Duality & Equivalence and the Quest for Unification

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- AEF, & Marco Matone, PLB 450 (1999) 34; ... ; IJMPA 15 (2000) 1869.

- G. Bertoldi, AEF & M. Matone, CQG 17 (2000) 3925.

- AEF & Marco Matone, PRL 78 (1997) 163

  related: E.R. Floyd 1982–2008

Quantum Trajectories Workshop, Los–Alamos, 27–30 July 2008

Formulating quantum mechanics from an equivalence principle

- Motivation – quantum gravity
- Legendre duality & 2^{nd} order diff. eq.
- EP $\not\Rightarrow$ CSHJE

  $\Rightarrow$ QSHJE $\hbar \neq 0$

  $\longrightarrow$ Schrödinger eq.
- EP $\longrightarrow$ Tunnel effect

  Energy quantization
- Cocycle condition & Möbius symmetry of QM
- Extensions to HD in E&M metrics
- Further highlights
- Conclusions

Motivation General Relativity: Covariance & Equivalence Principle

$\longrightarrow$ fundamental geometrical principle

Quantum Mechanics: No Such Principle

Axiomatic formulation ... $\mathrm{P} \sim |\Psi|^2$

However Quantum $+$ Gravity Theory

not known

Main effort: quantize GR; quantize space–time: $e.g.$ superstring theory

The main successes of string theory:

1) Viable perturbative approach to quantum gravity

2) Unification of gravity, gauge & matter structures

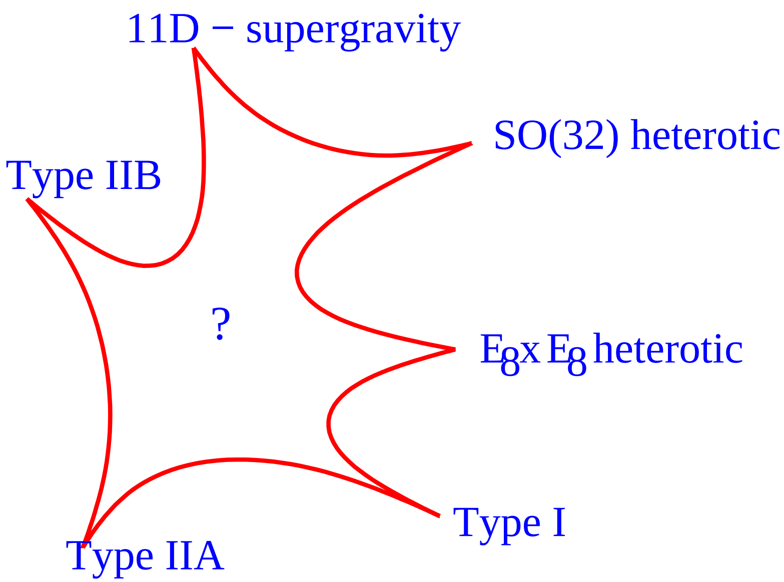
$i.e.$ construction of phenomenologically realistic models

$\longrightarrow$ relevant for experimental observation

State of the art: MSSM from string theory

(Cleaver,AEF, Nanopoulos, PLB 455 (1999) 135)

1995: String duality



Better understanding of string theory

However no rigorous formulation of quantum gravity

Our approach formulate quantum mechanics from
a principle of covariance and equivalence

In retrospect: the fundamental lesson from string dualities
Disconnected classically $->$ connected quantum mechanically

promote to a level of a fundamental principle

Start From:

1D CSHJE : $$\frac{1}{2m}\left(\frac{\partial \mathcal{S}_0}{\partial q}\right)^2 \; + \; V(q) \; - \; E \; = \; 0$$

define $W(q) = V(q) - E$

Equivalence Postulate:

For all $W(q)$ exist $q \; \longrightarrow \; \tilde{q} = \tilde{q}(q)$

such that $W(q) \; \longrightarrow \; \tilde{W}(\tilde{q}) = 0$

$\Longrightarrow$ Modification of the CSHJE

$$\longrightarrow \quad \frac{1}{2m}\left(\frac{\partial \mathcal{S}_0}{\partial q}\right)^2 \; + \; V(q) \; - \; E \; + \; Q(q) \; = \; 0$$

will show $Q(q) \; \longrightarrow$ quantum potential

$\longrightarrow$ Schrödinger equation

Generalization of HJ theory

$$H(q,p) \longrightarrow \tilde{H}(\tilde{q},\tilde{p}) = 0$$

$$\dot{q} = \frac{\partial H}{\partial p}\ ,\ \dot{p} = -\frac{\partial H}{\partial q} \qquad \longrightarrow \qquad \dot{\tilde{q}} = \frac{\partial \tilde{H}}{\partial \tilde{p}}\ ,\ \dot{\tilde{p}} = -\frac{\partial \tilde{H}}{\partial \tilde{q}}$$

$$H(q,p) \longrightarrow \tilde{H}(\tilde{q},\tilde{p}) = H(q,p) + \frac{\partial S}{\partial t} = 0 \Rightarrow \text{CSHJE}$$

The solution is the Classical Hamilton–Jacobi Equation

Formulate a similar question

Consider the transformations on

$$(\ q\ ,\ S_0(q)\ ,\ p = \frac{\partial S_0}{\partial q}) \longrightarrow (\ \tilde{q}\ ,\ \tilde{S}_0(\tilde{q})\ ,\ \tilde{p} = \frac{\partial \tilde{S}_0}{\partial \tilde{q}}\ )$$

Such that

$$W(q) \longrightarrow \tilde{W}(\tilde{q}) = 0$$

exist for all $W(q)$

$\Longrightarrow$ QHJE

$\longrightarrow$ Schrödinger equation

Legendre duality & 2^{nd} order diff. eqs.

intimate connection between $p-q$ duality & the equivalence postulate

Hamilton's Eqs. $\qquad \dot{q} = \frac{\partial H}{\partial p} \quad , \quad \dot{p} = -\frac{\partial H}{\partial q}$

invariant under $\quad p \longrightarrow -q$

breaks down once $V(q)$ is specified $e.g.$ $\quad \frac{1}{2m}p^2 + V(q) - E = 0$

Aim $\qquad$ Formulation with manifest $p-q$ duality

recall $\quad p = \frac{\partial S}{\partial q} \qquad$ define $\quad q = \frac{\partial T}{\partial p}$

$$S = p\frac{\partial T}{\partial p} - T \quad , \quad T = q\frac{\partial S}{\partial q} - S$$

Stationary Case: $\quad S(q,t) = S_0(q) - Et \quad , \quad T(p,t) = T_0(p) + Et$

Compute dS and dT $\quad \Rightarrow \quad \frac{\partial S}{\partial t} = -\frac{\partial T}{\partial t}.$

$$S_0 = p\frac{\partial T_0}{\partial p} - T_0 \quad , \quad T_0 = q\frac{\partial S_0}{\partial q} - S_0$$

Invariant under Möbius transformations:

$$q \longrightarrow q^v = \frac{Aq+B}{Cq+D},$$

$$p \longrightarrow p_v = \rho^{-1}(Cq+D)^2 p \ , \quad \rho = AD - BC$$

$$T_0 \longrightarrow T_0^v(p^v) = T_0(p) + \rho^{-1}(ACq^2 + 2BCq + BD)p.$$

Transformations: $q \to q^v = v(q)$ defined by $S_0^v(q^v) = S_0(q)$

(S_0 scalar function under v)

