

Propagating beliefs in spin-glass models

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(Received May 30, 2003)

We investigate the dynamics of an inference algorithm called belief propagation (BP) when employed in spin glass (SG) models and show that its macroscopic behavior can be traced by recursive updates of certain auxiliary field distribution, the stationary states of which reproduce the replica symmetric solution obtained by equilibrium analysis. We further provide a compact expression for the instability condition of the BP's fixed point which turns out to be identical to that of the instability for breaking the replica symmetry in equilibrium when the number of couplings per spin is infinite. This correspondence is extended to the case of finite connectivity in order to determine the phase diagram, which is validated numerically.

KEYWORDS: Spin-glasses, belief propagation, dynamics, AT stability

1. Introduction

Recently, there has been growing interest in the similarity between research on spin glass (SG) and that on information processing (IP).¹⁾ Since the employment of methods from SG theory have resulted in significant progress in solving several problems related to IP, including problems in machine learning,²⁾ error-correcting³⁻⁶⁾ and spreading codes,^{7,8)} it is natural to expect that the opposite might also be possible.

The purpose of the present paper is to provide one such example. More specifically, we show herein that the investigation of the dynamics of an iterative inference algorithm called belief propagation (BP), which has been developed in IP research,^{9,10)} provides a new understanding of the thermodynamical properties of SG when employed in SG models.

This paper is organized as follows. In the next section, we introduce BP to a family of SG models. This model family covers a variety of SG models that has been actively studied,^{11,12)} which is convenient for relating the results reported herein to the existing knowledge. In section 3, we investigate the macroscopic behavior of BP in the SG models. We show that the replica symmetric (RS) solution obtained in equilibrium analysis can be characterized as a *macroscopically* stationary state in BP. However, this does not imply that BP *microscopically* converges to a certain state. In section 4, we provide a compact expression of the microscopic instability condition around the fixed point in the BP dynamics, which is found to be identical to that of the instability for breaking the

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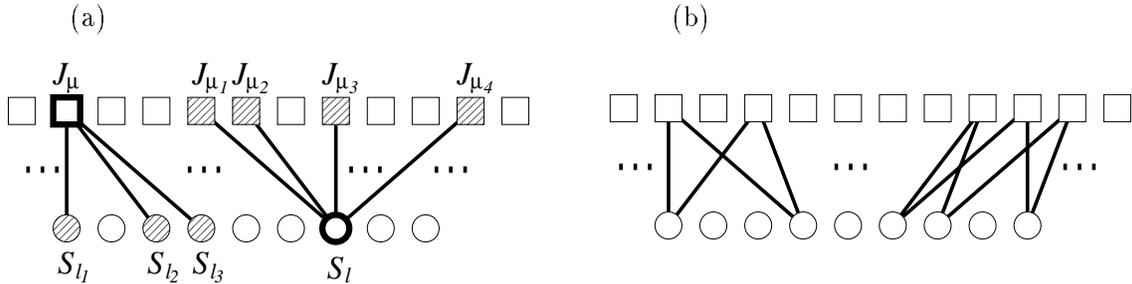


Fig. 1. (a): Graphical expression of SG models in the case of $K = 3$ and $C = 4$. In this expression, each spin S_l denoted as \bigcirc is linked to $C = 4$ couplings J_μ (\square), each of which is connected to $K = 3$ spins. $\mathcal{L}(\mu)$ and $\mathcal{M}(l)$ represent sets of indices of spins and couplings that are related to J_μ and S_l , respectively. In the figure, $\mathcal{L}(\mu) = \{l_1, l_2, l_3\}$ and $\mathcal{M}(l) = \{\mu_1, \mu_2, \mu_3, \mu_4\}$. (b): Cycles in a graph. A cycle is composed of multiple paths to link an identical pair of nodes. It is shown that BP can provide the exact spin averages in a practical time scale if a given graph is free from cycles.⁹⁾

replica symmetry in equilibrium, referred to as the de Almeida-Thouless (AT) instability,¹³⁾ when the connectivity per spin is infinite. The efficacy of this expression for a sparsely connected SG model is also numerically confirmed. The final section is devoted to a summary of the findings of the present study.

