# Stochastic oscillations of power systems

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# Lecture Outline

- Motivation

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- Value of solar energy in US: $\approx \$1\text{B/year}$
- Cost of power interruptions : $\gtrapprox \$100\text{B/year}$

Enormous opportunity in reliability improvement

# Power system security

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- Opportunities
  - Fast sensing - PMU
  - Fast non-local actuation - FACTS/HVDC
  - Algorithms and computing power

# Stochastic dynamics

**Why stochastic models ?**

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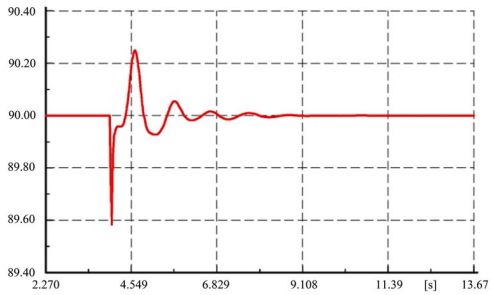
- Steady states are never reached



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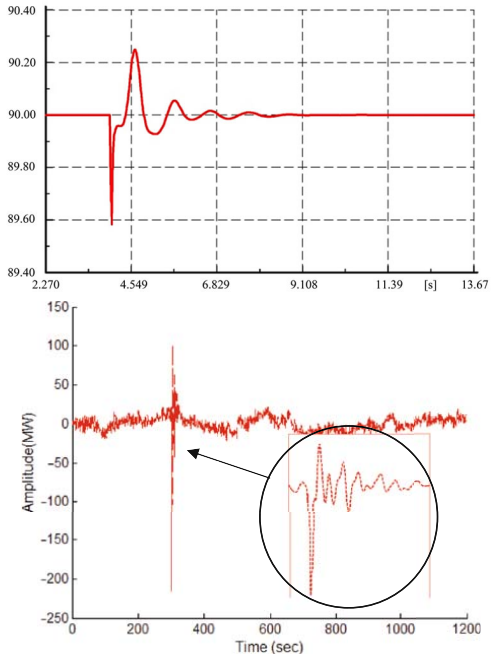
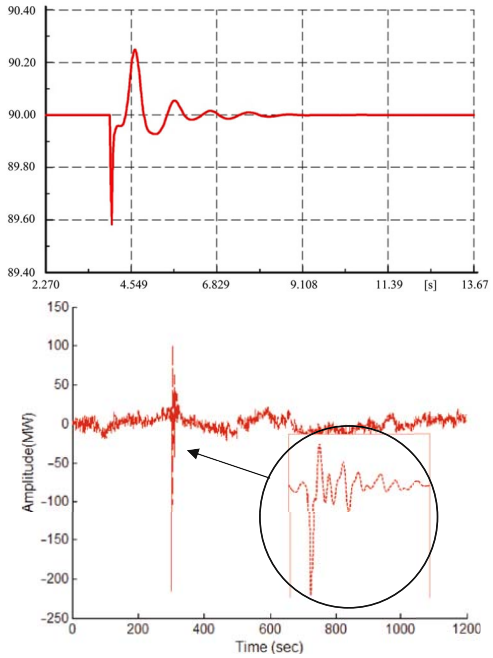



From: Messina, Inter-Area Oscillations in Power Systems

# Stochastic dynamics

## Why stochastic models ?

- Steady states are never reached
- System is inherently stochastic
- More problems in the future:
  - Intermittent Renewables
  - Operation closer to stability limits
  - Microgrids - more variability
  - More reliance on real-time sensing

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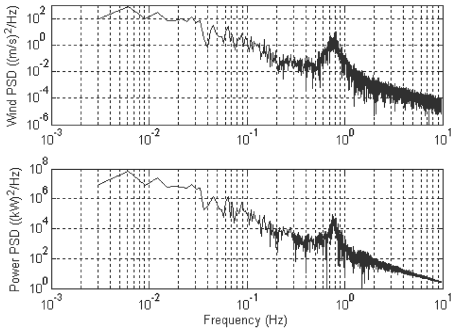
- Generators
  - Small-scale turbulence on blades
  - Cloud covering panels



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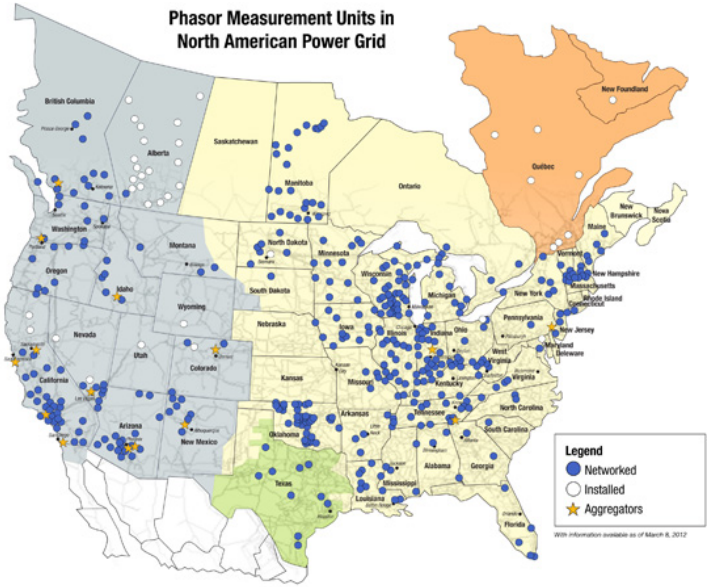
  - Real-time sensing and communication
  - Sub-cycle pulse sampling
  - Thousands units installed in US



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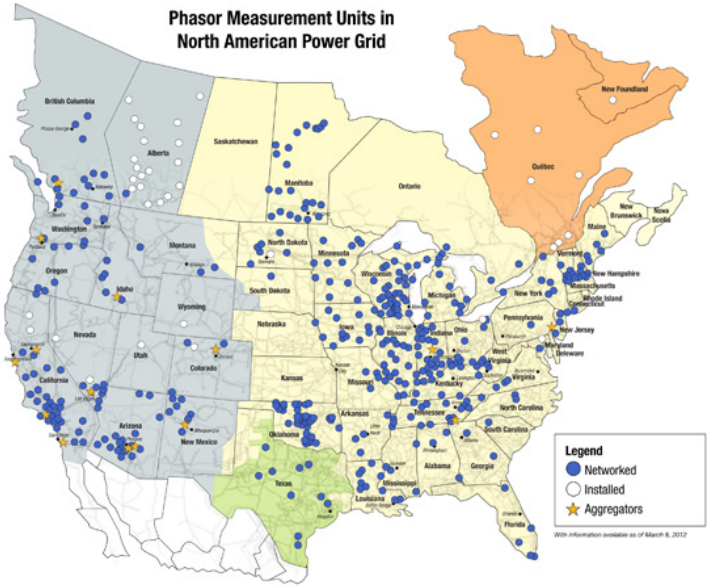
- Wide Area Monitoring/Control:
  - Security analysis, fault detection
  - Frequency/voltage control
  - Improved state estimation
  - Real-time dynamic equivalencing



# Disclaimers

- Long History of stochastic dynamics of power systems
- No general textbooks written
- Less research effort than there should be
- Works *I* found interesting/important written by:
  - Chris De Marco - first papers on stochastic models in 80s
  - Chika Nwanpka - some very advanced methods applied in 90s
  - Trudnowski, Messina - stochastic models for signal analysis
  - Paul Hines - security indicators, critical slowing down
  - Federico Milano - simulations of stochastic power systems

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- Stochastic process: ensemble of realizations of a given function $x(t)$
- Formally, think of $x(t, \sigma)$ with $\sigma \in S$ - realization of unknown parameters
- Most important operations on stochastic model:
  - *Sampling*: Generate realizations of the process: $x(t, \sigma_1), x(t, \sigma_2) \ldots$
  - *Averaging*: Integrate over possible realizations: $\sum_k F\{x(t, \sigma_k)\}$

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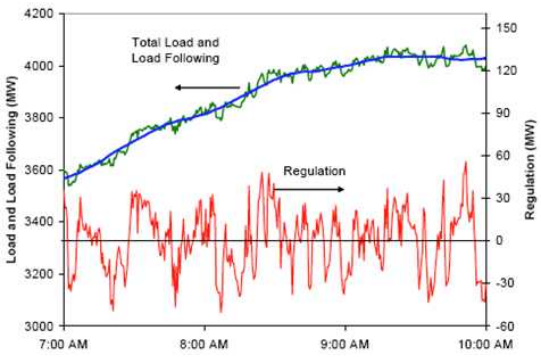

