A CDMA multiuser detection algorithm on the basis of belief propagation

Yoshiyuki Kabashima §
Department of Computational Intelligence and Systems Science, Tokyo Institute of Technology, Yokohama 2268502, Japan

Abstract. An iterative algorithm for the multiuser detection problem that arises in a code division multiple access (CDMA) systems is developed on the basis of Pearl's belief propagation (BP). We show that the BP-based algorithm exhibits nearly optimal performance in a practical time scale by utilising the central limit theorem and self-averaging property appropriately, whereas direct application of BP to the detection problem is computationally difficult and far from practical. We further present close relationships of the proposed algorithm to the Thouless-Anderson-Palmer approach and replica analysis known in spin-glass research.

PACS numbers: 89.70.+c,75.10.Nr,64.60.Cn

Submitted to: *J. Phys. A: Math. Gen.*

§ Email:kaba@dis.titech.ac.jp

1. Introduction

Code division multiple access (CDMA) is a core technology of today's wireless communication employing data transmission between multiple terminals and a single base station [1]. Although this technology is already in use, a strong demand still exists for improvements to respond to the ever-increasing use of mobile communication devices such as cellular phones and wireless LANs.

In the general scenario of a CDMA system, the binary signals of multiple users are modulated by spreading codes assigned to each user, and these modulated sequences are transmitted to a base station. The base station receives a mixture of the modulated sequences and possible noise. After that, a detector at the base station extracts the original binary signals from the received signals using knowledge of the users' spreading codes.

Multiuser detection is a scheme used in the detection stage [2]. By simultaneously detecting multiple user signals following the Bayesian framework, this scheme suppresses mutual interference and can provide optimal detection performance. However, as following the Bayesian approach exactly is computationally hard, the development of approximation algorithms is necessary for practical implementation.

The purpose of this paper is to answer such a demand. More specifically, we develop and analyse a practical multiuser detection algorithm using statistical mechanics. The algorithm is developed on the basis of Pearl's belief propagation (BP) [3] which is defined over graphically expressed statistical models. It is known that BP can be carried out at low computational cost if a given graph is sparse. Unfortunately, the graph for the multiuser detection problem is dense, which implies that the actual use of BP is still highly time-consuming. However, we show that one can derive an efficient algorithm of complexity proportional to the square of the number of users starting from BP and appropriately introducing the central limit theorem and self-averaging property, which are useful notions from statistical mechanics.

This paper is organised as follows. In the next section, the multiuser detection problem is formulated in the Bayesian framework. In section 3, a graphical expression is introduced. For the given graph, we offer a BP algorithm, which turns out to be time-consuming. In section 4, we derive a practical algorithm on the basis of the offered BP. Examining properties of the fixed point, it is shown that the derived BP-based algorithm provides a solution for the Thouless-Anderson-Palmer (TAP) mean field approach [4]. In section 5, we show that the macroscopic trajectory of this algorithm can be captured well by iterative updates of distributions of certain auxiliary fields, the stationary state of which turns out to reproduce the replica symmetric (RS) solution of the equilibrium state [5]. In addition, the microscopic instability condition of the fixed point turns out to coincide with the de Almeida-Thouless (AT) condition [6] for the RS solution. A comparison with other detection schemes is also presented to demonstrate the practical efficacy of the derived algorithm. The final section is devoted to a summary.

2. Multiuser detection

We focus on a K-user direct-sequence binary phase shift-keying (DS/BPSK) CDMA system using random binary spreading codes of the spreading factor N with unit energy over an additive white Gaussian noise (AWGN) channel. For simplicity, we assume that the signal powers are completely controlled to unit energy, but the extension to the case of distributed power is straightforward. In addition, we assume that chip timing as well as symbol timing are perfectly synchronised among users. Under these assumptions [5], the received signal can be expressed as

$$y_\mu = \frac{1}{\sqrt{N}} \sum_{k=1}^{K} s_{\mu k} b_k + \sigma_0 n_\mu, \tag{1}$$

where $\mu \in \{1, 2, \ldots, N\}$ and $k \in \{1, 2, \ldots, K\}$ are indices of samples and users, respectively. $s_{\mu k} \in \{-1, 1\}$ is the spreading code with unit energy independently generated from the identical unbiased distribution $P(s_{\mu k} = +1) = P(s_{\mu k} = -1) = 1/2$. b_k is the bit signal of user k, n_μ is a Gaussian white noise sample with zero mean and unit variance, and σ_0 is the standard deviation of AWGN. Using these normalisations, the signal to noise ratio is defined as $SNR = \beta/(2\sigma_0^2)$ where $\beta = K/N$‖. In the following, we assume a situation where both N and K are large, keeping β finite, which may not be far from practicality since a relatively large spreading factor up to $N = 256$ can be adopted in one of the third generation cellular phone systems "cdma2000" [7].