2. Belief propagation in spin-glass models

We herein take up a family of Ising SG models defined by the Hamiltonian

$$\mathcal{H}(\mathbf{S}|\mathbf{J}) = - \sum_{\mu=1}^M J_\mu \prod_{l \in \mathcal{L}(\mu)} S_l, \quad (1)$$

where $\mathcal{L}(\mu)$ denotes a set of indices which are connected to a quenched coupling J_μ . We assume that each coupling is independently generated from an identical distribution

$$P(J_\mu) = \frac{1 + J_0/(\sqrt{C}J)}{2} \delta\left(J_\mu - \frac{J}{\sqrt{C}}\right) + \frac{1 - J_0/(\sqrt{C}J)}{2} \delta\left(J_\mu + \frac{J}{\sqrt{C}}\right). \quad (2)$$

We further assume that for each μ , $\mathcal{L}(\mu)$ is composed of randomly selected $K \sim O(1)$ spin indices and that each spin index l is associated with C couplings, the set of which is denoted as $\mathcal{M}(l)$. $J_0 > 0$ and $J > 0$ are parameters to control the mean and the standard deviation of J_μ , respectively, which naturally links the current system (1) to the Sherrington-Kirkpatrick (SK) model¹¹⁾ in the case of $K = 2$ and $C \sim O(N)$, and to sparsely connected SG models^{12,14,15)} in general.

A major goal of statistical mechanics in the current system is to calculate the microscopic spin average $m_l = \text{Tr}_{\mathbf{S}} S_l \exp[-\beta\mathcal{H}(\mathbf{S}|\mathbf{J})] / \text{Tr}_{\mathbf{S}} \exp[-\beta\mathcal{H}(\mathbf{S}|\mathbf{J})]$ from the given Hamiltonian (1). This is formally identical to an inference problem for a posterior distribution $P(\mathbf{S}|\mathbf{J}) \propto \prod_{\mu=1}^M P(J_\mu|\mathbf{S})$ derived from a conditional probability $P(J_\mu|\mathbf{S}) = \exp[\beta J_\mu \prod_{l \in \mathcal{L}(\mu)} S_l] / \sum_{J_\mu = \pm J/\sqrt{C}} \exp[\beta J_\mu \prod_{l \in \mathcal{L}(\mu)} S_l]$ and a uniform prior, which can be expressed as a bipartite graph, as shown in Fig. 1 (a). In this expression, spins and couplings

are denoted as two different types of nodes and are linked by edges when they are directly connected. This is useful to explicitly represent statistical dependences between estimation variables (spins) and observed data (couplings).

BP is an iterative algorithm defined over the bipartite graph to calculate the spin average for a given set of couplings $\mathbf{J} = (J_\mu)$.^{9, 10)} In the current system, this is performed by passing *beliefs* (or messages) between the two types of nodes via edges at each update as

$$\hat{m}_{\mu l}^{t+1} = \tanh(\beta J_\mu) \prod_{k \in \mathcal{L}(\mu) \setminus l} m_{\mu k}^t, \quad (3)$$

$$m_{\mu l}^t = \tanh \left(\sum_{\nu \in \mathcal{M}(l) \setminus \mu} \tanh^{-1} \hat{m}_{\nu l}^t \right), \quad (4)$$

where beliefs $m_{\mu l}^t$ and $\hat{m}_{\mu l}^t$ are parameters to represent auxiliary distributions at the t th update as $P(S_l | \{J_{\nu \neq \mu}\}) = (1 + m_{\mu l}^t S_l)/2$ and $P(J_\mu | S_l, \{J_{\nu \neq \mu}\}) = \text{Tr}_{S_{k \neq l}} P(J_\mu | \mathbf{S}) P(\mathbf{S} | \{J_{\nu \neq \mu}\}) \propto (1 + \hat{m}_{\mu l}^t S_l)/2$, respectively. $\mathcal{L}(\mu) \setminus l$ denotes a set of spin indices which belong to $\mathcal{L}(\mu)$ excluding l , and similarly for $\mathcal{M}(l) \setminus \mu$. Calculating $\hat{m}_{\mu l}$ iteratively, the estimate of the spin average at the t th update is provided as

$$m_l^t = \tanh \left(\sum_{\mu \in \mathcal{M}(l)} \tanh^{-1} \hat{m}_{\mu l}^t \right). \quad (5)$$

BP is very similar to the transfer matrix method (TMM) and the Bethe approximation,^{16, 17)} which are frequently used in physics. For example, BP provides the exact spin averages by the convergent solution when the bipartite graph is free from cycles (Fig. 1 (b)),⁹⁾ which can be regarded as a generalization of a known property of TMM. In IP research, the process of convergence of BP has been investigated extensively,¹⁸⁾ whereas little has been examined for TMM in the study of SG. This strongly motivates us to examine the dynamical properties of BP in the current system, which we will focus on hereafter.