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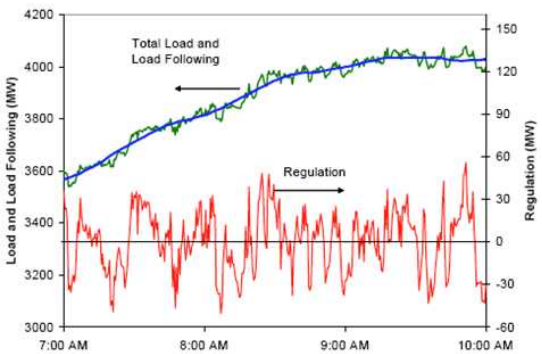


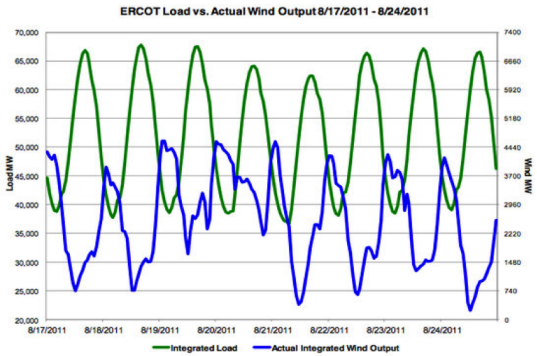

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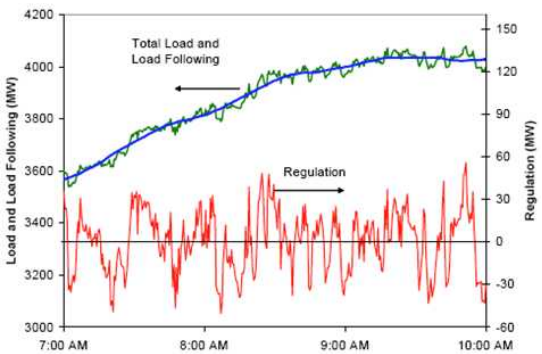


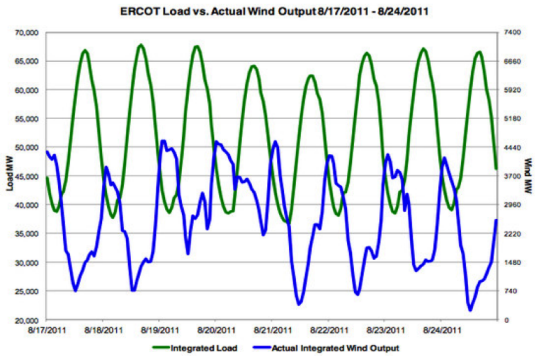


Load/Renewable variations: nearly stationary on seconds/minutes, cyclic on hours/months

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- For stationary processes

$$\mu_x(t) = \text{const}, \quad R_{xx}(t_1, t_2) = R_{xx}(t_1 - t_2)$$

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$$R_{xx}(t_1, t_2) = \mathbf{E}[x(t_1)x(t_2)] =$$

$$\int \frac{d\phi}{2\pi} \sin(\omega t_1 + \phi) \sin(\omega t_2 + \phi) =$$

$$\frac{1}{2} \cos \omega(t_1 - t_2)$$

# Wiener-Khinchin Theorem

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- Instantaneous power:

$$\mathbf{E}[x^2(t)] = \int \frac{d\omega}{2\pi} S_{xx}(j\omega)$$

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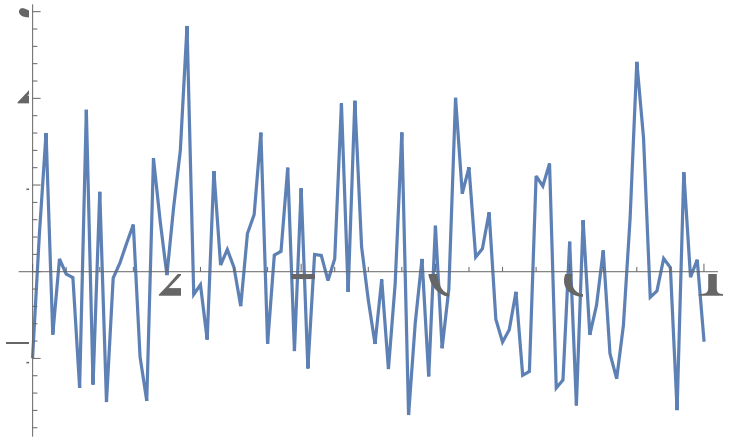
- Mean and correlation fully define Gaussian process
- Gaussian processes are invariant under linear transformations

# Process 1: White Noise

- White noise $\xi(t)$: building block of all random processes.
- No correlation in time: $R_{xx}(t) = \delta(t)$.



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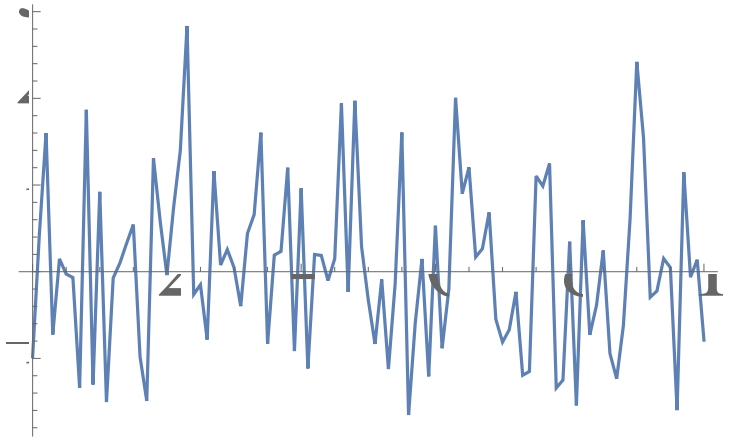

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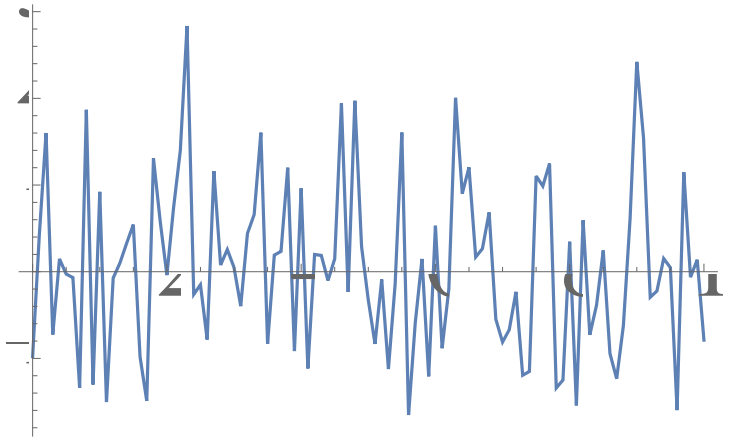

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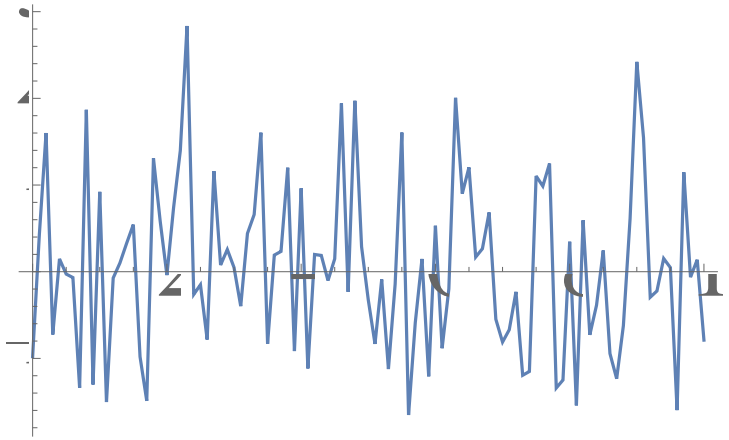
- Power system context:
  - Short surges of power
  - Starting of motors
  - Vibrations ?