Associate a 2^{nd}order diff. eq. with the Legendre transformation:

$$\left(\frac{\partial^2}{\partial S_0^2} + U(S_0)\right)\begin{pmatrix} q\sqrt{p} \\ \sqrt{p}\end{pmatrix} = 0$$

where $U(S_0) = \frac{1}{2}\{q, S_0\}$ $\qquad \frac{q'''}{q'} - \frac{3}{2}\left(\frac{q''}{q'}\right)^2$

We can derive this eq. in several ways

$$
\begin{aligned}
p &= \frac{\partial S_0}{\partial q} & \Rightarrow \qquad p\frac{\partial q}{\partial S_0} &= 1 \\
\frac{\partial}{\partial S_0} & & \Rightarrow \qquad \frac{\partial p}{\partial S_0}\frac{\partial q}{\partial S_0} + p\frac{\partial^2 q}{\partial S_0^2} &= 0
\end{aligned}
$$

rewritten as

$$
\frac{\partial^2_{S_0} q\sqrt{p}}{q\sqrt{p}} = \frac{\partial^2_{S_0}\sqrt{p}}{\sqrt{p}} = -U(S_0)
$$

or

$$
\frac{\partial^2}{\partial S_0^2} \quad : \qquad S_0(q) = \frac{1}{2}\sqrt{p}\frac{\partial T_0}{\partial\sqrt{p}} - T_0
$$

$$
\Longrightarrow \quad \left(\frac{\partial^2}{\partial S_0^2} + U(S_0)\right)\begin{pmatrix} q\sqrt{p} \\ \sqrt{p} \end{pmatrix} = 0
$$

manifest $p \leftrightarrow q$ – $S_0 \leftrightarrow T_0$ duality with

$$p = \frac{\partial S_0}{\partial q} \qquad\qquad q = \frac{\partial T_0}{\partial p}$$

$$S_0 = p\frac{\partial T_0}{\partial p} - T_0 \qquad\qquad T_0 = q\frac{\partial S_0}{\partial q} - S_0$$

$$\left(\frac{\partial^2}{\partial S_0^2} + U(S_0)\right)\begin{pmatrix} q\sqrt{p} \\ \sqrt{p} \end{pmatrix} = 0 \qquad\qquad \left(\frac{\partial^2}{\partial T_0^2} + \mathcal{V}(T_0)\right)\begin{pmatrix} p\sqrt{q} \\ \sqrt{q} \end{pmatrix} = 0$$

Involutive Legendre transformation $\leftrightarrow$ duality

Self–dual states

States with

$$pq \;=\; \gamma \;=\; const$$

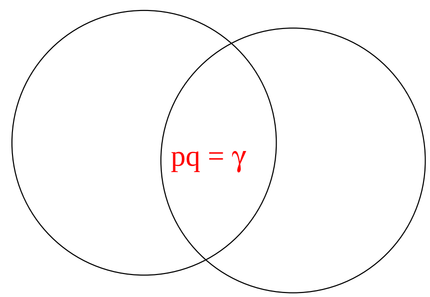
are simultaneous solutions of the two pictures with

$$S_0 \;=\; -\; T_0 \;+\; const$$

$$S_0(q) = \gamma \ln \gamma_q q \qquad\qquad T_0(p) = \gamma \ln \gamma_p p$$

$$S_0 \;+\; T_0 \;=\; pq \;=\; \gamma$$

where $\gamma_q \gamma_p \gamma = \mathrm{e}$



self–dual states

will show that $W^{sd} = W^0 = 0 \qquad \gamma^{sd} = \dfrac{\pm\hbar}{2i}$

Schwarzian derivative $\{h(x), x(y)\} = \left(\frac{\partial y}{\partial x}\right)^2 \{h(x), y\} - \left(\frac{\partial y}{\partial x}\right)^2 \{x, y\}.$

if $x = \frac{Ay+B}{Cy+D}$ then $\{x, y\} = 0$

$$U(S_0) \;=\; \frac{1}{2}\{q, S_0\} \;=\; \frac{1}{2}\{\frac{Aq+B}{Cq+D}, S_0\}$$