The goal of multiuser detection is to simultaneously detect bit signals $b_1, b_2, \ldots, b_K$ after receiving the signals $y_1, y_2, \ldots, y_N$. The Bayesian approach offers a useful framework for this. Assuming that the bit signals are independently generated from the unbiased distribution, the posterior distribution from the received signals is provided as

$$P(\boldsymbol{b}|\boldsymbol{y}) = \frac{\prod_{\mu=1}^{N} P(y_\mu|\boldsymbol{b})}{\sum_{\boldsymbol{b}} \prod_{\mu=1}^{N} P(y_\mu|\boldsymbol{b})}, \tag{2}$$

where

$$P(y_\mu|\boldsymbol{b}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}\left(y_\mu - \Delta_\mu\right)^2\right], \tag{3}$$

and $\Delta_\mu \equiv \frac{1}{\sqrt{N}} \sum_{k=1}^{K} s_{\mu k} b_k$. Here, the detector's noise parameter σ is introduced for the case when the true value σ_0 is not known.

Following the Bayesian framework, one can systematically derive the optimal detection strategy from the posterior distribution (2) for various cost functions assuming the posterior is correct [8, 9]. In particular, it can be shown that the bit error rate (BER), which is the cost function that is most frequently argued in CDMA research and we will focus on in this paper, is minimised by the maximiser of the posterior marginal (MPM) detector [5]

$$\hat{b}_k = \underset{b_k \in \{+1,-1\}}{\mathrm{argmax}} \sum_{b_{l \neq k}} P(\boldsymbol{b}|\boldsymbol{y}). \tag{4}$$

3. Graphical representation and belief propagation

Unfortunately, the necessary cost for exactly computing the MPM detector increases exponentially with respect to the number of users K in the current system, which implies that one has to resort to an approximation in practice. The belief propagation (BP), or the sum-product algorithm, is known as one of the most promising approaches

‖ Although *pattern ratio* $\tilde{\alpha} = N/K = \beta^{-1}$ and *inverse temperature* $\tilde{\beta} = \beta\sigma^{-2}$ are usually employed for characterising systems in statistical mechanics, we will follow notations frequently used in CDMA research.

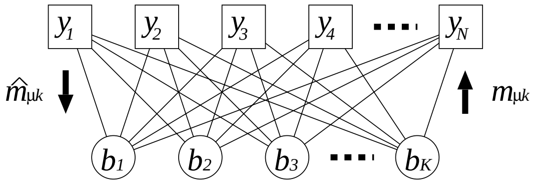

Figure 1. Graphical representation of the CDMA multiuser detection problem. Each edge corresponds to a component of spreading codes $s_{\mu k}$.

to such tasks. Recent research has revealed that this algorithm is closely related to the transfer matrix method and the Bethe approximation [10, 11], which are standard techniques in statistical mechanics, and exhibits excellent performance when a given statistical model is expressed by a sparse graph [12, 13, 14]. However, the properties of BP in dense graphs have not yet been sufficiently studied. In the following, we show that BP can also serve as an excellent approximation algorithm for dense graphs by appropriately utilising the randomness of the underlying model, and apply it to the current CDMA problem.

To introduce this algorithm to the current system, let us graphically denote the received and bit signals by two kinds of nodes, and connect them by an edge when they are related. The conditional probability of y_μ (3) depends on all of $b_1, b_2, \ldots, b_k$, implying that the posterior distribution (2) can be expressed as a (dense) complete bipartite graph, as shown in Figure 1.

The BP can then be defined as an algorithm that passes messages between the two kinds of nodes through edges as

$$P^{t+1}\left(y_\mu|b_k,\{y_{\nu\neq\mu}\}\right) \propto \hat{\alpha}^{t+1}_{\mu k}\sum_{b_{l\neq k}} P(y_\mu|\boldsymbol{b})\prod_{l\neq k} P^t(b_l|\{y_{\nu\neq\mu}\}), \tag{5}$$

$$P^t\left(b_k|\{y_{\nu\neq\mu}\}\right) = \alpha^t_{\mu k}\prod_{\nu\neq\mu} P^t\left(y_\nu|b_k,\{y_{\sigma\neq\nu}\}\right), \tag{6}$$

where $t = 1, 2, \ldots$ is an index for counting the number of updates, and $\hat{\alpha}^t_{\mu k}$ and $\alpha^t_{\mu k}$ are constants for normalisation constraints $\sum_{b_k=\pm 1} P^t\left(y_\mu|b_k,\{y_{\nu\neq\mu}\}\right) = 1$ and $\sum_{b_k=\pm 1} P^t\left(b_k|\{y_{\nu\neq\mu}\}\right) = 1$, respectively [3, 12]. The marginalised posterior at the tth update is given by $P^t\left(y_\mu|b_k,\{y_{\nu\neq\mu}\}\right)$ as $P^t(b_k|\boldsymbol{y}) = \alpha_k\prod_{\mu=1}^N P^t\left(y_\mu|b_k,\{y_{\nu\neq\mu}\}\right)$, where α_k is a normalisation constant.