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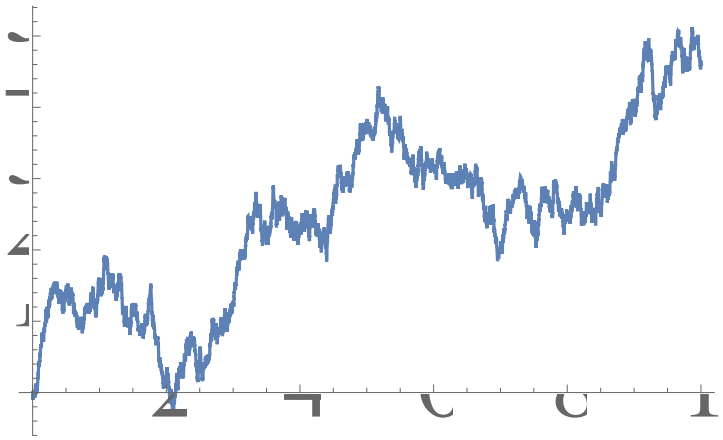

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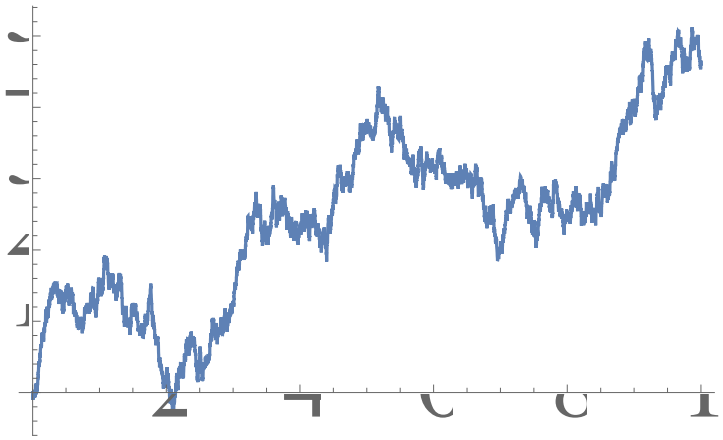

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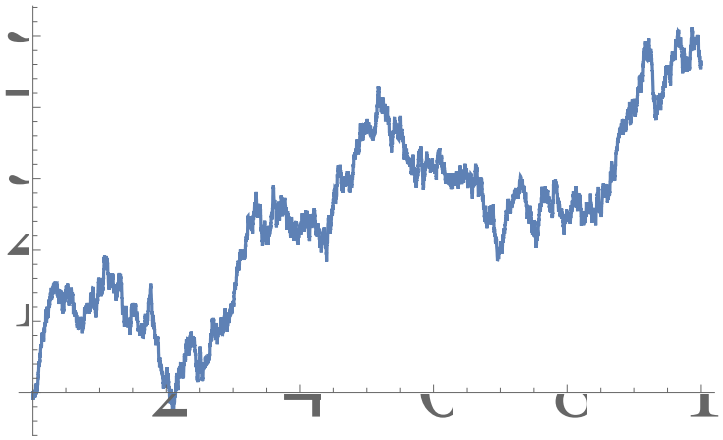
- Power systems context:
  - Gradual build-up of power
  - Small on/off switching events
  - Better model for short times



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- Like Brownian walk on small time-scales $\omega\tau \gg 1$
- Like white noise on long time-scales $\omega\tau \ll 1$

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- Nonlinear SDEs are more difficult
- Many asymptotic techniques developed in physics

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- Two-bus and multi-bus models
- Goal: characterize ambient fluctuations

# Two bus model

- Classical example of inter-area oscillations:

$$M\ddot{\delta}_1 + D\dot{\delta}_1 = \frac{P_1}{\omega_0} - \frac{P_{max}}{\omega_0}\sin(\delta_1 - \delta_2) \tag{1}$$

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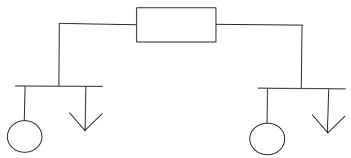

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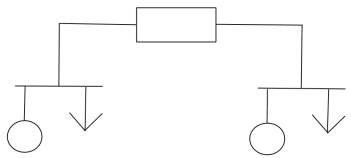

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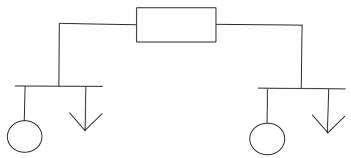

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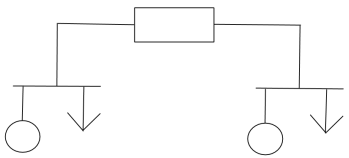
- $D$ - droop, primary frequency control
- $P_{1,2}$ - total production/generation levels



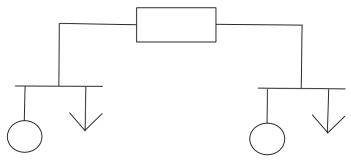
# Two bus model

- Classical example of inter-area oscillations:

$$M\ddot{\delta}_1 + D\dot{\delta}_1 = \frac{P_1}{\omega_0} - \frac{P_{max}}{\omega_0}\sin(\delta_1 - \delta_2) \qquad (1)$$

$$M\ddot{\delta}_2 + D\dot{\delta}_2 = \frac{P_2}{\omega_0} + \frac{P_{max}}{\omega_0}\sin(\delta_1 - \delta_2) \qquad (2)$$

- $M$ - aggregate turbine inertia
- $D$ - droop, primary frequency control
- $P_{1,2}$ - total production/generation levels
- $P_{max}$ - power transfer capacity of the line



# Normal mode variables

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- Independent dynamics

# Linearized dynamics

- For state $\delta_{12} = \delta_{12}^* + \delta$ close to equilibrium point:

$$\sin \delta_{12}^* = (P_1^0 - P_2^0)/P_{max}$$

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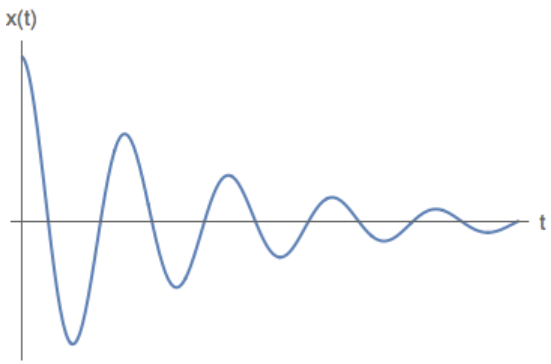
- Power fluctuations $p(t) = P_1 - P_1^0 - P_2 + P_2^0$
- Underdamped system:
  - Natural frequency $\Omega = \sqrt{\frac{2P_{max} \cos \delta_{12}^*}{\omega_0 M}}$
  - Damping rate: $\gamma = \frac{D}{2M}$



# Variability of angle

- For any random process $p(t)$ power spectral density of the angle difference is given by

$$S_{\delta\delta}(j\omega) = \frac{S_{pp}(j\omega)}{\omega_0^2 M^2} \frac{1}{(\omega^2 - \Omega^2)^2 + \gamma^2\Omega^2} = S_{pp}(j\omega)T(j\omega)$$

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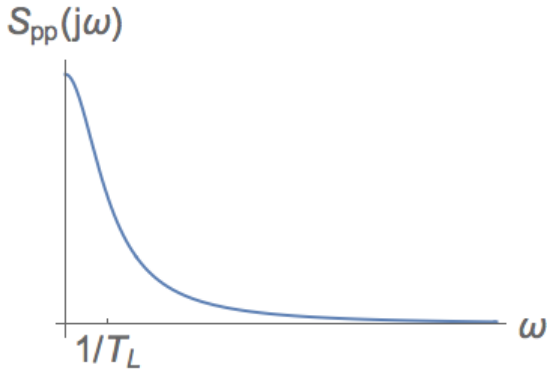
- Variance of angle fluctuations

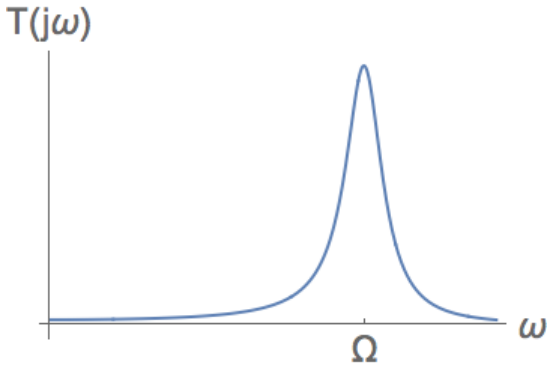
$$\mathbf{E}[\delta^2] = R_{\delta\delta}(0) = \int \frac{d\omega}{2\pi} S_{pp}(j\omega)T(j\omega)$$

# State fluctuations

- Two dominant contributions:

$$R_{\delta\delta}(0) = \int \frac{d\omega}{2\pi} S_{pp}(j\omega)T(j\omega) \approx \frac{R_{pp}(0)}{4P_{max}^2 \cos^2 \delta_{12}^*} + \frac{S_{pp}(j\Omega)}{2\omega_0 D P_{max} \cos \delta_{12}^*}$$