3. Macroscopic behavior and the replica symmetric solution

Let us first discuss the macroscopic behavior of BP dynamics (3) and (4). Although the current randomly constructed system is not free from cycles, the typical length of the cycles can be shown to grow as $O(\ln N)$ with respect to the system size N as long as C is $O(1)$,¹⁹⁾ which imply that the self-interaction caused by the past states is negligible in the thermodynamic limit. On the other hand, the self-interaction is also expected to be sufficiently small, even if C is large, since the strength of the coupling becomes weak as $O(C^{-1/2})$. This expectation together with eqs. (3) and (4) imply that the time evolution of the macroscopic distributions of beliefs $\pi^t(x) \equiv (1/NC) \sum_{l=1}^N \sum_{\mu \in \mathcal{M}(l)} \delta(x - m_{\mu l}^t)$ and $\hat{\pi}^t(\hat{x}) \equiv (1/NC) \sum_{l=1}^N \sum_{\mu \in \mathcal{M}(l)} \delta(\hat{x} - \hat{m}_{\mu l}^t)$ is likely to be well

captured by recursive equations

$$\hat{\pi}^{t+1}(\hat{x}) = \int \prod_{l=1}^{K-1} dx_l \pi^t(x_l) \left\langle \delta \left(\hat{x} - \tanh(\beta \mathcal{J}) \prod_{l=1}^{K-1} x_l \right) \right\rangle_{\mathcal{J}}, \quad (6)$$

$$\pi^t(x) = \int \prod_{\mu=1}^{C-1} d\hat{x}_\mu \hat{\pi}^t(\hat{x}_\mu) \delta \left(x - \tanh \left(\sum_{\mu=1}^{C-1} \tanh^{-1} \hat{x}_\mu \right) \right), \quad (7)$$

where $\langle \cdots \rangle_{\mathcal{J}}$ represents the average with respect to \mathcal{J} following distribution (2).

The validity of the current argument and its link to the replica symmetric (RS) ansatz in the equilibrium analysis have already been shown for finite C .^{20,21)} Here, we further show that these can be extended to the case of infinite C , even if the AT stability of the RS solution is broken in equilibrium.

When C becomes infinite, dealing with an auxiliary field of finite strength $h_{\mu l}^t \equiv \sum_{\nu \in \mathcal{M}(l) \setminus \mu} \tanh^{-1} \hat{m}_{\nu l}^t \simeq \sum_{\nu \in \mathcal{M}(l) \setminus \mu} \hat{m}_{\nu l}^t$ is more convenient than $\hat{m}_{\nu l}^t$ because $\hat{m}_{\nu l}^t$ becomes infinitesimal. Due to the central limit theorem, the distribution of the auxiliary field $\rho^t(h) \equiv (1/NC) \sum_{l=1}^N \sum_{\mu \in \mathcal{M}(l)} \delta(h - h_{\mu l}^t)$ can be regarded as Gaussian

$$\rho^t(h) = \int \prod_{\mu=1}^{C-1} d\hat{x}_\mu \hat{\pi}^t(\hat{x}_\mu) \delta \left(h - \sum_{\mu=1}^{C-1} \tanh^{-1} \hat{x}_\mu \right) \simeq \frac{1}{\sqrt{2\pi F^t}} \exp \left[-\frac{(h - E^t)^2}{2F^t} \right], \quad (8)$$

where E^t and F^t are the average and the variance, respectively, to parameterize the Gaussian distribution $\rho^t(h)$. The center expression implies $\pi^t(x) = \int dh \rho^t(h) \delta(x - \tanh(h))$. Plugging this into eq. (6) and recursively employing eq. (8), we obtain a compact expression for the update of E^t and F^t as

$$E^{t+1} = \beta J_0 (M^t)^{K-1}, \quad F^{t+1} = \beta^2 J^2 (Q^t)^{K-1}, \quad (9)$$

$$M^t = \int Dz \tanh(\sqrt{F^t} z + E^t), \quad Q^t = \int Dz \tanh^2(\sqrt{F^t} z + E^t), \quad (10)$$

where $Dz \equiv \exp[-z^2/2]/\sqrt{2\pi}$ and M^t and Q^t can be expressed as $M^t \simeq (1/N) \sum_{l=1}^N m_{\mu l}^t \simeq (1/N) \sum_{l=1}^N m_l^t$ and $Q^t \simeq (1/N) \sum_{l=1}^N (m_{\mu l}^t)^2 \simeq (1/N) \sum_{l=1}^N (m_l^t)^2$, respectively, due to the law of large numbers. Equations (9) and (10) serve as alternative expressions of eqs. (6) and (7).