As b_k is a binary variable, one can parameterise the above functions as $P^t\left(y_\mu|b_k,\{y_{\nu\neq\mu}\}\right) \propto (1+\hat{m}^t_{\mu k}b_k)/2$, $P^t\left(b_k|\{y_{\nu\neq\mu}\}\right) = (1+m^t_{\mu k}b_k)/2$ and $P^t(b_k|\boldsymbol{y}) = (1+m^t_k b_k)/2$ without loss of generality, which simplifies expressions (5) and (6) to

$$\hat{m}^{t+1}_{\mu k} = \frac{\sum_{\boldsymbol{b}} b_k P(y_\mu|\boldsymbol{b})\prod_{l\neq k}\left(\frac{1+m^t_{\mu l}b_l}{2}\right)}{\sum_{\boldsymbol{b}} P(y_\mu|\boldsymbol{b})\prod_{l\neq k}\left(\frac{1+m^t_{\mu l}b_l}{2}\right)}, \tag{7}$$

$$m^t_{\mu k} = \tanh\left(\sum_{\nu\neq\mu}\tanh^{-1}\hat{m}^t_{\nu k}\right). \tag{8}$$

Employing these variables, the approximated posterior average of b_k at the tth update can be computed as $m^t_k = \tanh\left(\sum_{\mu=1}^N \tanh^{-1}\hat{m}^t_{\mu k}\right)$. Because each received signal y_μ is connected to every bit signal b_k, evaluating eq. (7), unfortunately, produces a computational explosion when K is large, so that performing BP exactly is hopeless in the current system.

4. Propagating beliefs in a large complete bipartite graph

However, by appropriately applying the central limit theorem and the self-averaging property, which are useful notions from statistical-mechanical analysis [15], it becomes possible to approximately carry out the belief updates (7) and (8) in a practical time scale [16, 17, 10].

4.1. Gaussian approximation

Since $s_{\mu k} b_k/\sqrt{N}$ is small for large N, we can expand the conditional probability as

$$P(y_\mu|\boldsymbol{b}) \simeq \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(y_\mu - \Delta_{\mu k})^2}{2\sigma^2} + \frac{s_{\mu k}(y_\mu - \Delta_{\mu k})}{\sqrt{N}\sigma^2} b_k\right]$$
$$\simeq \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(y_\mu - \Delta_{\mu k})^2}{2\sigma^2}\right]\left(1 + \frac{s_{\mu k}(y_\mu - \Delta_{\mu k})}{\sqrt{N}\sigma^2} b_k\right), \quad (9)$$

where $\Delta_{\mu k} \equiv \sum_{l\neq k} s_{\mu l} b_l/\sqrt{N}$ in eq. (7). As the spreading codes are generated independently, $s_{\mu l}$ and $m_{\mu l}^t$ are statistically uncorrelated because $m_{\mu l}^t$ is regarded as the posterior average of b_l in a system from which y_μ and $s_{\mu k=1,\ldots,K}$ are excluded. Therefore, the correlation between $s_{\mu l}$ and b_l can be considered as sufficiently weak when b_l is generated from $P^t(b_l|\{y_{\nu\neq\mu}\}) = (1 + m_{\mu l}^t b_l)/2$ for typical realisations of codes and signals. This, in conjunction with the central limit theorem, implies that $\Delta_{\mu k} \equiv \sum_{l\neq k} s_{\mu l} b_l/\sqrt{N}$ obeys a Gaussian distribution $\mathcal{N}\left(\left\langle \Delta_{\mu k}^t \right\rangle_\mu, \beta(1 - Q_{\mu k}^t)\right)$, where $\left\langle \Delta_{\mu k}^t \right\rangle_\mu \equiv \sum_{l\neq k} s_{\mu l} m_{\mu l}^t/\sqrt{N}$ and $Q_{\mu k}^t \equiv (1/K)\sum_{l\neq k}(m_{\mu l}^t)^2$. Furthermore, the self-averaging property implies that the macroscopic variable $Q_{\mu k}^t$ typically converges to a certain value independently of each realisation of codes and signals for large K and N, and is highly likely to be well approximated by $Q^t \equiv (1/K)\sum_{k=1}^K (m_l^t)^2$. Substituting these, one can write eq. (7) as

$$\hat{m}_{\mu k}^{t+1} = A^t \left(\frac{y_\mu \boldsymbol{s}_\mu}{\sqrt{N}} - \beta\left(\mathbf{P}_\mu - \frac{\mathbf{I}}{K}\right)\boldsymbol{m}_\mu^t\right)_k, \quad (10)$$

where $\boldsymbol{s}_\mu \equiv (s_{\mu k})$, $\boldsymbol{m}_\mu^t \equiv (m_{\mu k}^t)$ and $A^t \equiv \left(\sigma^2 + \beta(1 - Q^t)\right)^{-1}$. $\mathbf{P}_\mu \equiv (1/K)(s_{\mu k} s_{\mu l})$ and $\mathbf{I} \equiv (\delta_{kl})$ are the projection and the identity matrices, and $(\cdots)_k$ denotes the kth component of the vector $\cdots$. Eq. (10) can be evaluated in $O(K)$ computations per pair (μk), which implies that a total of $O(NK^2)$ computations are required per update.