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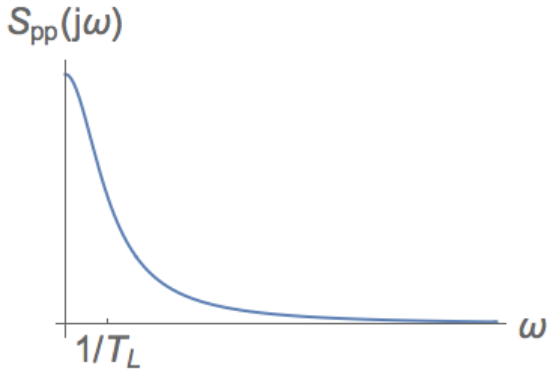
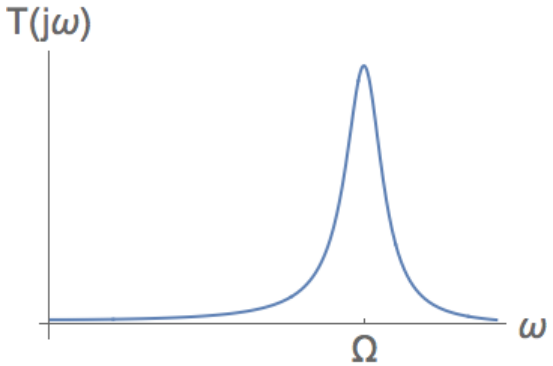



  - Slow drift of operating point



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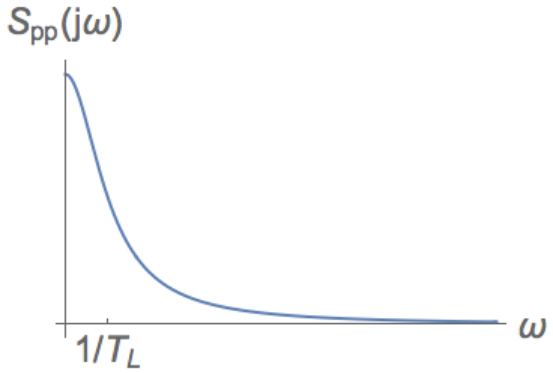


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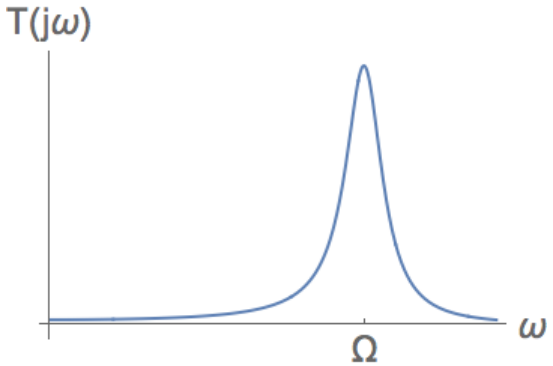

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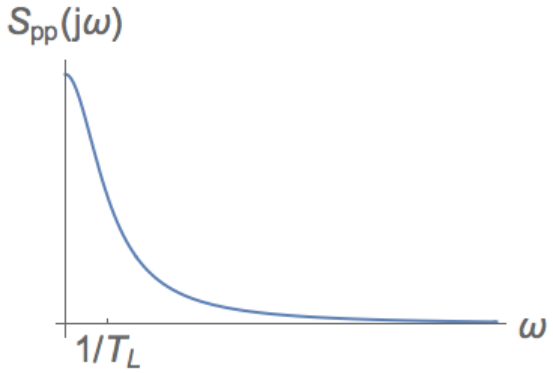


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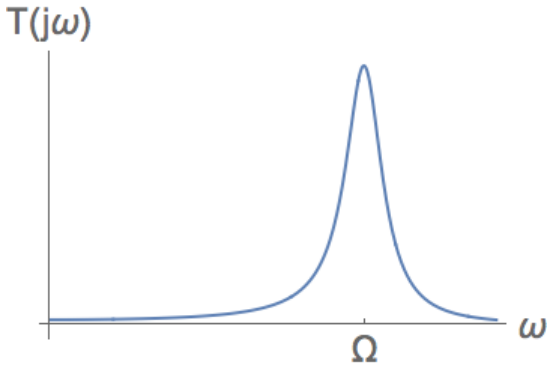


- Multi-scale decomposition of $\delta$:

$$\delta(t) = \hat{\delta}(\epsilon t) + \tilde{\delta}(t), \quad \epsilon = \frac{1}{\Omega T_p}$$

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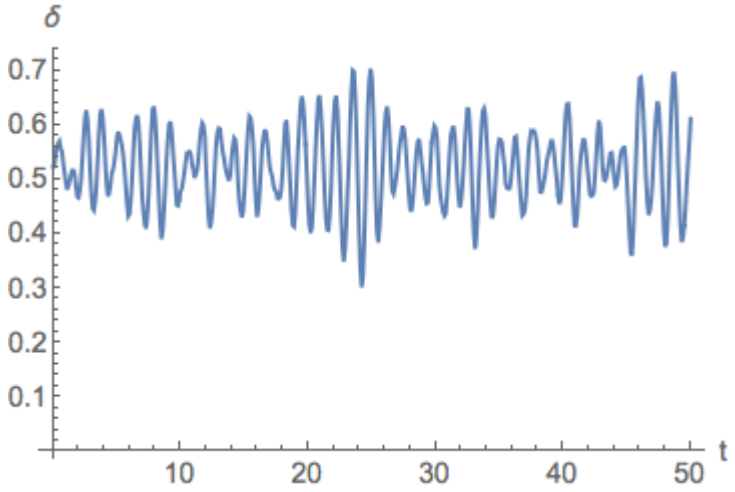
- Natural Security indicator (see works of P. Hines)
- Opportunity for better state estimation:
  - Deterministic algorithm become inaccurate
  - Fluctuations immediately give distance to collapse: $\cos\delta_{12}^*$
  - Extraction of directions possible but more difficult

# Simulations of 2 bus system



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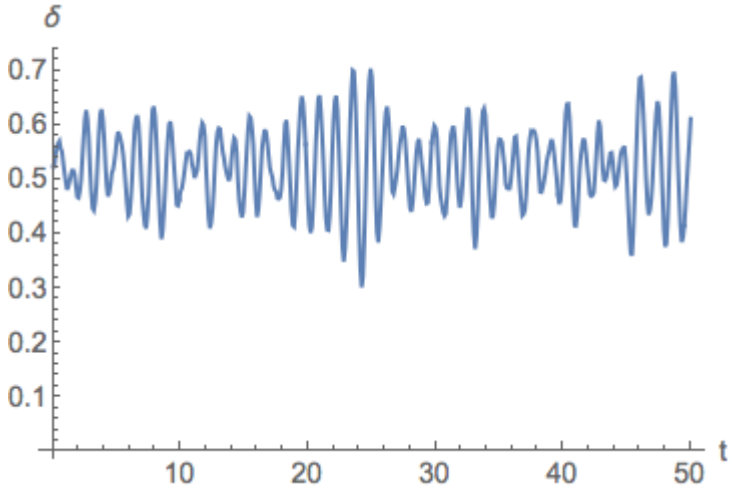


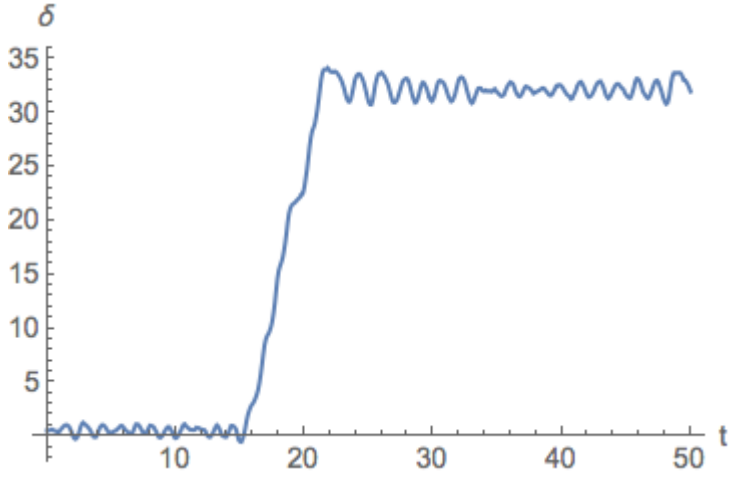

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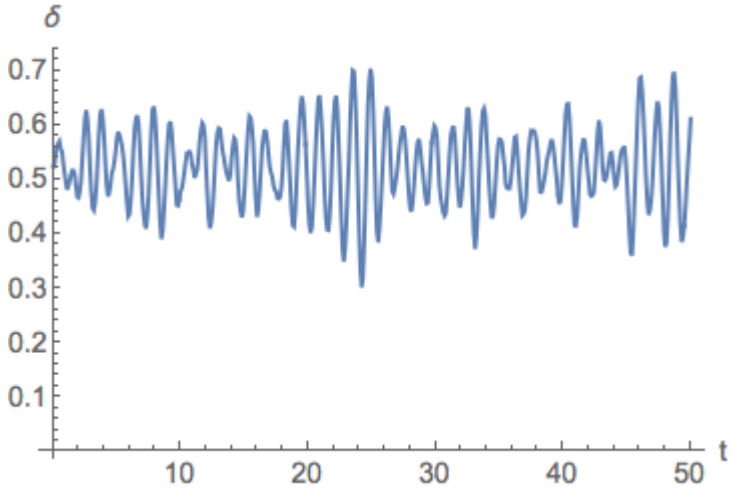