These equations can be regarded as the forward iteration of the saddle point equations to obtain the RS solution in the replica analysis of the multi-spin interaction infinite connectivity SG models¹⁾ and, in particular, of the SK model for $K = 2$.¹¹⁾ In order to confirm the validity of the above argument, we compared the time evolution of the belief update (3) and (4) (**BP**) with that of eqs. (9) and (10) (**RS**) for the SK ($K = 2$) model, which is shown in Fig. 2 (a) and (b). We also compared these values with the trajectory of the naive iteration of the BP's fixed-point condition

$$m_l = \tanh \left(\sum_{\mu \in \mathcal{M}(l)} \beta J_\mu \prod_{k \in \mathcal{L}(\mu) \setminus l} m_k - \sum_{\mu \in \mathcal{M}(l)} (\beta J_\mu)^2 \sum_{j \in \mathcal{L}(\mu) \setminus l} \left(\prod_{k \in \mathcal{L}(\mu) \setminus l, j} m_k \right)^2 (1 - m_j^2) m_l \right), \quad (11)$$

(**TAP**) which can be obtained by inserting $m_{\mu l} \simeq m_l - (1 - m_l^2)\hat{m}_{\mu l}$ at the fixed point of eqs. (3) and (4) $m_{\mu l}^t = m_{\mu l}$, $\hat{m}_{\mu l}^t = \hat{m}_{\mu l}$ and $m_l^t = m_l$. This becomes identical to the famous Thouless-Anderson-Palmer (**TAP**) equation of the SK model, in particular, for $K = 2$.²²⁾

The experiments were performed for $J_0 = 1.5, 0.5$ keeping $J = 1$ and $T = 0.5$, where the AT stability of the RS solution in equilibrium is satisfied for $J_0 = 1.5$, but is broken for $J_0 = 0.5$.¹³⁾ Figures 2 (a) and (b) show that **BP** and **RS** exhibit excellent consistency with respect to the macroscopic variables, irrespective of whether the AT stability is satisfied. This strongly validates the reduction from BP (3) and (4) to the macroscopic dynamics (9) and (10). On the other hand, **TAP** is considerably different from the others. This is natural because naively iterating eq. (11) is just one of the procedures for obtaining a solution and its trajectory in dynamics does not necessarily have any consistency with **BP** or **RS** whereas the BP's fixed point is correctly characterized by the TAP equation (11), which has a certain relationship to the RS solution in equilibrium, as shown in.²³⁾ These figures also imply that the dynamics of BP cannot be traced by a closed set of equations with respect to singly indexed variables m_l^t , even for $C \rightarrow \infty$, whereas the fixed point condition in this limit is provided as coupled equations of m_l (11). The necessity of keeping such extra variables for tracing the trajectory of m_l^t in the BP dynamics is also observed in a similar system of infinite connectivity.⁸⁾

4. Microscopic stability and the AT stability

Although Figures 2 (a) and (b) show that the macroscopic variables rapidly converge to those of the RS solution in BP, this does not imply that BP microscopically converges to a certain solution. In order to probe this microscopic convergence, we numerically examined the squared difference of the spin averages between successive updates $D^t \equiv (1/N) \sum_{l=1}^N (m_l^t - m_l^{t-1})^2$, the time evolution of which is shown in the insets of Figures 2 (a) and (b). These figures illustrate that the (microscopic) local stability of the BP's fixed point can be broken even if the macroscopic behavior appears to converge, which cannot be detected by simply examining the reduced macroscopic dynamics (9) and (10).

