITERATIVE CODING SYSTEMS

C. Measson
A. M.
R. Urbanke
T. Richardson

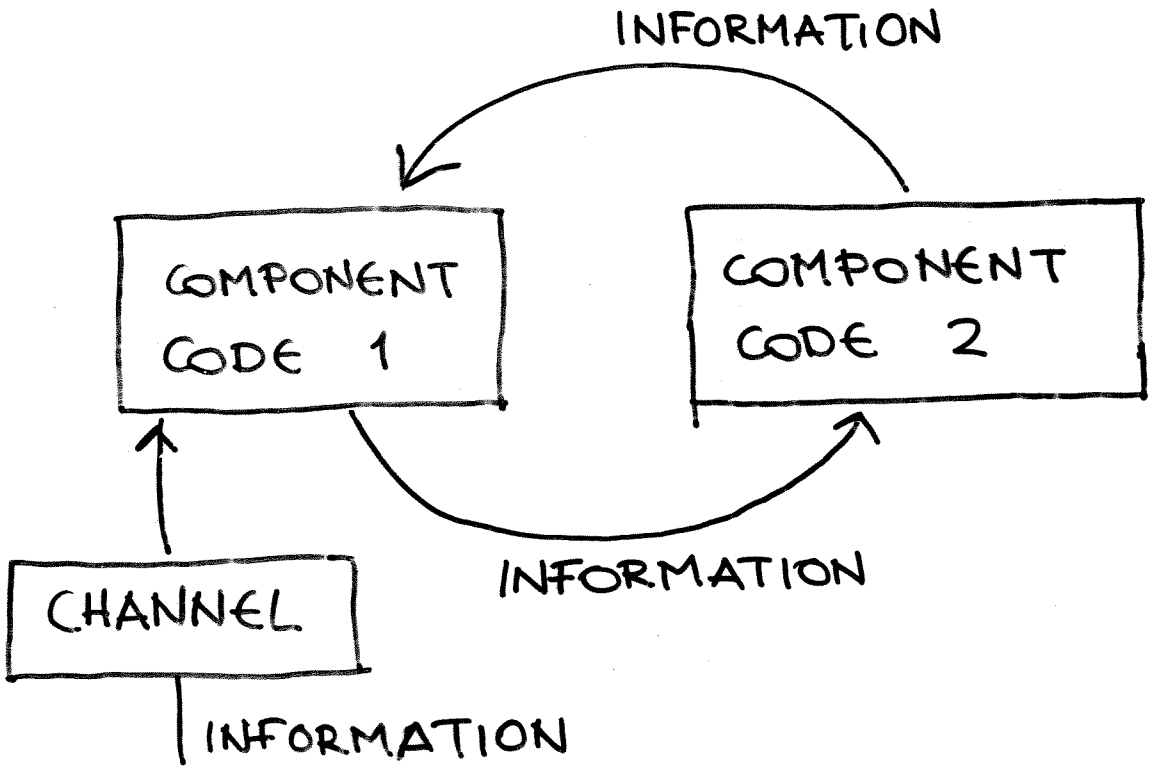
→ "HISTORICAL" INTRODUCTION

→ FROM EXIT TO GEXIT FUNCTIONS

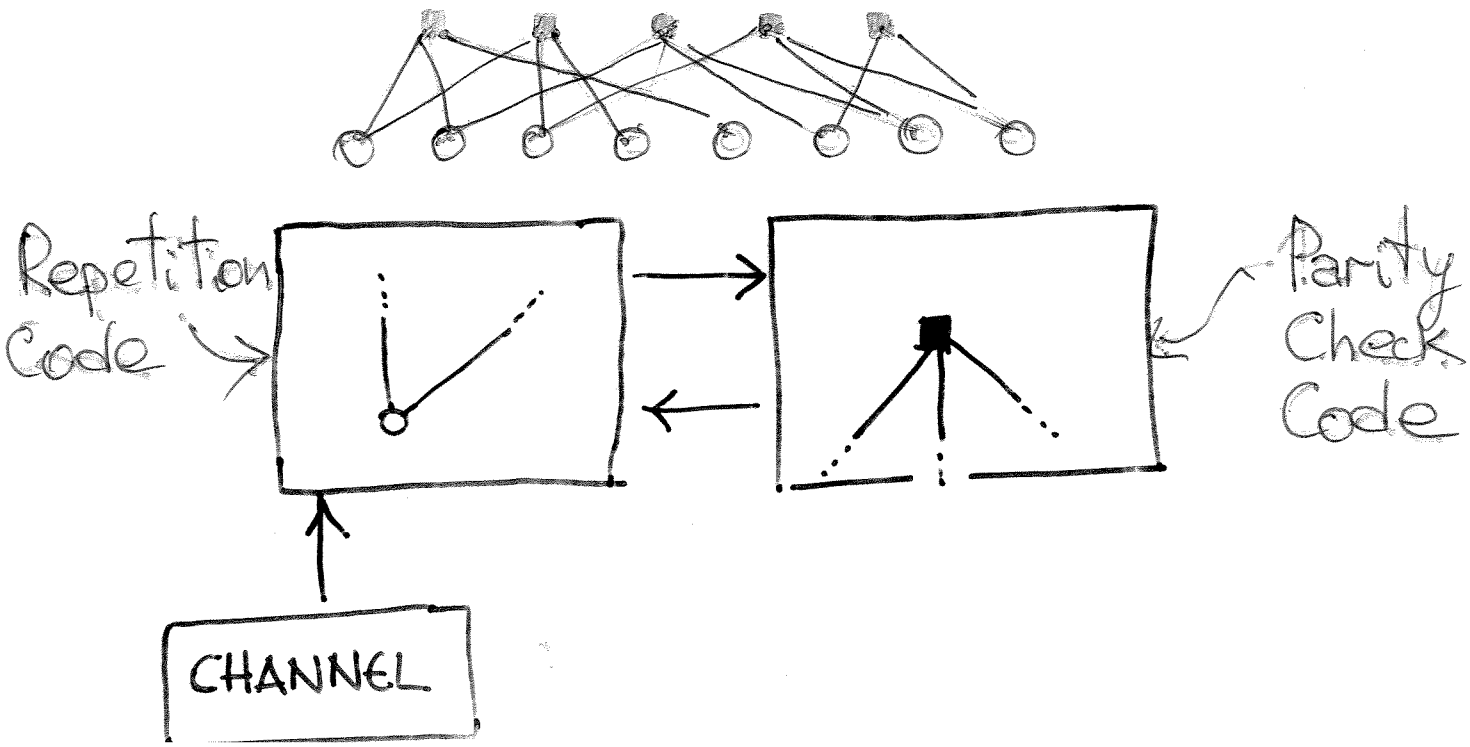
→ ALGORITHMIC INTERPRETATION OF AREA THEOREMS

Supported by EC-STIPCO

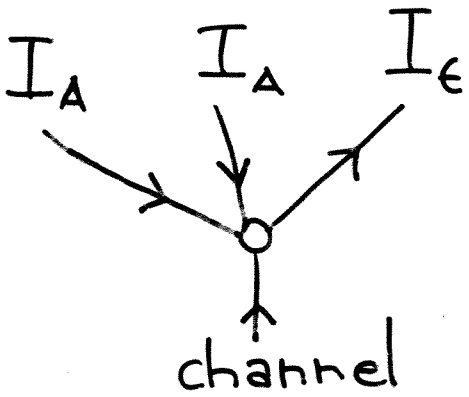
The engineer's point of view on iterative decoding.