Invariant under Möbius transformations

For general $q^v = v(q)$ $\Rightarrow$ $U(S_0^v(q^v)) \neq U(S_0(q))$

But $S_0^v(q^v) = S_0(q)$ ($\Rightarrow$ p transforms as $\frac{\partial}{\partial q}$ under $v(q)$)

By construction $\left(\frac{\partial^2}{\partial S_0^{v^2}} + U(S_0^v)\right)\phi^v(S_0^v) = 0$ is covariant

$\Rightarrow$ connect different potentials by coordinate transformations

$\Rightarrow$ Equivalence Postulate: $W(q) = V(q) - E$ connected

In particular $W \to W^0 \equiv 0$

The equivalence postulate is not consistent with classical mechanics

Consider the CSHJE
$$\frac{1}{2m}\left(\frac{\partial S^v(q^v)}{\partial q^v}\right)^2 \;+\; W^v(q^v) \;=\; 0$$

from $S_0^v(q^v) = S_0(q)$ we have
$$\frac{1}{2m}\left(\frac{\partial q^v}{\partial q}\right)^{-2}\left(\frac{\partial S(q)}{\partial q}\right)^2 + W^v(q^v) = 0$$

Covariance implies
$$W(q) \to W^v(q^v) = \left(\frac{\partial q^v}{\partial q}\right)^{-2} W(q)$$

$\Longrightarrow$ $W(q)$ should transform as a quadratic differential

Starting from the state $W^0(q^0) = 0$ we have
$$W^0(q^0) \to W^v(q^v) = \left(\frac{\partial q^v}{\partial q}\right)^{-2} W^0(q^0) = 0$$

W^0 is a fixed point in the space of all possible W,

and the equivalence postulate cannot be implemented

$\Longrightarrow$ Modify the CHJE

Requirements

1) Covariance

2) all $W \in \mathcal{H}$ are connected by $q^a \longrightarrow q^b$

3) $lim \longrightarrow$ CHJE

Modification

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 + W(q) + Q(q) = 0$$

consistency

$$W^v(q^v) + Q^v(q^v) = \left(\frac{\partial q^v}{\partial q}\right)^{-2} (W(q) + Q(q)) \, .$$

$W+Q \in \mathcal{Q} \longrightarrow$ space of functions transforming as quadratic differentials

and $W \notin \mathcal{Q}$ & $Q \notin \mathcal{Q}$

The most general transformations

$$W^a(q^a) = \left(\frac{\partial q^v}{\partial q}\right)^{-2} W(q) + (q^a; q^v),$$

$$Q^a(q^a) = \left(\frac{\partial q^v}{\partial q}\right)^{-2} Q(q) - (q^a; q^v),$$

with $q^a \longrightarrow q^v = v(q^a) \longleftrightarrow S_0^v(q^v) = S_0^a(q^a)$

For $a = 0$ we have $W^0(q^0) = 0$

and $W^v(q^v) = (q^0; q^v)$

All W–states are identified with the inhomogeneous term !

consider

$$W^b(q^b)$$

$$W^a(q^a) \qquad\qquad W^c(q^c)$$

We obtain the cocycle condition

$$(q^a; q^c) = \left(\frac{\partial q^b}{\partial q^c}\right)^2 \left[(q^a; q^b) - (q^c; q^b)\right],$$

$\Rightarrow$ Theorem $(q^a; q^c)$ invariant under Möbius transformations $\gamma(q^a)$

Theorem $\quad (q^a; q^c) \sim \{q^a; q^c\}$

The cocycle condition generalizes to higher dimensions with respect to the Euclidean and Minkowski metrics and is invariant under D–dimensional Möbius or conformal transformations

A natural way to represent a quadratic differential

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2+\frac{\beta^2}{4m}\left(\{f,q\}-\{g,q\}\right)=0$$