In order to characterize such instability, we next turn to the stability analysis of BP updates (3) and (4). Linearizing the updates with respect to the auxiliary field $h_{\mu l} = \tanh^{-1} m_{\mu l}$ around a fixed point solution $m_{\mu l}^t = m_{\mu l}$, we obtain the dynamics of the auxiliary field fluctuation $\delta h_{\mu l}^t$ as

$$\delta h_{\mu l}^{t+1} = \sum_{\nu \in \mathcal{M}(l) \setminus \mu} \frac{\tanh(\beta J_{\nu}) \prod_{k \in \mathcal{L}(\nu) \setminus l} m_{\nu k}}{1 - \left(\tanh(\beta J_{\nu}) \prod_{k \in \mathcal{L}(\nu) \setminus l} m_{\nu k} \right)^2} \times \sum_{j \in \mathcal{L}(\nu) \setminus l} \frac{1 - m_{\nu j}^2}{m_{\nu j}} \times \delta h_{\nu j}^t. \quad (12)$$

Analytically solving this linearized equation for a large graph is generally difficult. However, since the current system is constructed randomly, the self-interaction of $\delta h_{\mu l}^t$ from the past can be considered to be as small as those of the beliefs. This implies that the time evolution of the

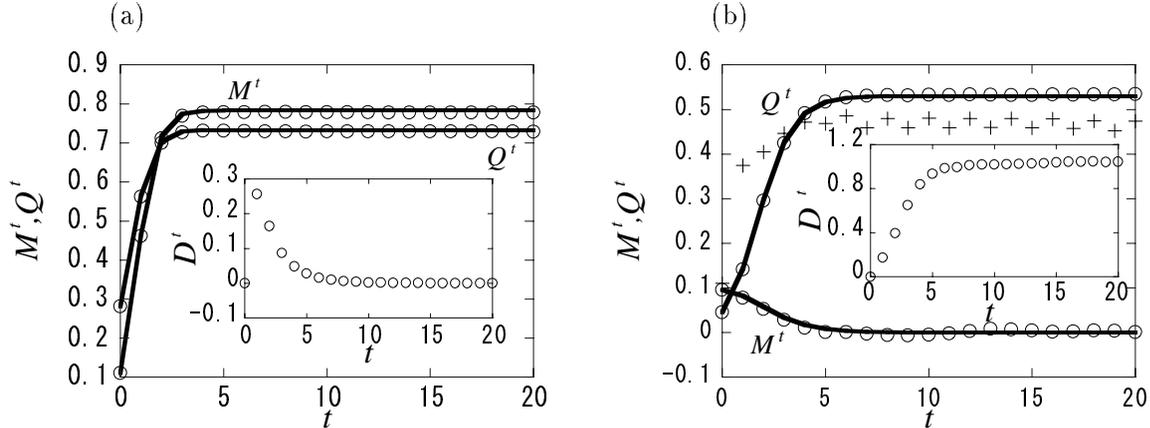


Fig. 2. Time evolution of macroscopic variables $M^t = (1/N) \sum_{i=1}^N m_i^t$ and $Q^t = (1/N) \sum_{i=1}^N (m_i^t)^2$ in the SK model for the BP updates (3) and (4) (**BP**: \circ), the reduced dynamics (9) and (10) (**RS**: lines) and the naive iteration of the TAP equation (11) (**TAP**: $+$) for (a) $J_0 = 1.5$ and (b) $J_0 = 0.5$ keeping $J = 1$ and $T = 0.5$. **TAP** is plotted only for Q^t in the case of $J_0 = 0.5$ in order to save space. Each marker is obtained from 100 experiments for $N = 1000$ systems. The AT stability is satisfied for $J_0 = 1.5$ but broken for $J_0 = 0.5$. Irrespective of the AT stability, the behavior of the macroscopic variables in the BP dynamics can be well captured by the reduced dynamics while the naive iteration of the TAP equation does not exhibit any convergence, even on a macroscopic scale. Insets: Squared deviation of spin averages between the successive updates $D^t = (1/N) \sum_{i=1}^N (m_i^t - m_i^{t-1})^2$ is plotted for the BP dynamics. The deviation vanishes to zero indicating convergence to a fixed point solution for $J_0 = 1.5$, but remains finite, showing the instability of the fixed point for $J_0 = 0.5$. The microscopic trajectory of the BP dynamics for $J_0 = 0.5$ exhibits not simple oscillatory but chaotic behavior, although D^t converges to a constant value. In experiments, such chaotic motion was observed even when control parameters were set much closer to the onset of the AT instability. The origin of this non-trivial behavior may be due to the (continuous) semi-circular eigenvalue distribution of the interaction matrix of the SK model for which many modes of fluctuations simultaneously become unstable at the critical point.