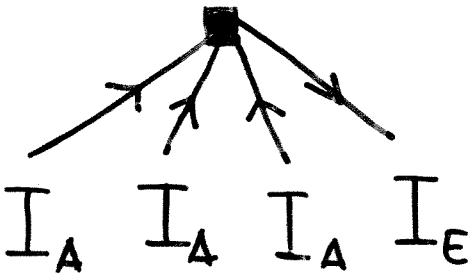


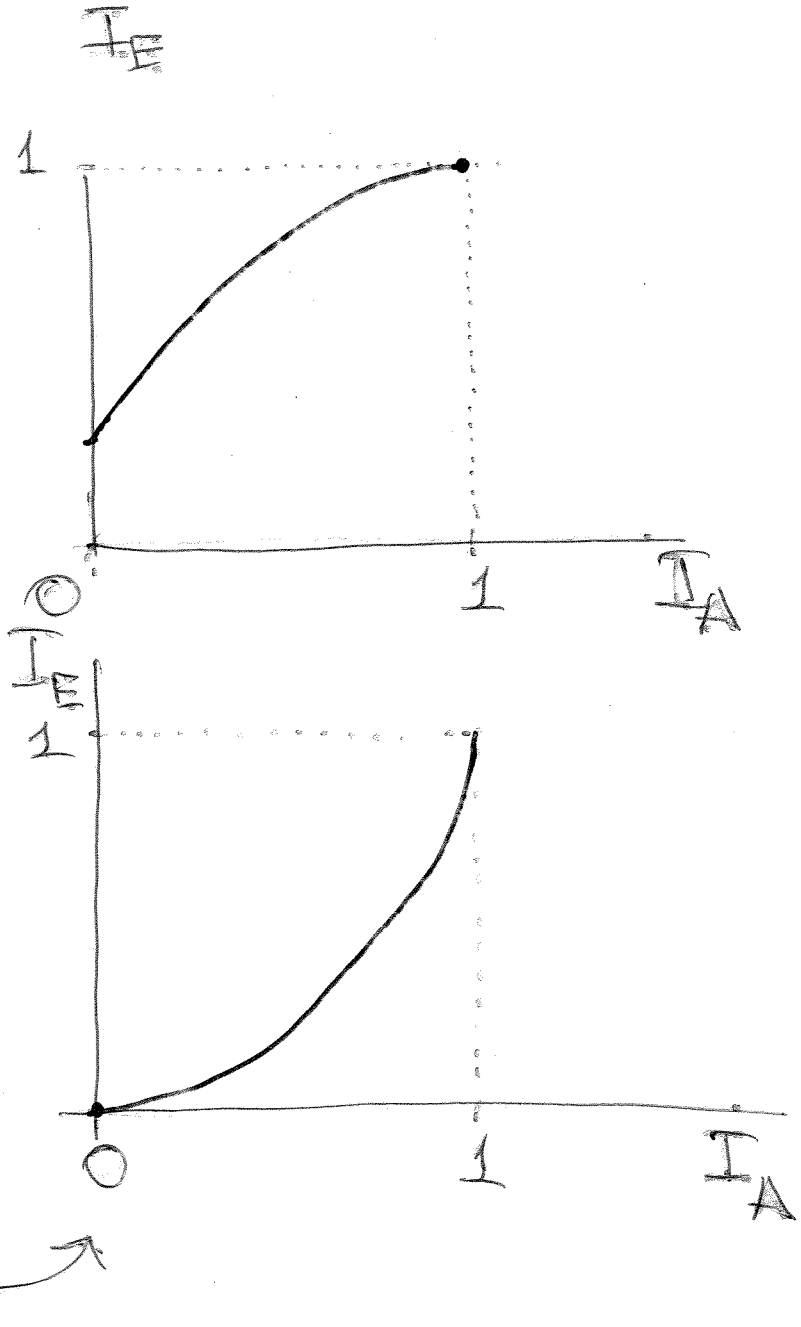
Example : LDPC codes



EXIT CHARTS .....

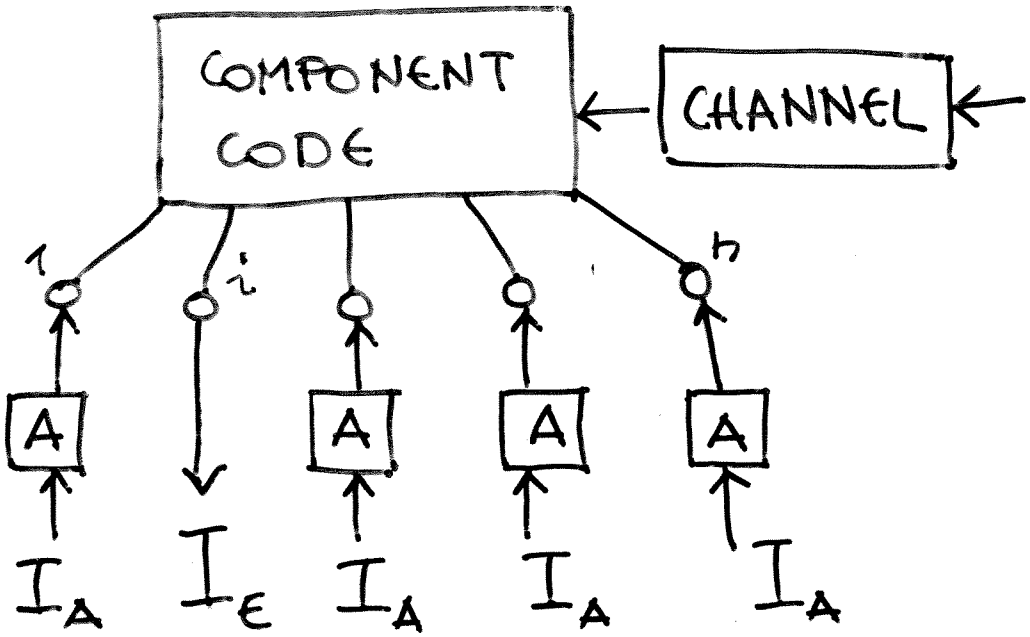






NOTICE: THESE CURVES ARE NOT YET WELL DEFINED

..... THEIR DEFINITION .....