Difference of Schwarzian derivatives is a quadratic differential

Identity
$$\left(\frac{\partial S_0}{\partial q}\right)^2=\frac{\beta^2}{2}\left(\{e^{\frac{i2S_0}{\beta}},q\}-\{S_0,q\}\right)$$

With $\quad f = e^{\frac{i2S_0}{\beta}} \qquad q = S_0$ up to Möbius transformations

Make the following identifications

$$W(q) = -\frac{\beta^2}{4m}\{e^{\frac{i2S_0}{\beta}},q\}=V(q)-E$$
$$Q(q) = \frac{\beta^2}{4m}\{S_0,q\}$$

which follows from the limit $\displaystyle\lim_{\beta\to 0}\frac{\beta^2}{4m}\left(\{f,q\}-\{g,q\}\right)=V(q)-E$

The Modified Hamilton–Jacobi Equation becomes

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 + V(q) - E + \frac{\beta^2}{4m}\{S_0, q\} = 0$$

in the limit $\beta \longrightarrow 0$ we get back the CSHJE and $S_0^{cl} = \lim_{\beta\longrightarrow 0} S_0$

The QSHJE gives back the Schrödinger Eq.

We identified $V(q) - E = -\frac{\beta^2}{4m}\{\mathrm{e}^{\frac{i2S_0}{\beta}}, q\}$

$V(q) - E$ is a potential of the 2^{nd}–order diff. Eq.

$$\left(-\frac{\beta^2}{2m}\frac{\partial^2}{\partial q^2} + V(q) - E\right)\Psi(q) = 0$$

The general solution $\Psi(q) = \frac{1}{\sqrt{S_0'}}\left(A\mathrm{e}^{+\frac{i}{\hbar}S_0} + B\mathrm{e}^{-\frac{i}{\hbar}S_0}\right)$

and $\mathrm{e}^{+\frac{i2S_0}{\hbar}} = \mathrm{e}^{i\alpha}\,\frac{w + i\bar{\ell}}{w - i\ell}$ $\qquad w = \frac{\psi_1}{\psi_2}$

$\ell = \ell_1 + i\,\ell_2 \qquad \ell_1 \neq 0 \qquad \alpha \in R \qquad \psi_1\,,\ \psi_2 \in R$

The equivalence transformation

$$W(q) \;=\; V(q) \;-\; E \quad\longrightarrow\quad \tilde{W}(\tilde{q}) \;=\; 0$$

always exists

We have to find $\quad q \;\rightarrow\; \tilde{q} \qquad$ take $\quad \tilde{q} \;=\; \frac{\psi_1}{\psi_2}$

then $$\left(-\frac{\beta^2}{2m}\,\frac{\partial^2}{\partial q^2} \;+\; V(q) \;-\; E\right)\Psi(q) \;=\; 0 \quad\rightarrow\quad \frac{\partial^2}{\partial \tilde{q}^2}\tilde{\psi}(\tilde{q}) \;=\; 0$$

where $$\tilde{\psi}(\tilde{q}) \;=\; \left(\frac{dq}{d\tilde{q}}\right)^{-\frac{1}{2}} \psi(q)$$

There is a subtle point if $S_0 = Aq + B \quad \longleftrightarrow \quad$ Free Particle

In this case $S_0 \ - \ T_0$ duality breaks down

This point correspond to $V(q) \ - \ E \ = 0$

$S_0 \ = \ const \ \Rightarrow$ A fixed point in $\mathcal{Q}$

$$\text{Classically} \ : \quad \frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 \qquad = \ 0 \quad \Longrightarrow \quad S_0 \ = \ const$$

$$\text{Q.M.} \ : \quad \frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 \ + \ \frac{\hbar^2}{4m}\{S_0, q\} \ = \ 0$$

which has the solutions

$$S_0 = \pm \frac{i}{2} \beta \ln q$$

that also follows from

$$V(q) - E \sim \{\mathrm{e}^{\frac{i2S_0}{\beta}}, q\} = 0$$

so for $V(q) - E = 0$

we set
$$\begin{aligned} S_0 &= const \quad \notin \mathcal{H} \\ S_0 &= \pm \frac{\beta}{2} \ln q \in \mathcal{H} \end{aligned}$$