fluctuation distribution $f^t(y) \equiv (1/NC) \sum_{l=1}^N \sum_{\mu \in \mathcal{M}(l)} \delta(y - \delta h_{\mu l}^t)$ can be provided by a functional equation

$$f^{t+1}(y) = \int \prod_{\mu=1}^{C-1} \prod_{l=1}^{K-1} dy_{\mu l} f^t(y_{\mu l}) \times \left\langle \delta \left(y - \sum_{\mu=1}^{C-1} \frac{\tanh(\beta \mathcal{J}_{\mu}) \prod_{k=1}^{K-1} x_{\mu k}}{1 - (\tanh(\beta \mathcal{J}_{\mu}) \prod_{k=1}^{K-1} x_{\mu k})^2} \times \sum_{l=1}^{K-1} \frac{1 - x_{\mu l}^2}{x_{\mu l}} \times y_{\mu l} \right) \right\rangle_{\mathcal{J}_{\mu}, x_{\mu l}}, \quad (13)$$

where $\langle \dots \rangle_{\mathcal{J}_{\mu}, x_{\mu l}}$ denotes the average over \mathcal{J}_{μ} and $x_{\mu l}$ according to eq. (2) and the stationary distribution of $\pi^t(x) = \pi(x)$, respectively, and the stability of the BP's fixed point can be characterized by whether the stationary solution $f^t(y) = f(y) = \delta(y)$ in update (13) is stable. This formulation makes analytical investigation possible to a certain extent.

In order to relate eq. (13) to the existing analysis, let us first investigate the limit $C \rightarrow \infty$, for which much more is known compared to the case of finite C . Due to the central limit theorem, the distribution of the field fluctuation can be assumed as a Gaussian $f^t(y) = (1/\sqrt{2\pi b^t}) \exp[-(y - a^t)^2/(2b^t)]$, where a^t and b^t are the mean and the variance of the distribution, respectively. Plugging this expression into eq. (13) offers update rules with respect to a^t and b^t as $a^{t+1} = (K-1)\beta J_0 M^{K-2}(1-Q)a^t$ and $b^{t+1} = (K-1)(\beta J)^2 Q^{K-2} \int Dz \left(1 - \tanh^2(\sqrt{F}z + E)\right)^2 b^t +$

$\left((K-1)(\beta J)^2 Q^{K-2} \int Dz \left(1 - \tanh^2(\sqrt{F}z + E) \right)^2 - (K-1)^2 (\beta J_0)^2 M^{2K-4} (1-Q)^2 \right) (a^t)^2$, where M, Q, E and F represent the convergent solutions of eqs. (9) and (10). In order to examine the stability of $f(y) = \delta(y)$, we linearize these equations around $a^t = b^t = 0$, which provides the critical condition of the instability with respect to the growth of b^t

$$(K-1)(\beta J)^2 Q^{K-2} \int Dz \left(1 - \tanh^2(\sqrt{F}z + E) \right)^2 = 1, \quad (14)$$

which becomes identical to that of the AT stability for the infinite range multi-spin interaction SG models and, in particular, for the SK model when $K = 2$.¹³⁾ Furthermore, in the case of the SK model ($K = 2$), the critical condition with respect to a^t around the paramagnetic solution $M = Q = 0$ corresponds to the para-ferromagnetic transition. Therefore, the two different phase transitions from the paramagnetic solution can be linked in a unified framework to the dynamic instabilities of BP by eq. (13).

When C is finite, one can numerically perform the stability analysis using eq. (13), the details of which will be reported elsewhere. In addition, analytical investigation also becomes possible for $K = 2$, as described below, since transitions from the paramagnetic solution in this case occur due to the local instability.