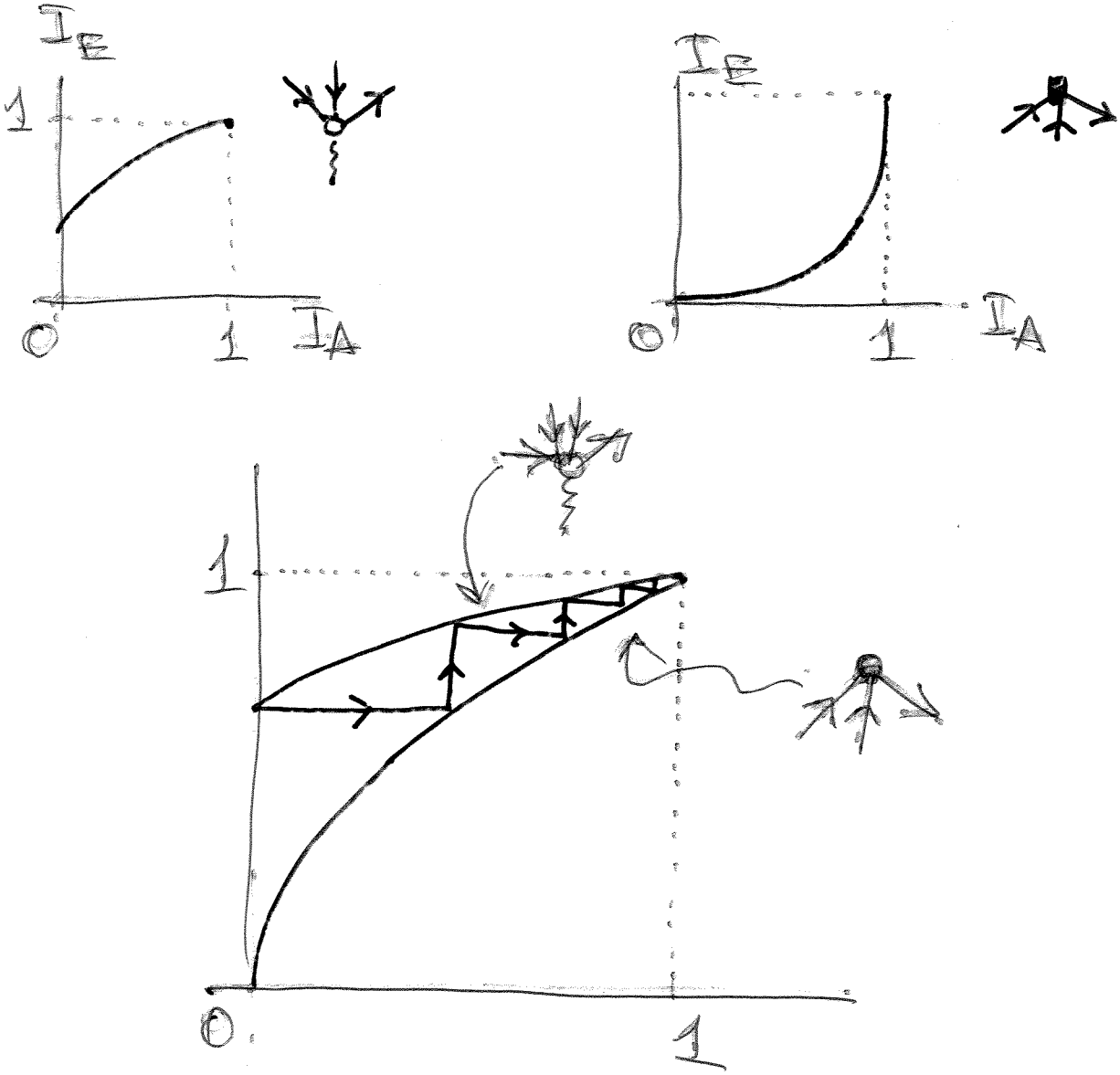


→ I_A bits of information on x_j, $j \neq i$ through a "fictitious" channel [A]

→ I_ϵ = 1 − entropy of x_i

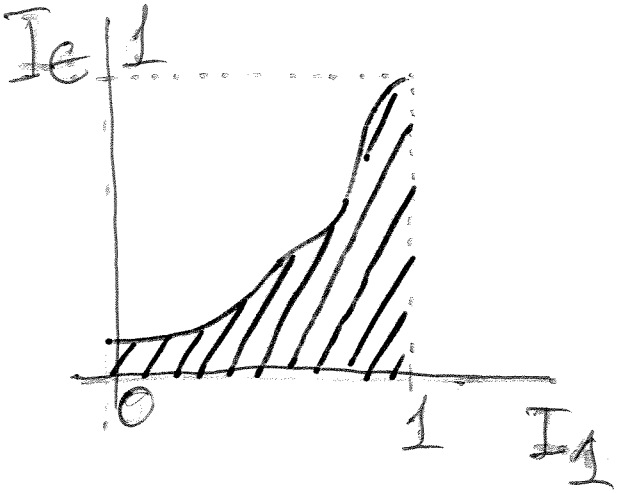
Notice that the choice of [A] is arbitrary.

..... AND THEIR USES



[ten Brink & others]

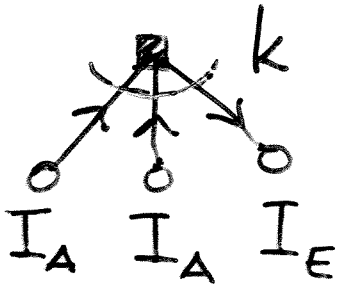
EMPIRICAL AREA RULES ....



$\cong$ 1 - rate

examples :

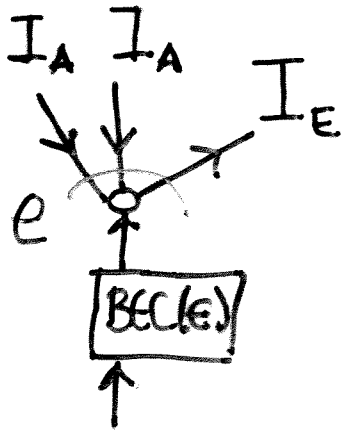
channel	=	BEC(ϵ)
A	=	BEC($1-I_A$)