4.2. Further reduction of computational cost

Computational cost can be further reduced to $O(K^2)$ when N is large by employing eq. (8). As $\hat{m}_{\mu k}^t$ typically scales as $O(N^{-1/2})$, eq. (8) can be expanded as $m_{\mu k}^t \simeq m_k^t - (\partial m_k/\partial \hat{m}_{\mu k}^t)\hat{m}_{\mu k}^t = m_k^t - (1 - (m_k^t)^2)\,\hat{m}_{\mu k}^t$. Substituting this into eq. (10) provides a recursive equation with respect to $\hat{\boldsymbol{m}}_\mu^t \equiv (\hat{m}_{\mu k}^t)$ as

$$\hat{\boldsymbol{m}}_\mu^{t+1} = A^t \frac{y_\mu \boldsymbol{s}_\mu}{\sqrt{N}} - \beta A^t \left(\mathbf{P}_\mu - \frac{\mathbf{I}}{K}\right)\boldsymbol{m}^t + \beta A^t \mathbf{P}_\mu \mathbf{C}^t \hat{\boldsymbol{m}}_\mu^t, \quad (11)$$

where $\mathbf{C}^t \equiv ((1-(m_k^t)^2)\delta_{kl})$. Using the relationships $\mathbf{P}_\mu \mathbf{C}^t \boldsymbol{s}_\mu = (1-Q^t)\boldsymbol{s}_\mu$ and $\mathbf{P}_\mu \mathbf{C}^t \mathbf{P}_\mu = (1-Q^t)\mathbf{P}_\mu$ and omitting negligible terms, the solution of eq. (11) can be expressed as

$$\hat{\boldsymbol{m}}_\mu^{t+1} = R^t \frac{y_\mu \boldsymbol{s}_\mu}{\sqrt{N}} - \boldsymbol{U}_\mu^t + \frac{1}{K}\beta A^t \boldsymbol{m}^t, \tag{12}$$

where R^t and $\boldsymbol{U}^t$ are obtained from recursive equations

$$R^t = A^t + A^t\beta(1-Q^t)R^{t-1}, \tag{13}$$

$$\boldsymbol{U}_\mu^t = A^t\beta\mathbf{P}_\mu \boldsymbol{m}^t + A^t\beta(1-Q^t)\boldsymbol{U}_\mu^{t-1}. \tag{14}$$

Since $\hat{m}_{\mu k}$ typically scales as $O(N^{-1/2})$, the posterior average can be expressed as $m_k^t = \tanh\left(\sum_{\mu=1}^N \tanh^{-1}\hat{m}_{\mu k}^t\right) \simeq \tanh\left(\sum_{\mu=1}^N \hat{m}_{\mu k}^t\right)$. This implies that the belief updates (5) and (6) are finally summarised as

$$\boldsymbol{h}^{t+1} = R^t\boldsymbol{h}^0 - \boldsymbol{U}^t + A^t\boldsymbol{m}^t, \tag{15}$$

$$\boldsymbol{U}^t = A^t\mathbf{W}\boldsymbol{m}^t + A^t\beta(1-Q^t)\boldsymbol{U}^{t-1}, \tag{16}$$

and eq. (13), where $m_k^t = \tanh(h_k^t)$, $\boldsymbol{h}^0 \equiv (h_k^0) \equiv (\sum_{\mu=1}^N y_\mu s_{\mu k}/\sqrt{N})$, $\boldsymbol{h}^t \equiv (h_k^t)$ and $\mathbf{W} \equiv (W_{kl}) \equiv \left(\sum_{\mu=1}^N s_{\mu k}s_{\mu l}/N\right) = \sum_{\mu=1}^N \beta\mathbf{P}_\mu$. From the posterior average m_k^t, the MPM detector at the tth update is evaluated as $\hat{b}_k^t = \mathrm{sign}(m_k^t)$ where $\mathrm{sign}(x) \equiv 1(x \geq 0),\ \ -1(\text{otherwise})$. Two points are worthy of note. Firstly, the most time-consuming operation in eqs. (13), (15) and (16) is $\mathbf{W}\boldsymbol{m}^t$, which requires a total of $O(K^2)$ computations. This implies that the computational cost for performing the current scheme is similar to that of conventional multistage detection [18]

$$\hat{b}_k^{t+1} = \mathrm{sign}\left(h_k^0 - \sum_{l\neq k} W_{kl}\hat{b}_l^t\right). \tag{17}$$