For the case $W(q) = -E \neq 0$ instead of $S_0 = \sqrt{2mE}q$

we have the solutions $S_0 = -\frac{i}{2} \ln \left(\frac{A\mathrm{e}^{\frac{2i}{\hbar}\sqrt{2meq}} + B}{C\mathrm{e}^{\frac{2i}{\hbar}\sqrt{2meq}} + D} \right)$

we have that $S_0 \neq Aq + B$ always !!!

Allows: Equivalence postulate for all states

Full $S_0 - T_0$ duality

with the self–dual point $\gamma = \pm \frac{i}{2}\hbar$

The trivializing coordinate is solution of

$$\tilde{q} \;=\; \frac{\psi_1}{\psi_2} \;=\; \mathrm{e}^{\frac{2iS_0}{\beta}}$$

$$-\frac{\beta^2}{4m}\{\mathrm{e}^{\frac{2iS_0}{\beta}}, q\} \;=\; V(q) \;-\; E$$

or

$$\{\tilde{q}, q\} \;+\; \frac{4m}{\hbar^2}(V(q) \;-\; E) \;=\; 0$$

Tunneling:

The fundamental equation in our approach

$$\frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2 + V(q) - E + \frac{\beta^2}{4m}\{S_0, q\} = 0$$

Quantum Hamilton–Jacobi equation

which is equivalent to

$$W(q) = V(q) - E = -\frac{\hbar^2}{4m}\left\{ e^{\frac{2iS_0}{\beta}}, q \right\}$$

whose solution is

$$e^{\frac{2iS_0}{\beta}} = \frac{\psi_1}{\psi_2}$$

where ψ_1 and ψ_2 are the linearly independent solutions

of the corresponding Schrödinger equation

$\Rightarrow$ Schrödinger equation $\rightarrow$ Linearization of the QHJE

More Generally:

Due to Mobiüs invariance of $\left\{ e^{\frac{2iS_0}{\beta}}, q \right\}$

We have $e^{\frac{2iS_0}{\beta}} = \frac{Aw+B}{Cw+D}$ with $w = \frac{\psi_1}{\psi_2}$ and $AD-BC \neq 0$

We can set

$$e^{\frac{2iS_0}{\beta}} = e^{i\alpha}\frac{w+i\bar{\ell}}{w-i\ell}$$

$\alpha \in R$ $\ell = \ell_1 + i\,\ell_2$ $\ell_1 \neq 0$ and we get

$$p = \frac{\partial S_0}{\partial q} = \frac{\hbar(\ell+\bar{\ell})}{2|\psi_2 - i\ell\psi_1|^2} \sim \frac{1}{|\phi|^2}$$

Classically $$p = \pm\sqrt{2m(E-V)}$$

$$\Rightarrow \quad p \notin R \text{ for } q \in \Omega$$

where

$$\Omega = \{q \in R \mid V(q) - E > 0\}$$

Q.M. $$p = \pm\sqrt{2m(E - V - Q)}$$

we found that

$$p = \frac{\epsilon}{|\phi|^2} \qquad \epsilon = \pm 1$$

$$\Rightarrow p \in R \qquad \forall\, q \in R$$

$\Rightarrow$ no forbidden regions

except for the infinitely deep potential well

Energy quantization:

Probability: $\Longrightarrow$ (Ψ, Ψ') continuous ; $\Psi \in L^2(R)$

$\Longrightarrow$ quantization, bound states

What are the conditions on the trivializing transformations?