→ 

$$\text{rate} = \frac{k-1}{k}$$

$$I_\epsilon = (I_A)^{k-1}$$

$$\int_0^1 I_A^{k-1}\, dI_A = \frac{1}{k}$$

→ 

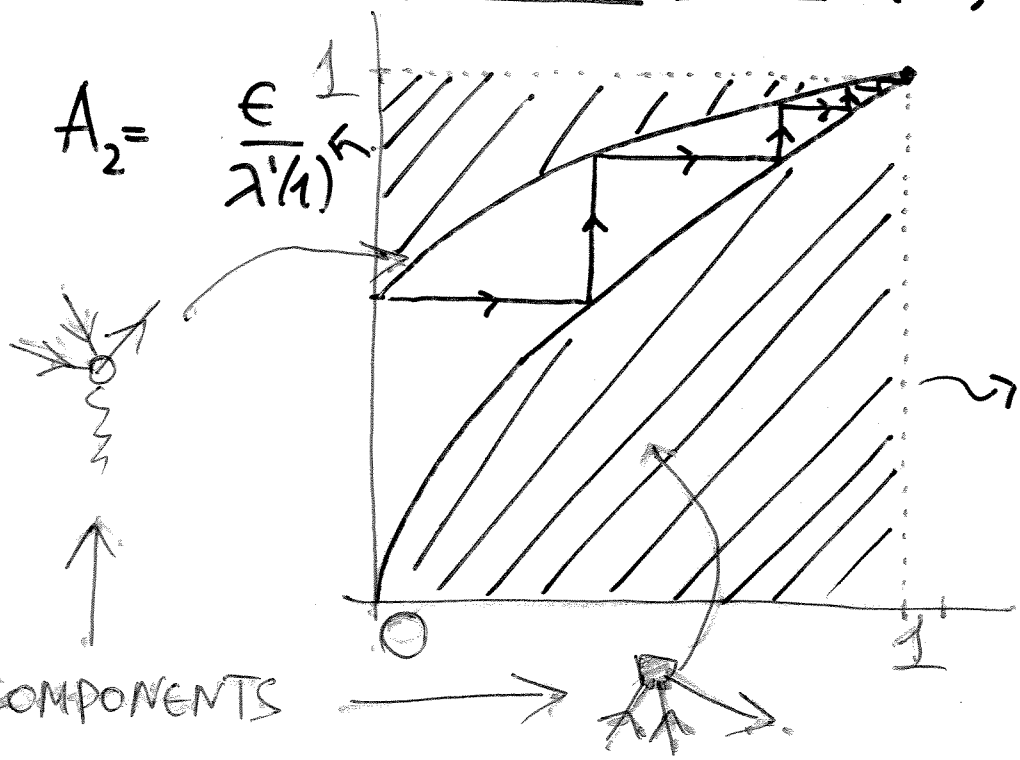
$$\text{rate} = \frac{\epsilon}{e}$$

$$I_\epsilon = 1 - \epsilon(1-I_A)^{e-1}$$

$$\int_0^1 \left[1 - \epsilon(1-I_A)^{e-1}\right] dI_A = 1 - \frac{\epsilon}{e}$$

.... AND THEIR USES (1)

$A_2 = \frac{\epsilon}{\lambda'(1)}$

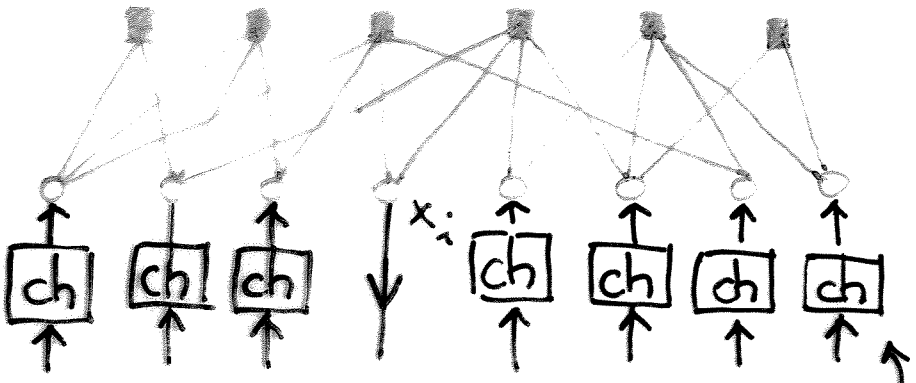


$\rightsquigarrow A_1 = 1 - \frac{1}{\rho'(1)}$

$$\boxed{A_1 + A_2 < 1} \Rightarrow \boxed{\frac{\epsilon}{\lambda'(1)} + 1 - \frac{1}{\rho'(1)} < 1} \Rightarrow \boxed{\frac{\lambda'(1)}{\rho'(1)} > \epsilon}$$

$R_e < C$

.... AND THEIR USES (2)