Secondly, as the fixed point condition, coupled nonlinear equations

$$m_k = \tanh\left[\sigma^{-2}\left(h_k^0 - \sum_{l\neq k} W_{kl}m_l\right) - \frac{\beta(1-Q)m_k}{\sigma^2(\sigma^2+\beta(1-Q))}\right], \tag{18}$$

are obtained from our update scheme, where $Q = (1/K)\sum_{k=1}^k m_l^2$. This is identical to the Thouless-Anderson-Palmer (TAP) equation for the current system developed in statistical mechanics [4, 19]. However, it should be emphasised here that the naive iteration of eq. (18) does not serve as a useful detection algorithm, as finding the fixed point from a practically reasonable initial state turns out to be difficult. This will be illustrated by numerical experiments in the next section.

5. Macroscopic analysis

5.1. Density evolution and the replica symmetric solution

Density evolution is a framework for analysing the dynamical behaviour of BP pursuing a macroscopic distribution of messages [20, 21]. In the current system, this analysis is considerably simplified since the distribution of the gauged field $b_k h_{\mu k}^t$, where b_k is kth user's true binary signal and $h_{\mu k}^t \equiv \tanh^{-1} m_{\mu k}^t = \sum_{\nu\neq\mu}\tanh^{-1}\hat{m}_{\nu k}^t \simeq \sum_{\nu\neq\mu}\hat{m}_{\nu k}^t$, is likely to be well approximated by a Gaussian as a result of the central limit theorem.

Let us assume that $b_k h_{\mu k}^t$ is independently sampled from a Gaussian distribution with an average and variance E^t and F^t, respectively. Notice that we assumed that E^t and F^t are independent of index μ due to the self-averaging property. The self-averaging property also leads us to expect that macroscopic variables $\sum_{k=1}^K b_k m_{\mu k}^t/K \equiv M_\mu^t$ and $\sum_{k=1}^K (b_k m_{\mu k}^t)^2/K = \sum_{k=1}^K (m_{\mu k}^t)^2/K = Q_\mu^t$ are independent of μ as $M_\mu^t \simeq M^t \equiv \sum_{k=1}^K b_k m_k^t/K$ and $Q_\mu^t \simeq Q^t$. These indicate that macroscopic variables M^t and Q^t can be evaluated as

$$M^t = \int Dz \tanh(\sqrt{F^t}z + E^t), \quad Q^t = \int Dz \tanh^2(\sqrt{F^t}z + E^t), \tag{19}$$

where $Dz \equiv dz \exp[-z^2/2]/\sqrt{2\pi}$. Since the MPM detector is given as $\hat{b}_k^t = \mathrm{sign}(m_k^t) \simeq \mathrm{sign}(m_{\mu k}^t) = \mathrm{sign}(h_{\mu k}^t)$, BER is given by $P_b^t = (1/K)\sum_{k=1}^K (1 - \mathrm{sign}(b_k m_k^t))/2 \simeq (1/K)\sum_{k=1}^K (1 - \mathrm{sign}(b_k h_{\mu k}^t))/2 = \int_{-\infty}^{-E^t/\sqrt{F^t}} Dz$.

On the other hand, as $b_k h_{\mu k}^t$ is independently sampled, $\boldsymbol{s}_\mu \cdot \boldsymbol{m}_\mu^t/\sqrt{N}$ in the right hand side of eq. (10) becomes an uncorrelated Gaussian random number with respect to index μ due to the central limit theorem, since $h_{\mu k}^t$ is composed of $\hat{m}_{\nu(\neq\mu)k}^t$, which has a sufficiently small correlation with the randomly generated code $\boldsymbol{s}_\mu$. This, in conjunction with statistical uniformness with respect to indices μ and k, implies that the average and variance at the $t+1$st update are given by $E^{t+1} = \sum_{\mu=1}^N (1/K)\sum_{k=1}^K b_k \hat{m}_{\mu k}^{t+1} = (1/K)\sum_{\mu=1}^N \boldsymbol{b}\cdot\hat{\boldsymbol{m}}_\mu^t$ and $F^{t+1} = \sum_{\mu=1}^N \left[(1/K)\sum_{k=1}^K \left(b_k \hat{m}_{\mu k}^t\right)^2 - (1/K^2)\left(\sum_{k=1}^K b_k \hat{m}_{\mu k}^t\right)^2\right] \simeq (1/K)\sum_{\mu=1}^N \hat{\boldsymbol{m}}_\mu^t \cdot \hat{\boldsymbol{m}}_\mu^t$, respectively. Evaluating these using eqs. (1) and (10), E^{t+1} and F^{t+1} become

$$E^{t+1} = \frac{1}{\sigma^2 + \beta(1-Q^t)}, \quad F^{t+1} = \frac{\beta(1 - 2M^t + Q^t) + \sigma_0^2}{[\sigma^2 + \beta(1-Q^t)]^2}. \tag{20}$$

Eqs. (19) and (20) express the density evolution of the current algorithm.