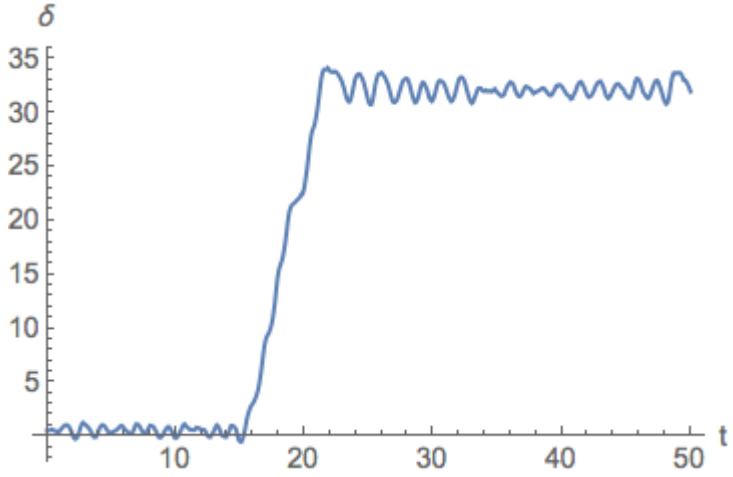


- Swing instability can be triggered by random fluctuations.
- How do we estimate the probability ?

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- Natural security indicator - probability of collapse
- Extensions to multi-dimensional case possible (Podolsky, KT '13)

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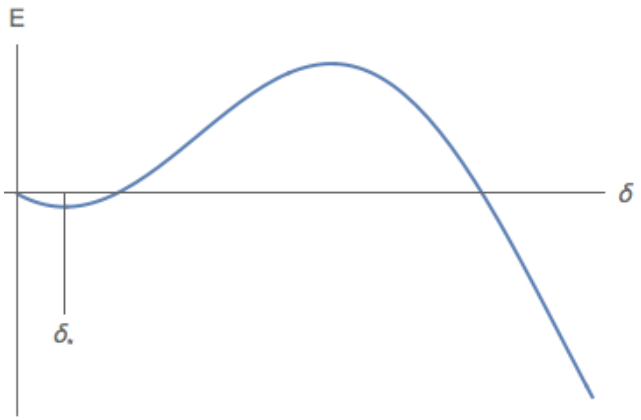

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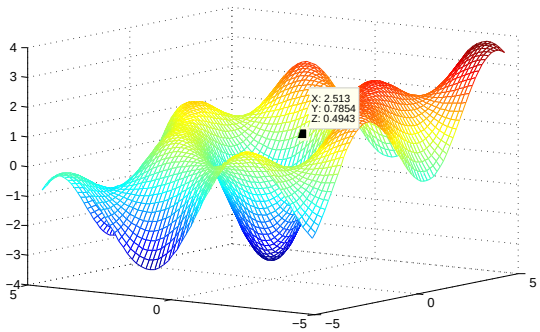


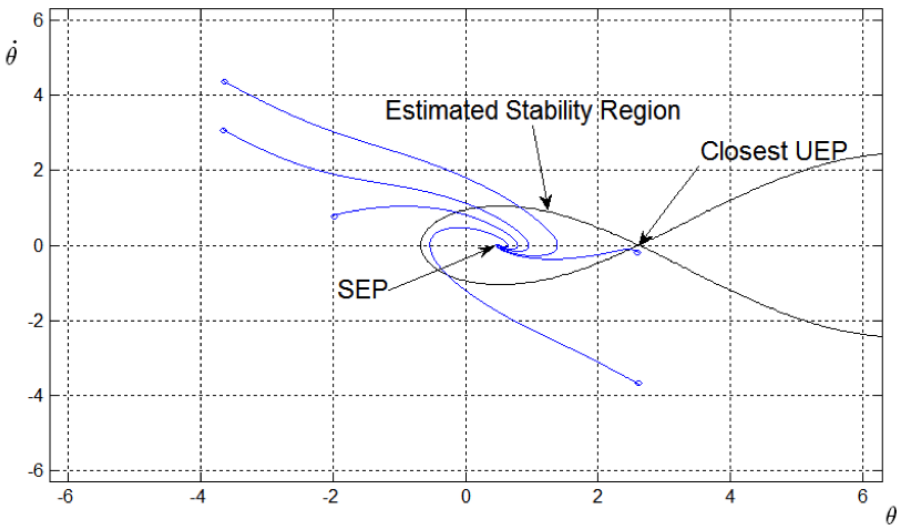

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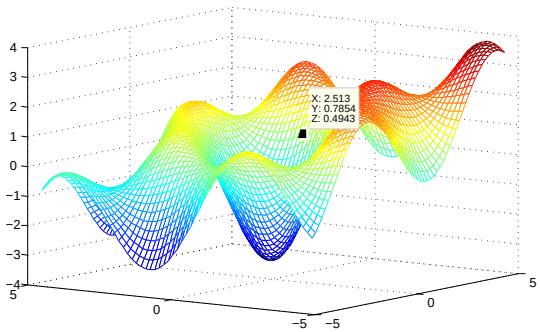


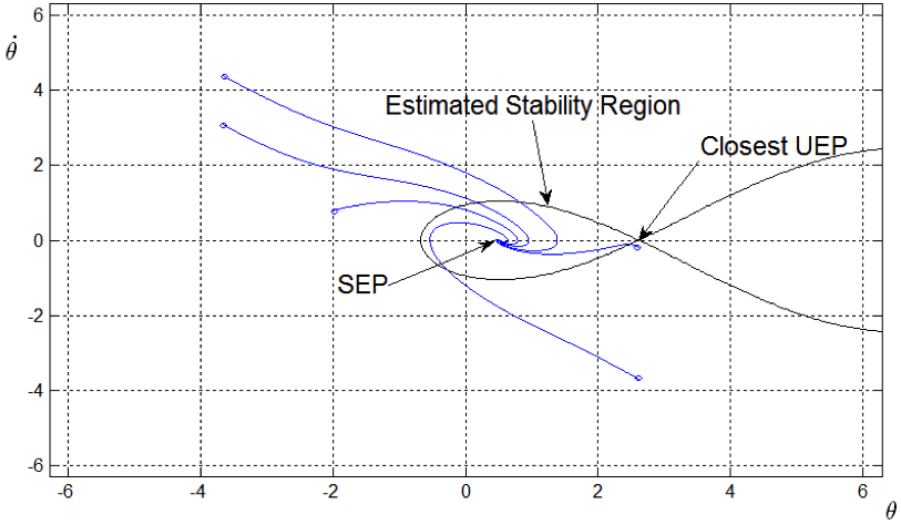


- Fast transient stability certificate

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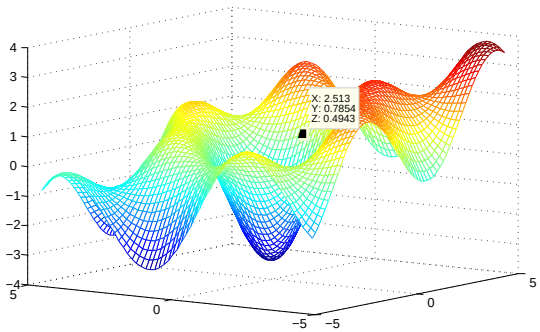