$$q^0 \;=\; w \;=\; \frac{\psi_1}{\psi_2} \;=\; \frac{\psi^D}{\psi}$$

we have $$\{w, q\} \;=\; -\frac{4m}{\hbar^2}(V(q) - E)$$

$\Rightarrow \quad w \neq const \;;\; w \in C^2(R) \;\; and \;\; w''$ differentiable on R

In addition from the properties of $\{,\} \;\rightarrow\; \{w, q^{-1}\} \;=\; q^4\{w, q\}$

$\Rightarrow \quad w \neq const \;;\; w \in C^2(\hat{R}) \;\; and \;\; w''$ differentiable on $\hat{R}$

where $\hat{R} \;=\; R \cup \{\infty\}$

$$\Rightarrow \;\; w(-\infty) \;=\; \begin{cases} w(+\infty), & for \;\; w(-\infty) \;\neq\; \pm\infty, \\ -w(+\infty), & for \;\; w(-\infty) \;=\; \pm\infty \end{cases}$$

$$\Longrightarrow$$

Equivalence postulate $\Longrightarrow$ continuity of (ψ^D, ψ) and $(\psi^{D'}, \psi')$

Theorem:

$$if \qquad V(q) - E = \begin{cases} P_-^2 > 0 & for \quad q < q_- \\ P_+^2 > 0 & for \quad q > q_+ \end{cases}$$

then the ratio $w = \psi^D/\psi$ *is continuous on* $\hat{R}$ *iff*

the corresponding Schrödinger equation admits

an $L^2(R)$ *solution*

1) $\psi \in L^2(R) \Rightarrow \psi^D \notin L^2(R)$

$$w = \frac{A\psi_D + B\psi}{C\psi_D + D\psi} \Rightarrow \lim_{q \to \pm\infty} = \frac{A}{C}$$

$$\Rightarrow w(-\infty) = w(+\infty)$$

2) ...

Potential Well:

$$V(q) = \begin{cases} 0 & |q| \leq L \\ V_0 & |q| > L \end{cases}$$

$$k = \frac{\sqrt{2mE}}{\hbar} \qquad K = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

$$\begin{array}{lll} |q| \leq L & \Psi_1^1 = \cos kq & \Psi_2^1 = \sin kq \\ q > L & \Psi_1^2 = \mathrm{e}^{-Kq} & \Psi_2^2 = \mathrm{e}^{Kq} \end{array}$$

The solution at $q < L$ is fixed by parity

four possibilities $(1,1)\ (2,1)\ /\ (1,2)\ (2,2)$

take $(1,1)$: $\Psi,\ \Psi'$ continuous $\Rightarrow$ $k \tan kL = K$

Use $\Psi^D = c\,\Psi \int_{q_0}^{q} dx\, \Psi^{-2}(x) + d\,\Psi$

$$\Rightarrow \quad w = \frac{1}{[k \sin(2kL)]} \begin{cases} \cos(2kL) - \mathrm{e}^{-2K(q+L)} & q < -L \\ \sin(2kL)\tan(kq) & |q| \leq L \\ \mathrm{e}^{2K(q-L)} - \cos(2kL) & q > L \end{cases}$$

$$\lim_{q\to\pm} \frac{\psi^D}{\psi} = \pm\infty \qquad \Longrightarrow \qquad E_n(k\ \tan kL\ =\ K)$$ are admissible solutions

take $(1,2)$: $\quad \Psi,\ \Psi'$ continuous $\quad \Rightarrow \quad k\ \tan(kl)\ =\ -K$

$$\Rightarrow \quad w = \frac{1}{[k\sin(2kL)]} \begin{cases} \cos(2kL) - \mathrm{e}^{-2K(q+L)} & q\ <\ -L \\ \sin(2kL)\tan(kq) & |q|\ \leq\ L \\ \mathrm{e}^{2K(q-L)} - \cos(2kL) & q\ >\ L \end{cases}$$

$$\lim_{q\to\pm} \frac{\psi^D}{\psi}\ =\ \mp\frac{1}{k}\cot(2kL) \qquad \Longrightarrow \qquad w(-\infty)\ \neq\ w(+\infty)$$

($k^{-1}(\cot 2kL) = 0$ is not compatible with $k\tan(kL) = -K$)

$\Rightarrow \qquad E_n(k\ \tan kL\ =\ -K)$ are not admissible solutions

$\Longrightarrow$

We can understand the

$$\Psi \in L^2(R)$$

condition

$+$ existence of bound states

with quantized energy eigenvalues

as a consequence of the

postulated equivalence principle.