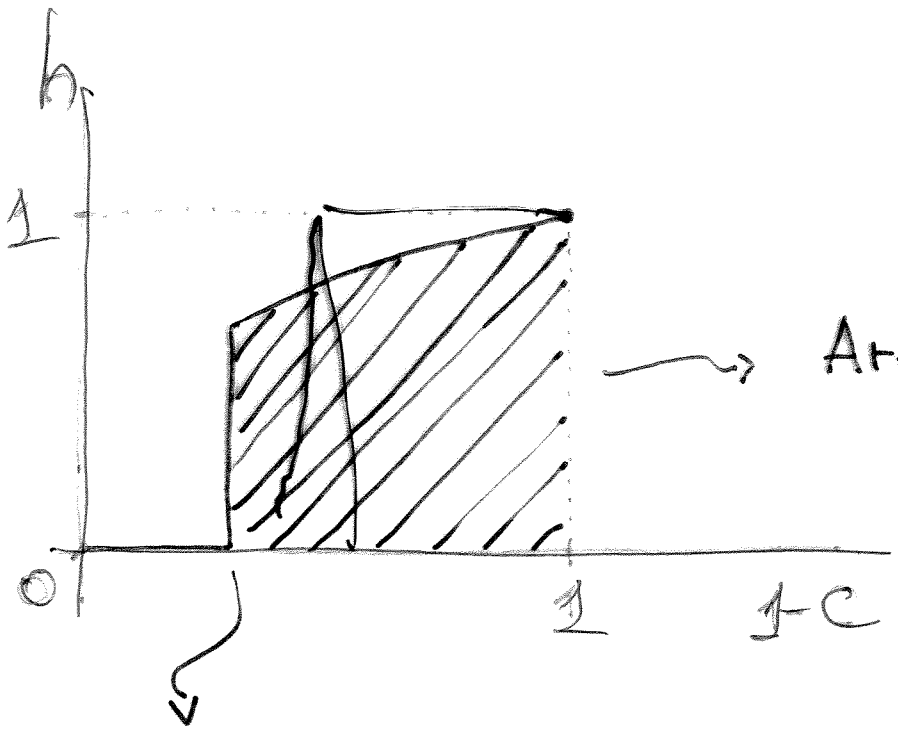


complete code : rate R

$h_i = H(X_i | Y_{\sim i})$

communication channel : capacity C

$X_{1..} X_n \rightarrow [ch] \rightarrow Y_1 .. Y_n$



Area = R

MAXIMUM LIKELIHOOD THRESHOLD

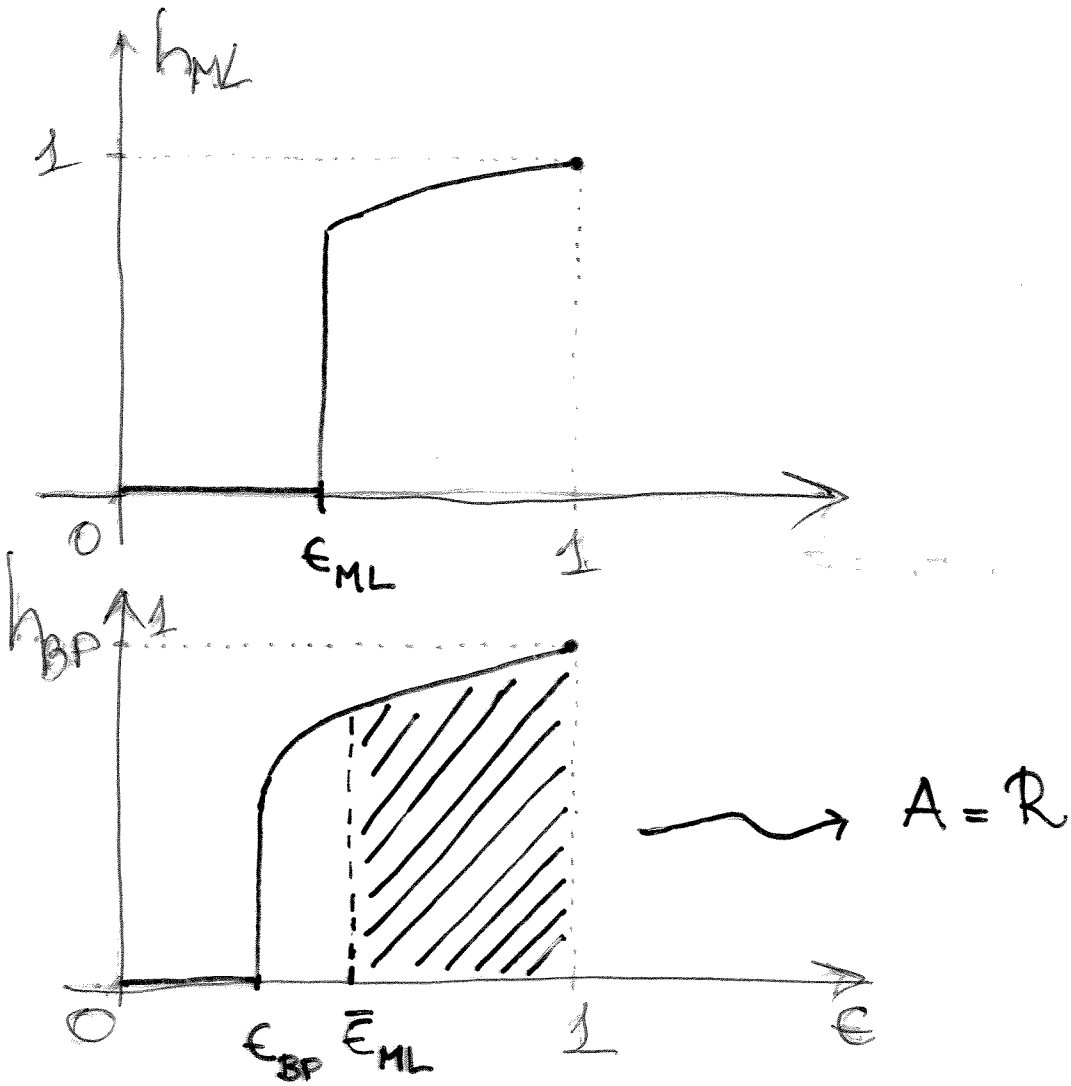
$\Rightarrow R < C$

Ashikhmin, Kramer, ten Brink [2002]

AREA THEOREMS ARE EXACT FOR THE BEC

Méasson, Urbanke [2003]

UPPER BOUNDS ON ML THRESHOLDS



$$h_{ML}(\epsilon) \leq h_{BP}(\epsilon) \quad \Rightarrow \quad \epsilon_{ML} \leq \bar{\epsilon}_{ML}$$

FROM EXIT TO GEXIT FUNCTIONS

→ A NEW PROOF FOR THE BEC.

$$X_i \longrightarrow \boxed{\text{BEC}(\epsilon_i)} \longrightarrow Y_i$$

conditional entropy : $H(X|Y)$

Theorem:

$$dH(X|Y) = \sum_{i=1}^{n} \frac{\partial H(X_i|Y)}{\partial \epsilon_i} \, d\epsilon_i \qquad \square$$

Proof:

$$\begin{aligned} H(X|Y) &= H(X_i|Y) + H(X_{\sim i}|X_i Y) \\ &= H(X_i|Y) + H(X_{\sim i}|X_i Y_{\sim i}) \end{aligned}$$

$\square$

Lemma: If $X_i \rightarrow \boxed{BEC(\epsilon_i)} \rightarrow Y_i$ then

$$\frac{\partial H(X_i|Y)}{\partial \epsilon_i} = H(X_i|Y_{\sim i}) \equiv EXIT_i$$

□

Proof:

$$H(X_i|Y) = Pr[Y_i=0] \cdot H(X_i|Y_i=0, Y_{\sim i}) + \quad (=0)$$
$$+ Pr[Y_i=1] \cdot H(X_i|Y_i=1, Y_{\sim i}) +$$
$$+ Pr[Y_i=?] \cdot H(X_i|Y_i=?, Y_{\sim i})$$
$$= \epsilon_i \cdot H(X_i|Y_{\sim i})$$

□

Cotollary

$$\int_0^1 \frac{1}{n} \sum_{i=1}^{n} EXIT_i \, d\epsilon = R$$

□

$\epsilon_1 = \epsilon_2 = \dots = \epsilon_n \equiv \epsilon$

Proof

$$\int_0^1 \dots d\epsilon = \int_0^1 \frac{dH(X|Y)}{d\epsilon} d\epsilon = H(\epsilon=1) - H(\epsilon=0$$
$$= R - 0$$

BUT the Theorem does not assume

$$X_i \longrightarrow \boxed{BEC(\epsilon_i)} \longrightarrow Y_i$$

IDEA :

$$EXIT_i \longrightarrow GEXIT_i$$

$$\| \qquad\qquad \|$$

$$H(X_i|Y_{\sim i}) \longrightarrow \frac{\partial H(X_i|Y)}{\partial \epsilon_i}$$

here ϵ_i is any (smooth) parametrizati of the channel

we have just to compute a derivative!