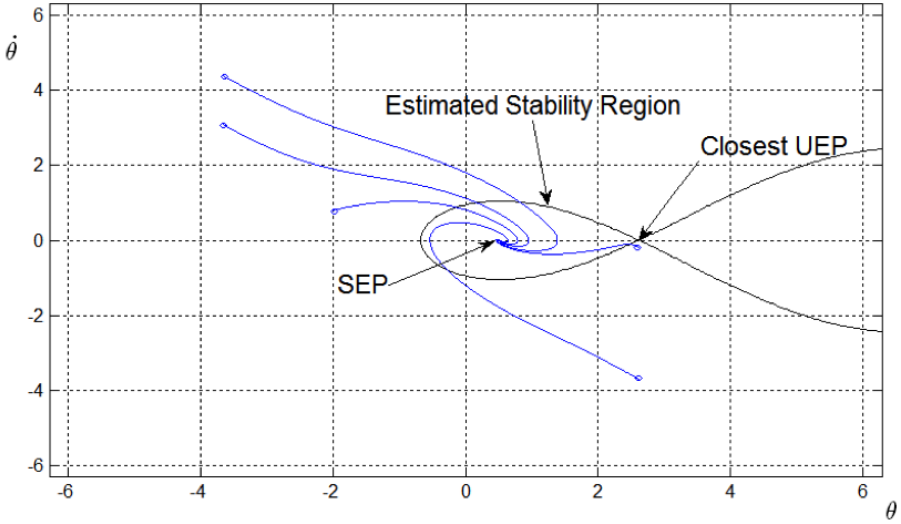


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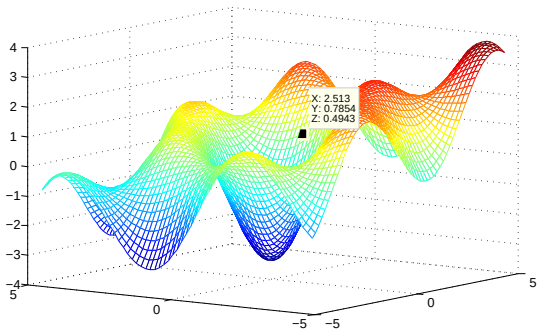


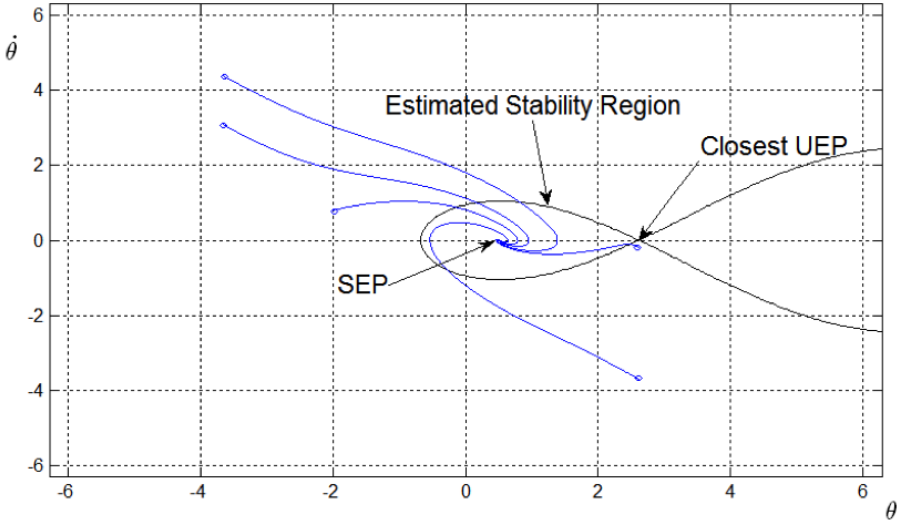


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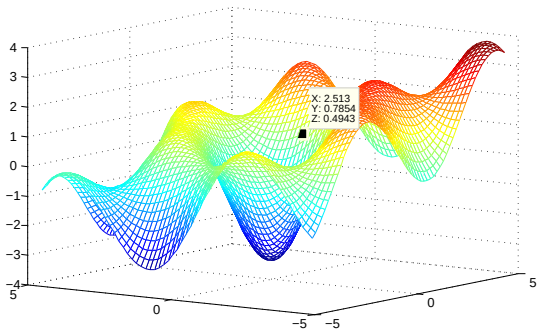


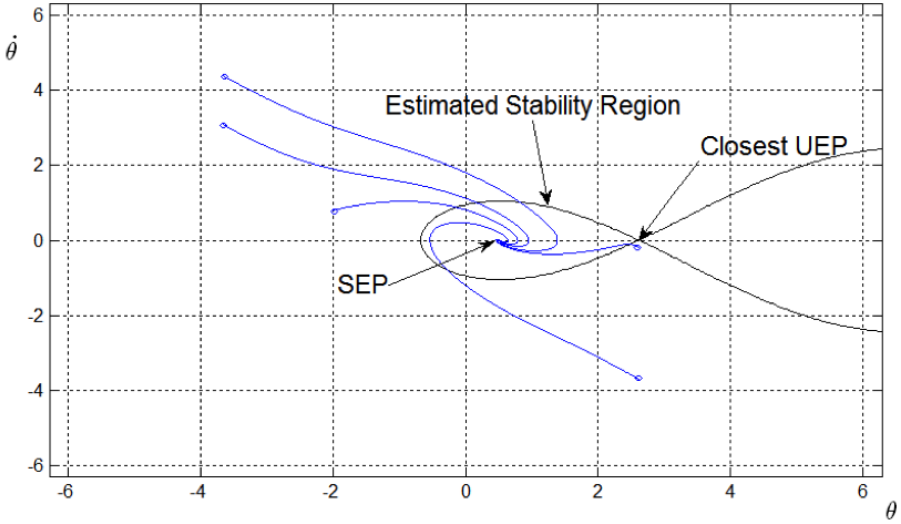


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- For typical systems $\Delta f \ll 1$ Hz.

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- Can become very large!
- Need secondary frequency control to stabilize

# Secondary frequency control

- Simple ACE signal model: $P_{1,2} \to P_{1,2} - ku(t)$

$$\dot{u} = -\frac{u}{T} + \dot{\delta}_{ave} \Longrightarrow u(t) = \int_0^\infty \dot{\delta}_{ave}(t-\tau)\exp\left(-\frac{\tau}{T}\right)d\tau$$

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- Slow variations of loads are naturally damped:

$$S_{\dot{\delta}_{ave}\dot{\delta}_{ave}} = \int_0^\infty \frac{d\omega}{2\pi\omega_0^2}\frac{S_{pp}(j\omega)}{\left|D - j\omega M + \frac{kT}{1-j\omega T}\right|^2}$$

# Secondary frequency control

- Simple ACE signal model: $P_{1,2} \to P_{1,2} - ku(t)$

$$\dot{u} = -\frac{u}{T} + \dot{\delta}_{ave} \Longrightarrow u(t) = \int_0^\infty \dot{\delta}_{ave}(t-\tau)\exp\left(-\frac{\tau}{T}\right)d\tau$$

- Slow variations of loads are naturally damped:

$$S_{\dot{\delta}_{ave}\dot{\delta}_{ave}} = \int_0^\infty \frac{d\omega}{2\pi\omega_0^2}\frac{S_{pp}(j\omega)}{\left|D - j\omega M + \frac{kT}{1-j\omega T}\right|^2}$$

- Performance metric for Secondary Frequency controls.

# Lecture Outline

- Motivation
- Introduction to Stochastic systems
- Fluctuations in two - bus system
- Fluctuations in multi-bus system
- Why Energy methods [are doomed to] fail
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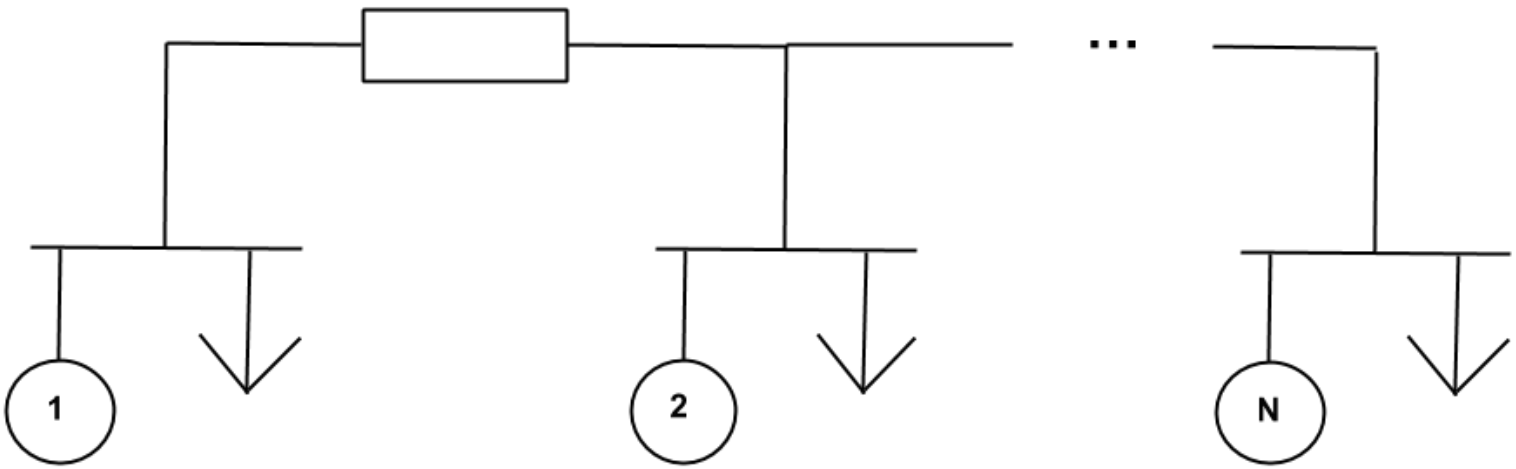
# Multi-bus model




$$M\ddot{\delta}_k + D\dot{\delta}_k + \frac{P_{max}}{\omega_0}\left[\sin(\delta_{k-1} - \delta_k) + \sin(\delta_{k+1} - \delta_k)\right] = \frac{1}{\omega_0}P_k$$