Generalizations:

Cocycle condition $\longrightarrow$ D-dimensional E&M metrics

invariant under D–dimensional

Mobiüs (conformal) trans.

Quadratic differential:

$$\alpha^2(\nabla S_0)\cdot(\nabla S_0) = \frac{\Delta(Re^{\alpha S_0})}{Re^{\alpha S_0}} - \frac{\Delta R}{R} - \alpha\left(2\frac{\nabla R\cdot\nabla S_0}{R} + \Delta S_0\right),$$

or

$$\alpha^2(\partial S)\cdot(\partial S) = \frac{\partial^2(Re^{\alpha S})}{Re^{\alpha S}} - \frac{\partial^2 R}{R} - \alpha\left(2\frac{\partial R\cdot\partial S}{R} + \partial^2 S\right),$$

or

$$\alpha^2(\partial S - eA)\cdot(\partial S - eA) = \frac{D^2(Re^{\alpha S})}{Re^{\alpha S}} - \frac{\partial^2 R}{R} - \frac{\alpha}{R^2}\partial\cdot\left(R^2(\partial S - eA)\right),$$

$$D^\mu \;=\; \partial^\mu - \alpha e A^\mu$$

Further highlights

1. Planck length from the equivalence postulate
   (AEF, Marco Matone, Phys. Lett. **B445** (1999) 77)

$$\frac{\partial^2\Psi}{\partial q^2} = 0 \quad \Rightarrow \qquad \psi_1 = 1 \quad ; \quad \psi_2 = q$$

$$\Rightarrow \qquad \text{duality implies a length scale}$$

2. Equivalence classes of the wave–function

$$\Psi_E(\delta) = \frac{1}{\sqrt{S_0'(\delta)}}\left(A\mathrm{e}^{-\frac{i}{\hbar}S_0(\delta)} + B\mathrm{e}^{\frac{i}{\hbar}S_0(\delta)}\right)$$

$$\delta = \{\alpha, \ell\} \to \delta'\{\alpha', \delta'\}$$

$$\Psi_E\{\delta'\} = \Psi_E\{\delta\}$$

but $p = \frac{\partial S_0}{\partial q}$ changes

however $\neq$ Bohm!!!

1. $p \neq m\dot{q}$
2. Bohm: $\Psi = R\mathrm{e}^{\frac{i}{\hbar}S_0}$

   Ψ for bound states $\Rightarrow S_0 = 0$

   $\Rightarrow$ classical limit in Bohm's approach ?!

   we have $S_0 \neq Aq + B$ Always

conclusions :

The equivalence postulate $\Rightarrow$ QHJE $\Rightarrow$ $\hbar \neq 0$

p-q duality & Equivalence Postulate $\Rightarrow$ $S_0 \neq A\,q + B$ Always

Equivalence Postulate $\Rightarrow$

$$\Psi(q) = \frac{1}{\sqrt{S_0'}}\left(A\mathrm{e}^{+\frac{i}{\hbar}S_0} + B\mathrm{e}^{-\frac{i}{\hbar}S_0}\right)$$

Tunnel effect

Energy quantization & $\Psi \in L^2(R)$

Generalizes to Higher Dimensions in E & M metrics

Outlook

T–duality as phase–space duality in compact space

Generalize to curved space; generalised geometry

Develop EP approach to quantum gravity

plus fundamental issues in QG