It should be noted that the expression obtained for the density evolution directly links the proposed algorithm to the replica analysis of the equilibrium state presented in [5] since eqs. (19) and (20) can be regarded as the naive iteration dynamics of the saddle point equations provided by the replica method under the replica symmetric (RS) ansatz [22]. This implies that our algorithm can practically calculate the MPM detector (4) in $O(K^2)$ computations, obtaining the fixed point solution when K is large as replica analysis is likely to give an exact evaluation for $K \to \infty$.

5.2. Microscopic stability and the de Almeida-Thouless condition

Although eqs. (19) and (20) indicate that the distribution of the gauged field $b_k h_{\mu k}^t$ converges to a stationary state provided by the RS solution, this does not necessarily imply the convergence of the microscopic variables, such as $h_{\mu k}^t$ (or $m_{\mu k}^t$) and m_k^t. To examine whether BP dynamics converge to a certain solution microscopically, we perform stability analysis.

For this, the update of $h_{\mu k}^t = \sum_{\nu\neq\mu} \tanh^{-1} \hat{m}_{\nu k}^t \simeq \sum_{\nu\neq\mu} \hat{m}_{\nu k}^t$ is linearised around a fixed point solution $h_{\mu k}$ using eqs. (8) and (10), yielding

$$\delta h_{\mu k}^{t+1} = -\frac{1}{\sigma^2 + \beta(1-Q)} \sum_{\nu\neq\mu}\sum_{j\neq k} \frac{s_{\nu k} s_{\nu j}}{N}(1 - m_{\nu j}^2)\delta h_{\nu j}^t, \tag{21}$$

where Q and $m_{\nu j}$ are the fixed point values of Q^t and $m_{\nu j}^t$, respectively. The following two remarks are useful for analysing the linear stability. Firstly, since spreading codes

$\boldsymbol{s}_\nu$ are generated randomly, the summation in the right hand side gives a Gaussian random number due to the central limit theorem, as the $\delta h_{\nu j}^t$ are uncorrelated. Secondly, as both of the indices μ and k are excluded from the summation in the right hand side, correlations of $\delta h_{\mu k}^{t+1}$ with respect to indices μ, k and t become negligible. This makes it possible to analyse the stability of the fixed point by examining whether the first and second moments of the fluctuations $\delta h_{\mu k}^t$ grow or not with each update (21).

Since the average of $s_{\nu k}$ is zero, the first moment becomes negligible after single application of eq. (21) when K and N are large. However, squaring eq. (21) to examine the time evolution of the second moment yields a non-trivial equation,

$$\begin{aligned}(\delta h_{\mu k}^{t+1})^2 &\simeq \frac{1}{[\sigma^2+\beta(1-Q)]^2}\overline{\left(\sum_{\nu\neq\mu}\sum_{j\neq k}\frac{s_{\nu k}s_{\nu j}}{N}(1-m_{\nu j}^2)\delta h_{\nu j}^t,\right)^2} \\ &= \frac{\beta}{N\left[\sigma^2+\beta(1-Q)\right]^2}\sum_{\nu\neq\mu}\left(\frac{1}{K}\sum_{j\neq k}(1-m_{\nu j}^2)^2(\delta h_{\nu j}^t)^2\right) \\ &\simeq \frac{\beta}{N\left[\sigma^2+\beta(1-Q)\right]^2}\sum_{\nu\neq\mu}\left(\frac{1}{K}\sum_{j\neq k}(1-m_{\nu j}^2)^2\right)\left(\frac{1}{K}\sum_{j\neq k}(\delta h_{\nu j}^t)^2\right)\end{aligned} \quad (22)$$

where $\overline{(\cdots)}$ denotes average over the spreading codes $s_{\nu j}$ and we replaced a sample average of products $\frac{1}{K}\sum_{j\neq k}(1-m_{\nu j}^2)^2(\delta h_{\nu j}^t)^2$ with a product of sample averages $\left(\frac{1}{K}\sum_{j\neq k}(1-m_{\nu j}^2)^2\right)\times\left(\frac{1}{K}\sum_{j\neq k}(\delta h_{\nu j}^t)^2\right)$, which is valid when K is large since $m_{\nu j}$ and $\delta h_{\nu j}^t$ are uncorrelated. Further, it can be expected that due to the self-averaging property, the macroscopic variable $\frac{1}{K}\sum_{j\neq k}(1-m_{\nu j}^2)^2$ can be expressed as