For a small β , the paramagnetic solution $\pi(x) = \hat{\pi}(x) = \delta(x)$ ($m_{\mu l} = \hat{m}_{\mu l} = 0$) expresses the correct stable fixed point of the BP dynamics. Inserting this into eq. (13) does not provide a closed set of equations with respect to a finite number of parameters since $f^t(y)$ is no longer a Gaussian. However, assuming $f^t(y) \simeq \delta(y)$, the stability analysis can be reduced to coupled equations with respect to the mean and the variance of $f^t(y)$ as $a^{t+1} = (C-1) \langle \tanh(\beta \mathcal{J}) \rangle_{\mathcal{J}} a^t$ and $b^{t+1} = (C-1) \left(\langle \tanh^2(\beta \mathcal{J}) \rangle_{\mathcal{J}} b^t + \left(\langle \tanh^2(\beta \mathcal{J}) \rangle_{\mathcal{J}} - \langle \tanh(\beta \mathcal{J}) \rangle_{\mathcal{J}}^2 \right) (a^t)^2 \right)$. Linearizing these around $a^t = b^t = 0$ provides the critical conditions with respect to the growth of a^t and b^t as

$$(C-1) \langle \tanh(\beta \mathcal{J}) \rangle_{\mathcal{J}} = 1, \quad (15)$$

and

$$(C-1) \langle \tanh^2(\beta \mathcal{J}) \rangle_{\mathcal{J}} = 1, \quad (16)$$

respectively. These critical conditions around the *simple* paramagnetic solution have already been obtained in similar systems using perturbation methods.²⁴⁻²⁶⁾ However, the current scheme may be superior to previously employed methods because expression (13) is compact and therefore can be easily applied to the stability analysis of the *non-trivial* ferromagnetic solution with the aid of numerical methods (Δ in Fig. 3), even in the case of multi-spin interaction ($K \geq 3$), whereas such extension in the other schemes requires higher-order expansion and becomes highly complicated.

Equations (15) and (16) may correspond to the para-ferromagnetic and the para-SG phase transitions, respectively, since this is the case for $C \rightarrow \infty$. In order to examine this possibility, we

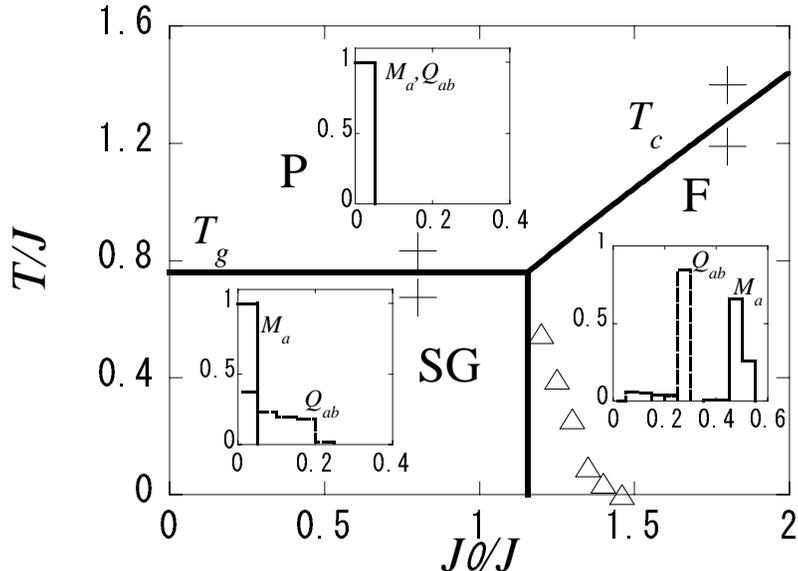


Fig. 3. Phase diagram for $K = 2$ and $C = 4$ suggested by eqs. (15) and (16). P, F and SG denote the paramagnetic, ferromagnetic and spin glass phases, respectively. The boundary between F and SG is based on conjecture. In order to examine the validity of this diagram, 100 Monte Carlo experiments were performed for $N = 2000$ systems at conditions denoted by $+$. For each condition, frequencies of macroscopic magnetizations $M_a = (1/N) \sum_{i=1}^N m_i^a$ and overlaps $Q_{ab} = (1/N) \sum_{i=1}^N m_i^a m_i^b$ ($a > b$) were evaluated, where m_i^a is the average of S_i obtained from 20000 Monte Carlo steps per spin for experiments $a, b = 1, 2, \dots, 100$ (insets). For both of the two conditions in P, all M_a and Q_{ab} fall into the first bin. On the other hand, sharp peaks indicate the order to be the ferromagnetic state in F, and a broad distribution of Q_{ab} indicates the breaking of the replica symmetry in SG. Markers (Δ) in the ferromagnetic phase represent the critical condition with respect to the stability of the BP's fixed point numerically obtained from eq. (13), which may correspond to the AT instability in equilibrium of the non-trivial ferromagnetic solution.

performed numerical experiments for $N = 2000$ and $C = 4$. Although further investigation may be necessary, the data obtained from 100 experiments of 20000 Monte Carlo steps per spin exhibit good consistency with analytical expressions (15) and (16), indicating that the correspondence between the phase transitions in equilibrium and the dynamic instabilities of BP holds for finite C (Fig. 3).