Theorem: If $X_i \to \boxed{Q(y|x)} \to Y_i$, then

$$GEXIT_i = \sum_{x_i y_i z_i} P(x_i) P(z_i|x_i) \frac{dQ}{d\epsilon}(y_i|x_i) \cdot \cdot \log\left\{ \sum_{x_i'} \frac{P(x_i'|z_i)\, Q(y_i|x_i')}{P(x_i|z_i)\, Q(y_i|x_i)} \right\}$$

extrinsic information at i [phys: local field]

□

Corollary: For a linear code on a BMS channel:

density of extrinsic LLR at X_i

$$GEXIT_i = \int_{-\infty}^{+\infty} a^{(i)}(\ell)\, k_L(\ell)\, d\ell$$

$$k_L(\ell) \equiv \sum_y \frac{dQ(y|0)}{d\epsilon} \cdot \log\left(1 + e^{-L_Q(y) - \ell}\right)$$

□

$$L_Q(y) = \log \frac{Q(y|0)}{Q(y|1)}$$

[$EXIT_i$ is obtained by

$$k_L(\ell) \to \tilde{k}_L(\ell) = \log\left(1 + e^{-\ell}\right) \quad]$$

example 1 : BEC(ε)

$$k_L(\epsilon) = \sum_y \frac{dQ(y|0)}{d\epsilon} \log\left[1 + \frac{Q(y|1)}{Q(y|0)} e^{-\epsilon}\right]$$

$$y \in \{ \; 0, \quad 1, \quad ? \; \}$$

$Q(y\|0) =$	$1-\epsilon$	0,	ϵ
$\frac{dQ(y\|0)}{d\epsilon} =$	-1	0	1

$$k_L(\epsilon) = -\log\left[1 + \frac{Q(0|1)}{Q(0|0)} e^{-\epsilon}\right] +$$

$$+ \log\left[1 + \frac{Q(?|1)}{Q(?|0)} e^{-\epsilon}\right] =$$

$$= \log\left[1 + e^{-\epsilon}\right] = \tilde{k}_L(\epsilon)$$

GEXIT = EXIT !

example 2 : BSC(p)

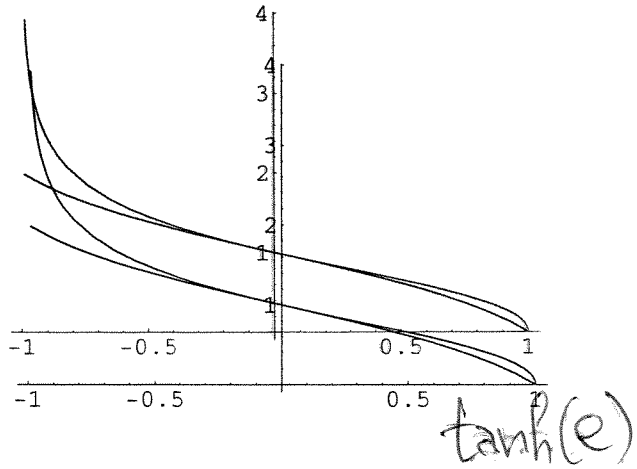
$$k_L(\ell) = \sum_y \frac{dQ(y|0)}{dp} \log\left[1 + \frac{Q(y|1)}{Q(y|0)} e^{-\ell}\right]$$

$$y \in \{ \quad 0 \quad , \quad 1 \quad \}$$

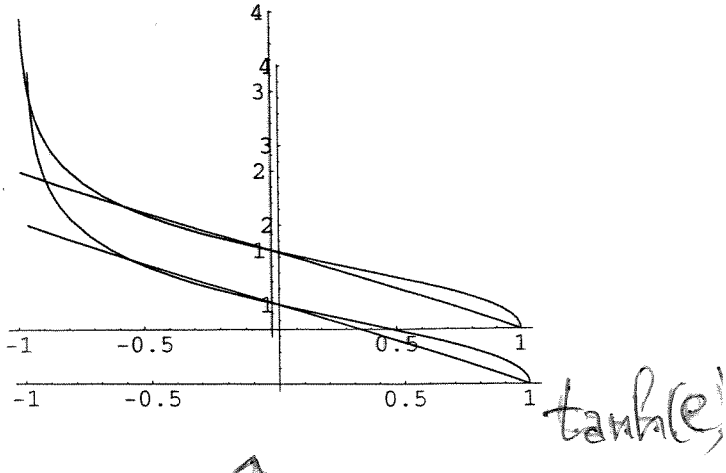
$Q(y\|0) =$	$1-p$	p
$\frac{dQ(y\|0)}{dp} =$	-1	1

$$k_L(\ell) = \log\left(1 + \frac{1-p}{p} e^{-\ell}\right) - \log\left(1 + \frac{p}{1-p} e^{-\ell}\right)$$

$$\left[\neq \tilde{k}_L(\ell) = \log\left(1+e^{-\ell}\right)\right]$$


4
4
3
3
2
2
1
1
-1
-0.5
0.5
1
-1
-0.5
0.5
1
tanh(e)

p=0.1


4
4
3
3
2
2
1
1
-1
-0.5
0.5
1
-1
-0.5
0.5
1
tanh(e)

p=0.3

Upper bounds through DENSITY EVOLUTION

Lemma

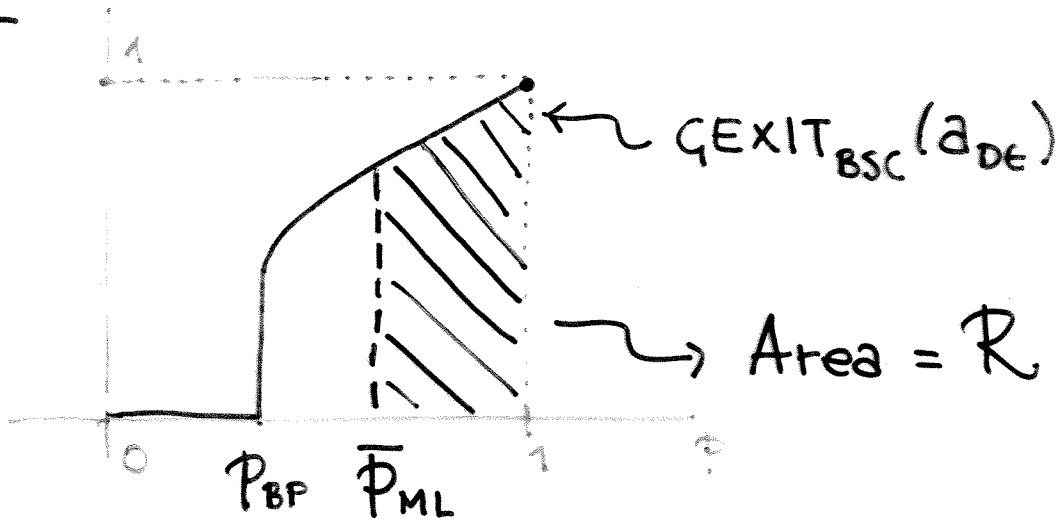
$$GEXIT_{BSC} \leq GEXIT_{BSC}(a_{DE})$$

$$\equiv \int_{-\infty}^{\infty} K_L^{BSC}(e)\, a_{DE}(e)\, de$$

density evolution fixed point..