$$\frac{1}{K}\sum_{j\neq k}(1-m_{\nu j}^2)^2 \simeq \int Dz\left(1-\tanh^2(\sqrt{F}z+E)\right)^2 \quad (23)$$

independently of ν, where F and E are the fixed point values of F^t and E^t, and $\frac{1}{K}\sum_{j\neq k}(\delta h_{\nu j}^t)^2$ coincides with the second moment of the fluctuation which does not depend on ν for large K and N. This means that the second moment is enlarged through the belief update and, therefore, the fixed point solution becomes unstable if

$$\frac{\beta}{[\sigma^2+\beta(1-Q)]^2}\times\int Dz\left(1-\tanh^2(\sqrt{F}z+E)\right)^2 > 1. \quad (24)$$

It should be emphasised here that this is nothing but the de Almeida-Thouless (AT) condition for the RS solution of the current system [5, 6]. Similar correspondence between the microscopic stability of BP and the AT condition for the RS solution has been also pointed out in a family of spin-glass models [23]. Although it is known that several non-replica-based approaches to the equilibrium state also provide certain critical conditions which are equivalent to that of AT [15, 24], this, as well as the link between the density evolution and the RS solution, might offer a non-trivial bridge between a dynamical analysis of BP and a fully static theory on the basis of the replica method in spin-glass research.

5.3. Method comparison

To validate the results obtained so far, we performed numerical experiments in systems of $N = 2000$. Figure 2 shows the time evolution of BER in the case of $\beta = K/N = 0.5$

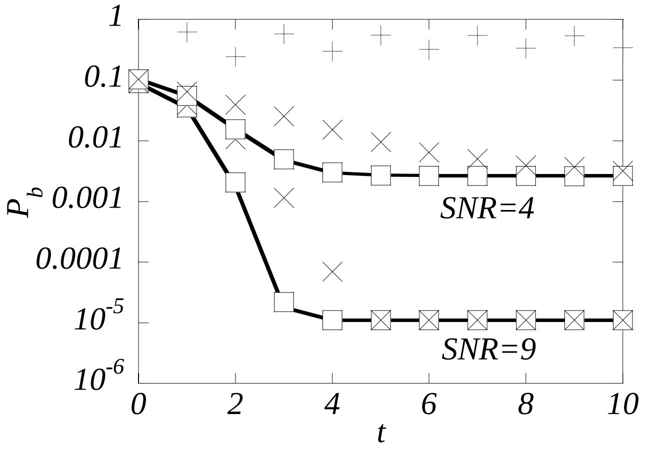

Figure 2. Time evolution of BER for the proposed algorithm (PA:□), conventional multistage detection (MSD:×), naive iteration of the TAP equation (TAP:+) and density evolution (DE: lines) in the case of $N = 2000$, $\beta = 0.5$ and $SNR = 4, 9$ (data for TAP are shown only for $SNR = 9$). Each marker represents the averaged BER at the tth update evaluated from 10000 experiments. In the experiments, the correct noise parameter $\sigma = \sigma_0$ was used and the initial condition for $t = 0$ was set as $m_k^0 = \tanh(h_k^0/\sigma^2)$ for PA and TAP, and $m_k^0 = \mathrm{sign}(h_k^0)$ for MSD. Values of macroscopic variables induced by PA-TAP initial conditions were provided for DE. PA exhibits the fastest convergence and excellent consistency with DE.

obtained from 10000 experiments for the proposed algorithm (eqs. (13), (15) and (16): PA), conventional multistage detection (eq. (17): MSD), iteration of the TAP equation (eq. (18): TAP) and density evolution (eqs. (19) and (20): DE). In the experiments, the noise parameter σ was set to the correct value σ_0.

Firstly, it is clear that PA converges to the fixed point considerably faster than MSD, which is a highly desirable property in practical use. In [21], it is shown that self-reaction from the past states works to disturb the convergence in the conventional multistage detection. On the other hand, such reaction is appropriately cancelled at each update in the proposed algorithm by the last terms of the right hand side of eqs. (15) and (16), which may serve as an intuitive explanation for the superiority of PA. Secondly, PA and DE exhibit excellent consistency as we speculated, which implies that application of the central limit theorem and the self-averaging property in deriving eqs. (19) and (20) is fully validated. Finally, TAP does not serve as a useful detection algorithm. This is because iteration of eq. (18) does not correctly approximate BP and, therefore, subtraction of a diagonal term $\beta(1-Q)m_k/\sigma^2(\sigma^2+\beta(1-Q))$ does not provide appropriate cancellation of the self-reaction in the transient dynamics, whereas it does provide the correct fixed point condition in the stationary state.