Boundary conditions $\delta_0 = \delta_1$ and $\delta_N = \delta_{N+1}$

# Linearized dynamics

- Linearized around steady state:

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- Only gradients of angles affect the flows

# Electromechanical waves

- Consider uniform flow solution $\Delta_{k,k+1} = P_{max} \cos \delta^* / \omega_0$
- Normal mode solutions have simple structure

$$\delta_k^{(n)}(t) = \exp\left[j\frac{\pi n}{N}\left(k \pm c_n t\right) - Dt\right]$$

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- Similar solutions whenever $\Delta$ has Laplacian structure
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- Equipartition principle: same average energy for each degree of freedom (assume $\delta_k^* = 0$)

$$\mathbf{E}\left[\frac{M\dot{\delta}_k^2}{2}\right] = \frac{T}{2}, \quad \mathbf{E}\left[\frac{P_{max}}{\omega_0}(1 - \cos\delta_{k,k+1})\right] = \frac{T}{2}$$

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- Can air molecules in this room break the wall ?

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   - Passivity, Integral Quadratic constraints ?
3. Use other Lyapunov functions
   - Less general, but more appropriate

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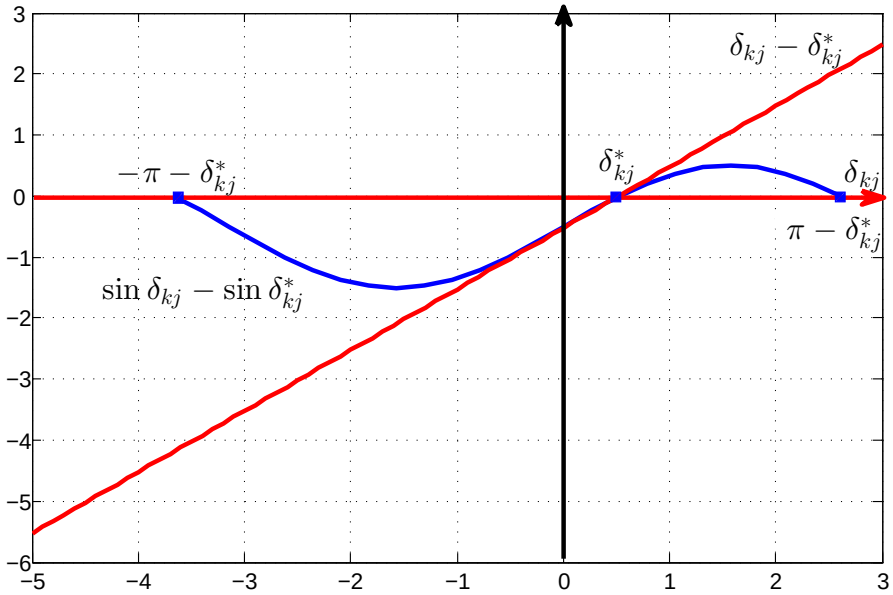
# Bounding nonlinearity



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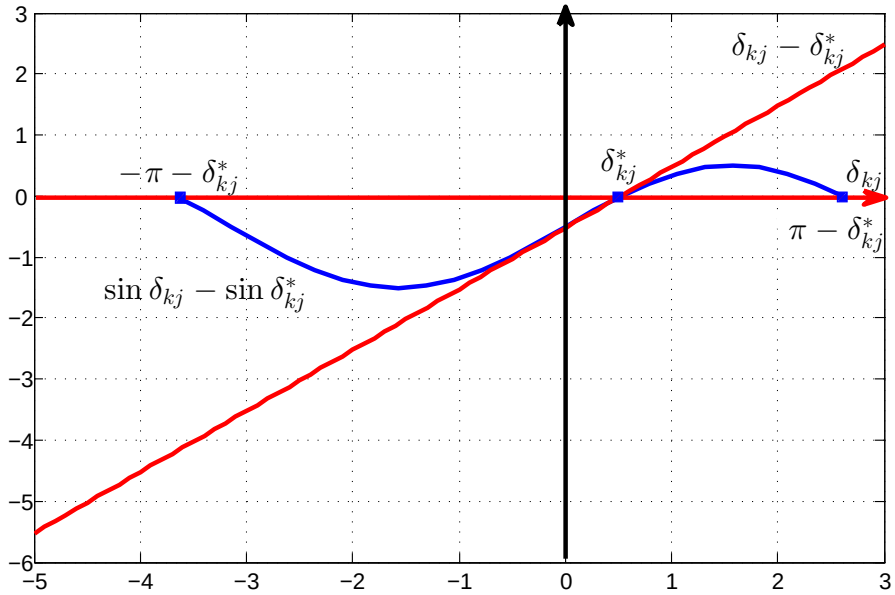


- Sector bound on nonlinearity
  - $0 \leq (\delta_{kj} - \delta^*_{kj})(\sin\delta_{kj} - \sin\delta^*_{kj}) \leq (\delta_{kj} - \delta^*_{kj})^2$
  - In polytope $\mathscr{P} : \{\delta, \dot{\delta} : |\delta_{kj} + \delta^*_{kj}| < \pi\}$ nonlinear terms are bounded

# Lyapunov function

$$V(x) = \frac{1}{2}x^T Q x - \sum_{\{k,j\}\in\mathscr{E}} K_{\{k,j\}} \left( \cos\delta_{kj} + \delta_{kj} \sin\delta^*_{kj} \right)$$

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- Generalizes the kinetic and potential terms
- Decay $\dot{V} < 0$ certified only in the polytope $\mathscr{P}$
- Matrices $K$ and $Q$ are not arbitrary

# Linear Matrix inequality

$$\begin{bmatrix} A^T Q + QA & QB - C^T H - (KCA)^T \\ B^T Q^T - H^T C - (KCA) & -2H \end{bmatrix} \leq 0$$

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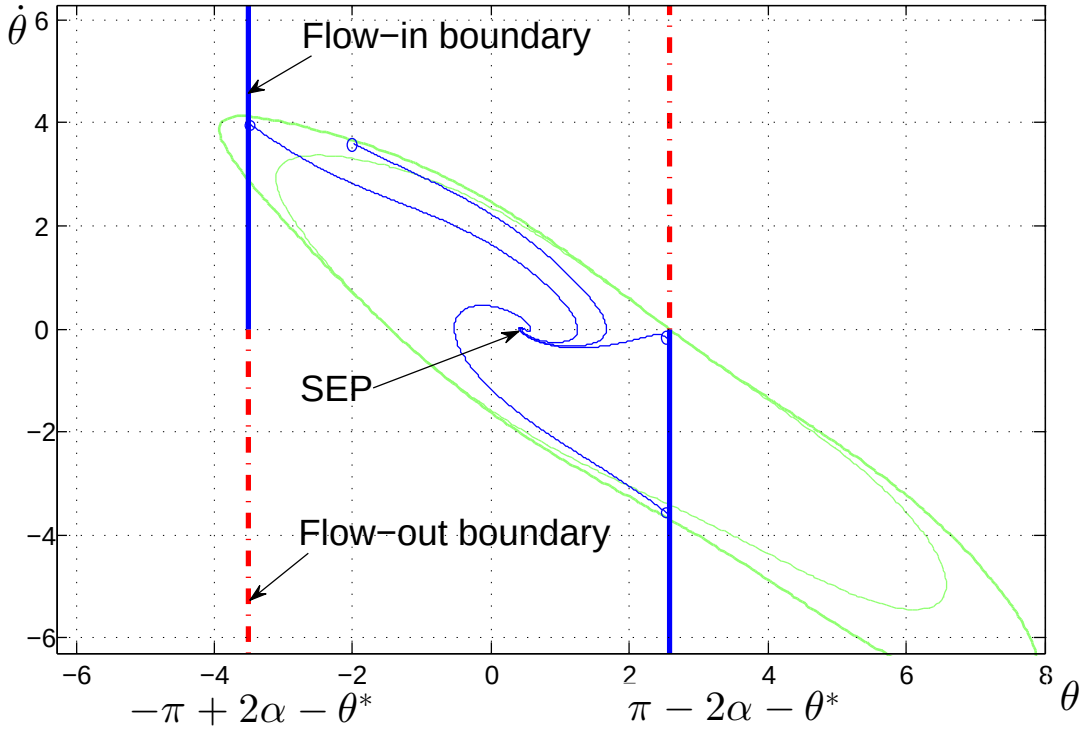
- Convex optimization problem
- Powerful SDP methods