□

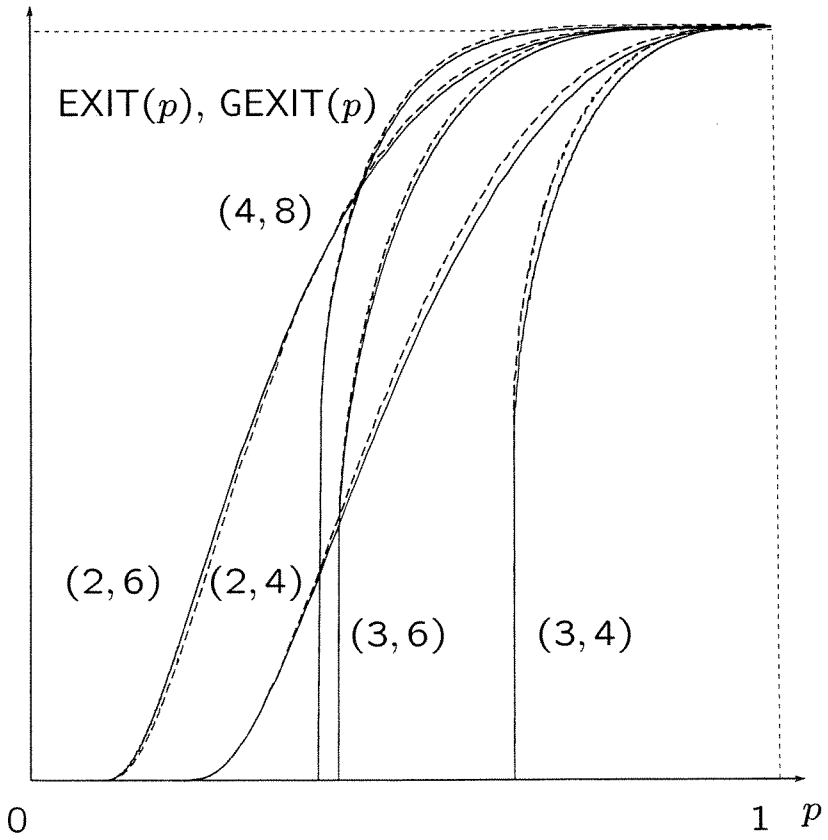
Corollary



$$p_{ML} \leq \bar{p}_{ML}$$

this coincides with the result from replicas or cavity

ex: (6,3) GALLAGER

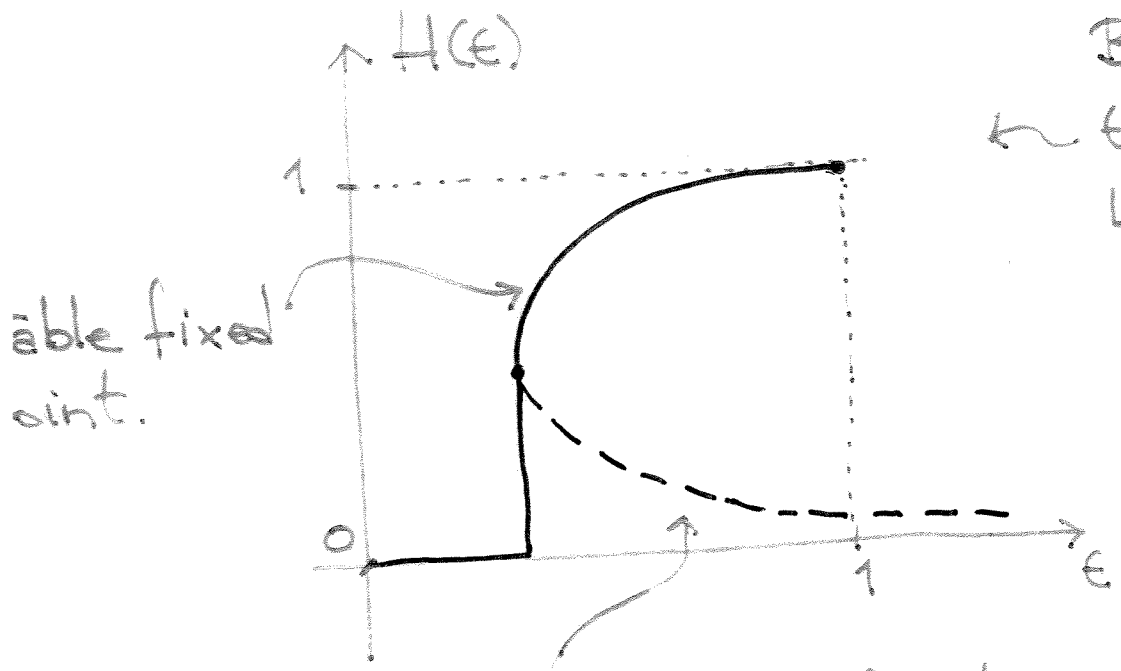

EXIT(p), GEXIT(p)
(4, 8)
(2, 6)
(2, 4)
(3, 6)
(3, 4)
0
1
p

example 3 : AWGN

$$\text{GEXIT} = \frac{1}{2}\,\text{MMSE}$$

[cf. Guo, Verdu, Shamai] [2004]

BEC : new area theorems .....

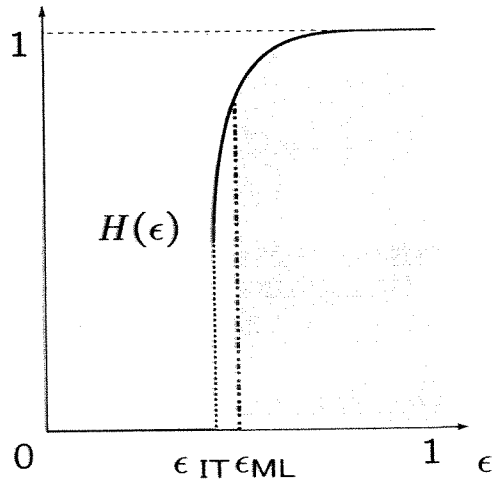


Belief Propagation
EXIT function
LDPC code.

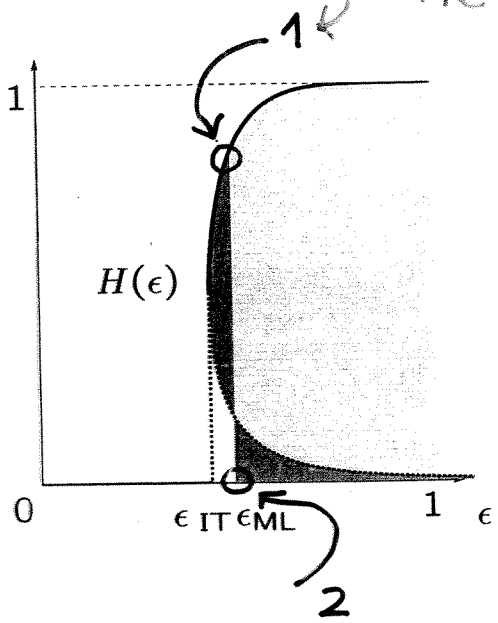
able fixed
oint.