5. Summary

In summary, we have investigated the dynamic behavior of BP when employed in SG models. We have shown that the time evolution of macroscopic variables can be well captured, even in the transient stage and even when the replica symmetry is broken in equilibrium, by recursive updates of auxiliary field distributions, which is identical to the forward iteration of the saddle point equations under the RS ansatz in the replica analysis. We have further shown that the dynamic instability of the BP's fixed point is closely related to the AT instability of the RS solution, which has been numerically supported.

The relationship between the current scheme and an existing AT analysis for finite connectivity SG models²⁷⁾ that generally requires complicated calculation and is not frequently employed in

practice is under investigation. Extending the current framework to the local stability analysis of the replica symmetry breaking solution^{28,29} is a challenge that will be taken up in future studies.

This study was partly supported by Grants-in-Aid from MEXT, Japan, Nos. 13680400, 13780208 and 14084206.

- 1) H. Nishimori, *Statistical Physics of Spin Glasses and Information Processing – An Introduction*, Oxford Press (Oxford), (2001).
- 2) TLH. Watkin, A. Rau and M. Biehl, *Rev. Mod. Phys.* **65**, 499 (1993).
- 3) N. Sourlas, *Nature* **339**, 693 (1989).
- 4) Y. Kabashima, T. Murayama and D. Saad, *Phys. Rev. Lett.* **84**, 1355 (2000).
- 5) N. Nishimori and KYM. Wong, *Phys. Rev. E* **60**, 132 (1999).
- 6) A. Montanari and N. Sourlas, *Eur. Phys. J. B* **18**, 107 (2000).
- 7) T. Tanaka, *Europhys. Lett.* **54**, 540 (2001).
- 8) Y. Kabashima, *cond-mat/0210535* (2002).
- 9) J. Pearl, *Probabilistic Reasoning in Intelligent Systems: Network of Plausible Inference*, Morgan Kaufmann (San Francisco), (1988).
- 10) DJC. MacKay, *IEEE Trans. IT*, **45**, 399 (1999).
- 11) D. Sherrington and S. Kirkpatrick, *Phys. Rev. Lett.* **35**, 1792 (1975).
- 12) KYM. Wong and D. Sherrington, *J. Phys. A* **20** L793 (1987).
- 13) JRL. de Almeida and DJ. Thouless, *J. Phys. A* **11**, 983 (1978).
- 14) Y. Kabashima and D. Saad, *Europhys. Lett.* **45**, 97 (1999).
- 15) T. Murayama and M. Okada, *cond-mat/0207637* (2002).
- 16) HA. Bethe, *Proc. R. Soc. London, Ser A* **151**, 552 (1935).
- 17) Y. Kabashima and D. Saad, *Europhys. Lett.* **44**, 668 (1998).
- 18) A. Yuille, *Neural Computation* **14**, 1691 (2002).
- 19) R. Vicente, D. Saad and Y. Kabashima, *J. Phys. A* **33**, 6527 (2000)
- 20) T. Richardson and R. Urbanke, *IEEE Trans. IT*, **47**, 599 (2001).
- 21) R. Vicente, D. Saad and Y. Kabashima, *Europhys. Lett.* **51**, 698 (2000).
- 22) DJ. Thouless, PW. Anderson and RG. Palmer, *Phil. Mag.* **35**, 593 (1977).
- 23) M. Mezard, G. Parisi and MA. Virasoro, *Spin Glass Theory and Beyond*, World Scientific (Singapore), (1986).
- 24) MW. Klein, LJ. Schowalter and P. Shukla, *Phys. Rev. B* **19**, 1492, (1979).
- 25) DJ. Thouless, *Phys. Rev. Lett.* **56**, 1082, (1986).
- 26) M. Mezard and G. Parisi, *Europhys. Lett.* **3**, 1067, (1987).
- 27) P. Mottishaw and C. De Dominicis, *J. Phys. A* **20**, L375, (1987).
- 28) G. Parisi, *J. Phys. A* **13**, 1101 (1980).
- 29) M. Mezard and G. Parisi, *Eur. Phys. J. B* **20**, 217 (2001).