Figure 3 shows the influence of noise parameter σ on PA. This indicates that discrepancies between the detector's noise parameter σ and the correct value σ_0 degrade the detection performance. It has been shown that performance of inference is generally optimised by the correct parameter choice, which corresponds to the Nishimori condition [25] known in spin-glass research, when exact evaluation of MPM

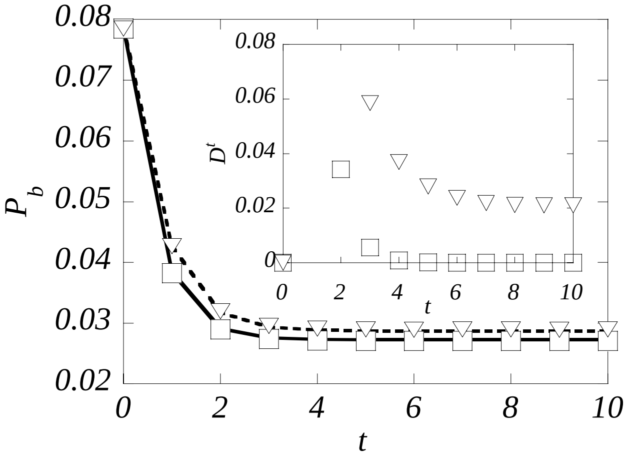

Figure 3. Influence of the noise parameter σ on the proposed algorithm. 10000 experiments were performed to evaluate BER in the case of $N = 2000$, $\beta = 0.25$ and $\sigma_0 = 1$. Markers indicate experimentally obtained BER for $\sigma = \sigma_0 = 1$ (the correct choice: $\square$) and for $\sigma = 1/3$ (a smaller value: ∇), and lines represent trajectories of the density evolution for each case. These data show that performance becomes optimal when the true parameter $\sigma = \sigma_0$ is used and errors in this parameter choice degrade the detection performance. Although BER seems to converge in both cases, this does not necessarily mean that the algorithm is microscopically attracted to a certain solution. To probe such microscopic convergence, the time evolution of the squared difference of the posterior averages between successive times $D^t \equiv (1/K)\sum_{k=1}^{K}(m_k^t - m_k^{t-1})^2$ is plotted in the inset. It is shown that D^t for $\sigma = 1/3$ (∇) does not vanish and converges to a finite value, indicating the microscopic instability of a fixed point while rapid decay to zero is observed for $\sigma = \sigma_0 = 1$ ($\square$). In the numerical data, residual motion for $\sigma = 1/3$ is not simple but seems chaotic. The left hand side of eq. (24) becomes $1.404(> 1)$ and $0.165(< 1)$ for $\sigma = 1/3$ and $\sigma = \sigma_0 = 1$, respectively, which is consistent with the behaviour observed in experiments.

estimator is possible [8, 9]. The current result, in conjunction with the previously mentioned relationship between the density evolution and replica analysis, implies that the optimality of the Nishimori condition also holds for the proposed approximation algorithm.

The inset shows that the microscopic stability of the fixed point can be broken due to condition (24) when σ is sufficiently smaller than σ_0, even though macroscopic trajectory seems to converge. Such microscopic instability does not occur for $\sigma > \sigma_0$, whereas the performance is also degraded due to mismatch of the parameter. This implies that it may not be easy to adjust σ to σ_0 only by monitoring the microscopic behaviour of the algorithm.

In the regime of instability, non-trivial chaotic motion was observed numerically even when control parameters were set close to the critical values. This may be due to highly degenerated eigenvalues of the interaction matrix $\mathbf{W}$ for which many modes of fluctuations would become unstable simultaneously at the critical condition.

6. Summary

In summary, we have developed a novel algorithm for CDMA multiuser detection from belief propagation by appropriately applying the central limit theorem and self-averaging property. The new algorithm exhibits considerably faster convergence than conventional multistage detection without increasing computational cost significantly, and is likely to provide a nearly optimal MPM detector when the spreading factor N is large. We have also clarified the relationship between the obtained algorithm and the existing equilibrium analysis presented in [5] using the density evolution scheme. Finally, we have shown a non-trivial link between microscopic stability of BP dynamics and the AT condition of replica analysis.

After completing this work, another detection algorithm of $O(K^2)$ computational cost was proposed by other authors [26]. The algorithm is obtained by subtracting self-reaction terms from conventional multistage detection and can be carried out at about a half cost by a serial computer. However, the current algorithm still exhibits faster convergence in terms of necessary updates and, therefore, would be preferable when implemented in an electrical circuit.

Extension of the current scheme to cases of non-random code generation and small system size [27] is under way. Besides this, performance evaluation for data encoded by error-correcting codes is a challenging and practically important future work.

Acknowledgements

The author thanks Toshiyuki Tanaka for his useful discussion. Support from Grant-in-Aid, MEXT, Japan, Nos. 13680400, 13780208 and 14084206 is acknowledged.

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