unstable fixed point of
DENSITY EVOLUTION

And their algorithmic interpretation



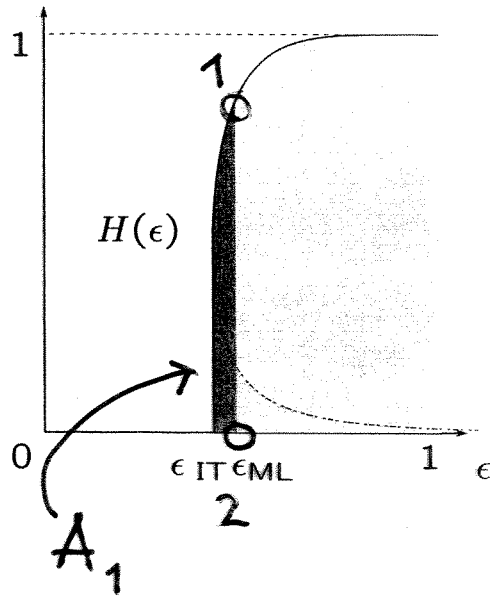
The information required for decoding is available here

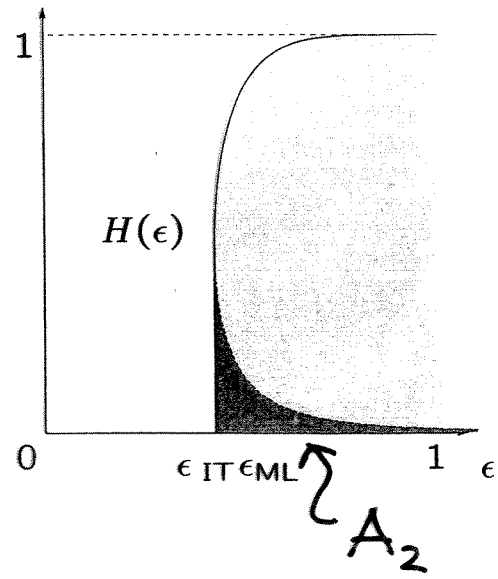


~ Maxwell construction in statistical mechanics

IDEA : an algorithm which goes from ① to ② without net exchange of information.

MAXWELL DECODER





→ At (1) BP gets stucked

→ It begins making guesses

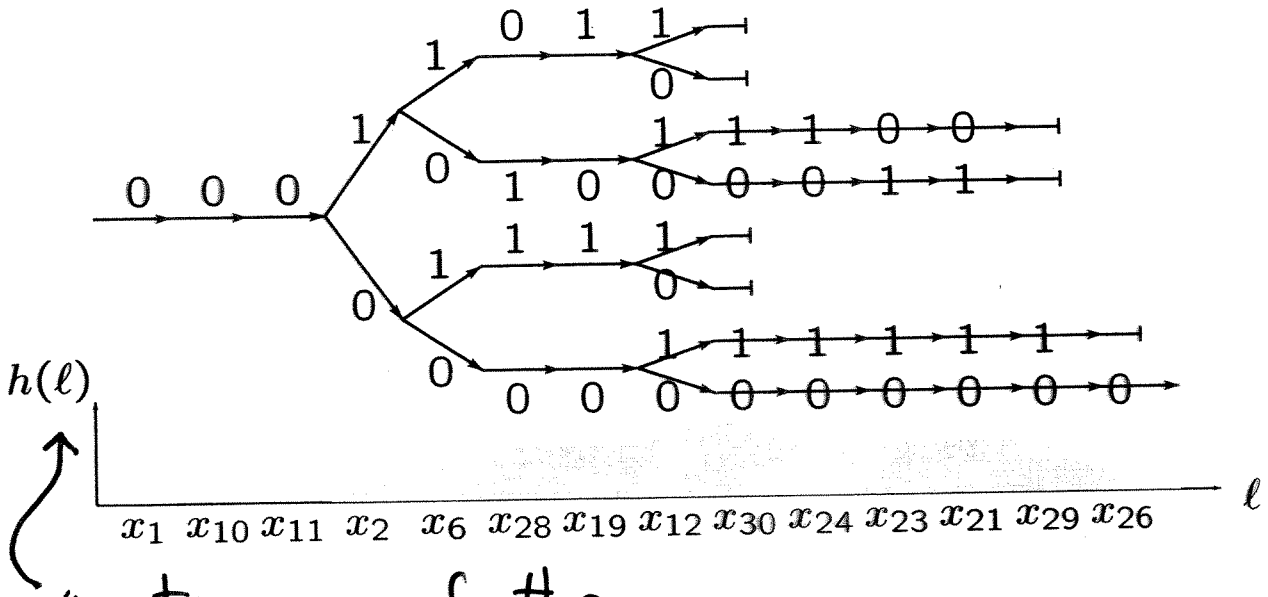
[# guesses ≅ n · A_1]

→ It eventually encounters contradictions

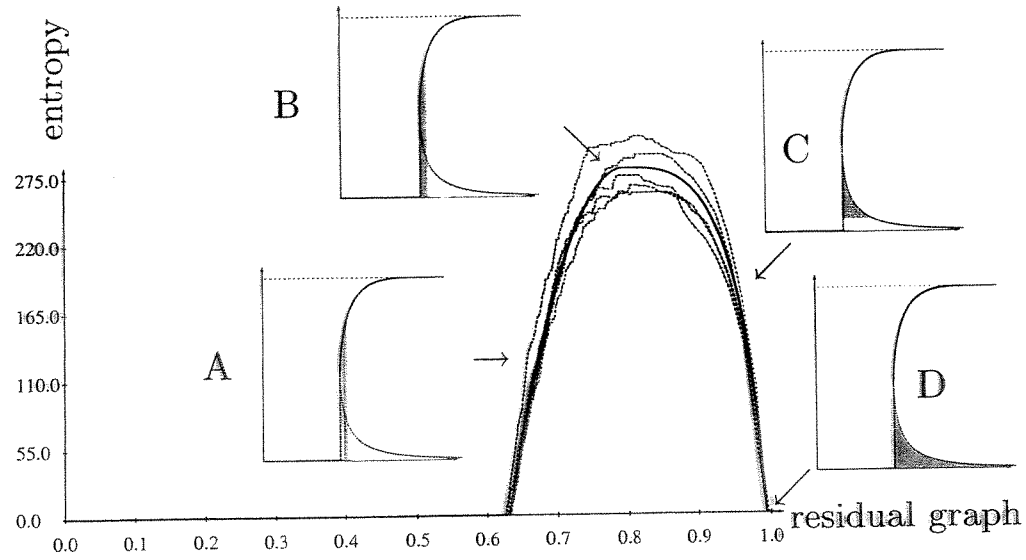
[# contradictions ≅ n · A_2]

→ At the ML threshold a unique sequence of guesses survives

$$[A_1 = A_2]$$



"entropy" of the algorithm.



CONCLUSIONS

→ "Thermodynamic" structure emerges naturally.

[$H(X|Y)$ $\leftrightarrow$ free energy F

EXIT $\leftrightarrow$ $\frac{\partial F}{\partial \lambda}$?] parameter of the model

→ New point of view on classical ideas of physics

[reversible transformation $\Delta L = 0$

is decoder without exchange of information]