# Stability region



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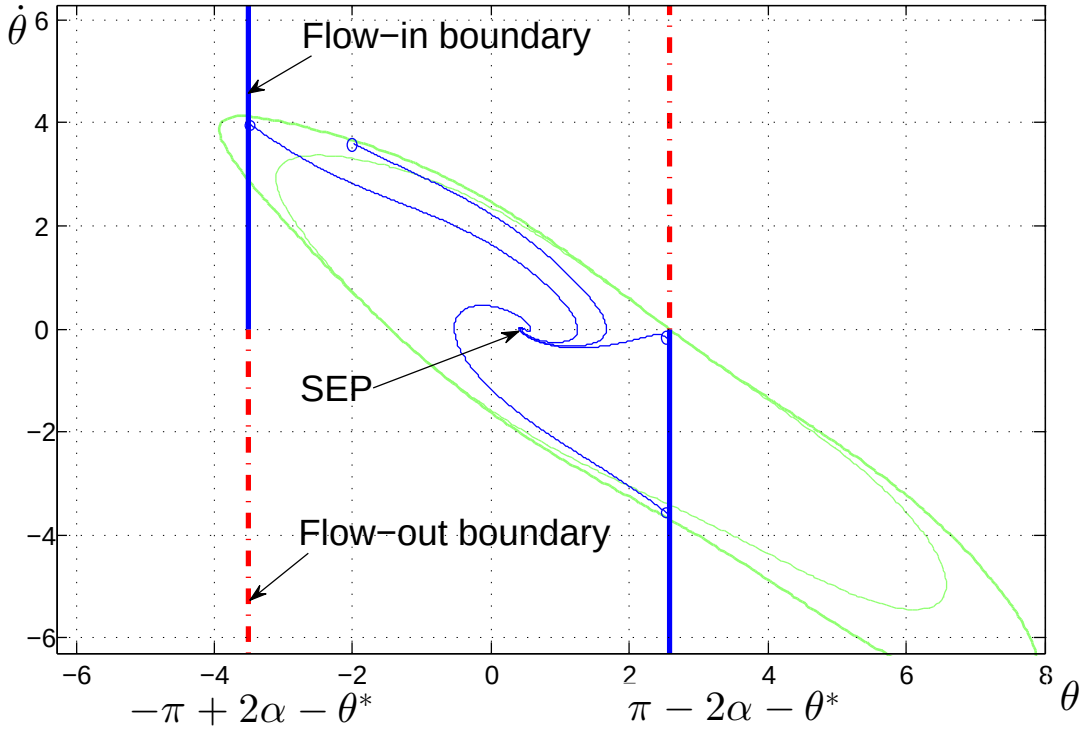


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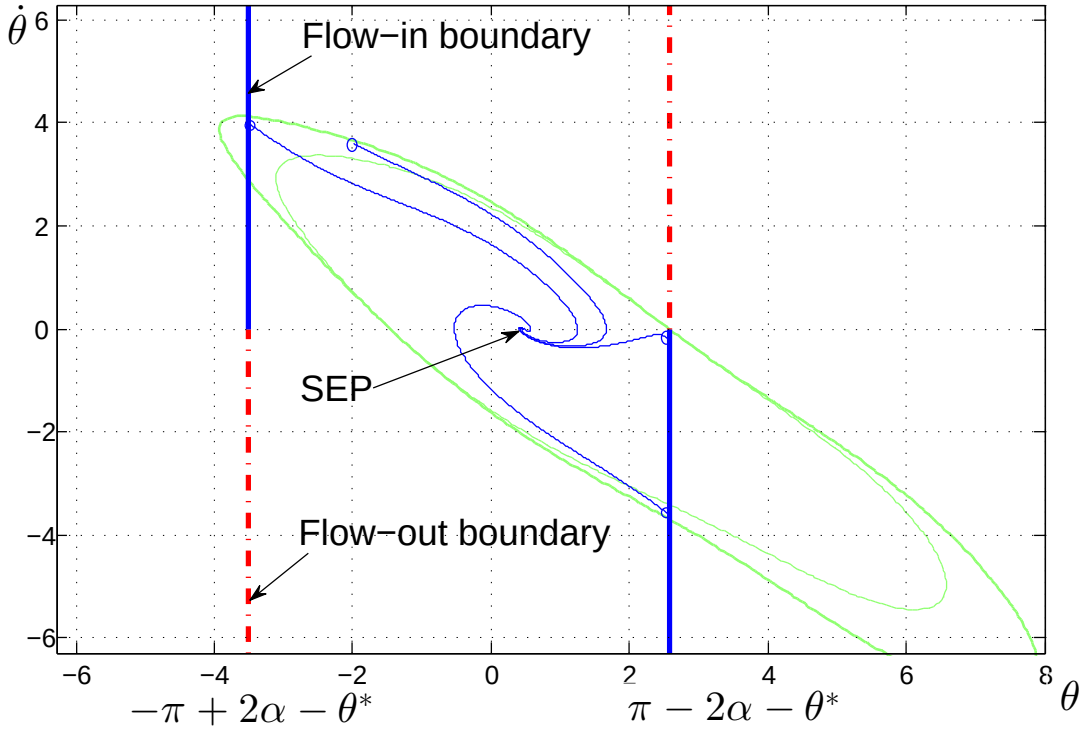


- Stability certified only as long as trajectory stays in $\mathscr{P}$
- Level set of Lyapunov function should not intersect the boundaries of $\mathscr{P}$ with outgoing flow

# Stability certificate

System is stable as long as $V(x(0)) < V_{\min}$ with

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- Least conservative certificate

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Consider a subset of phase space

$$\mathcal{Q} = \{x : |\delta_{kj}| < \pi/2\}$$
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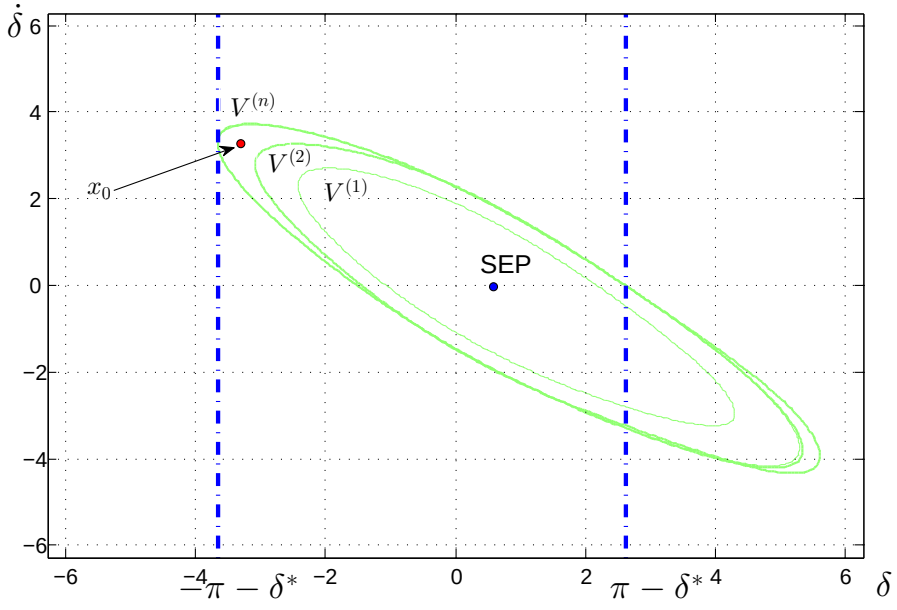
- Other convex constructions possible based on algebraic bounds of $V(x)$

# Adapting Lyapunov functions



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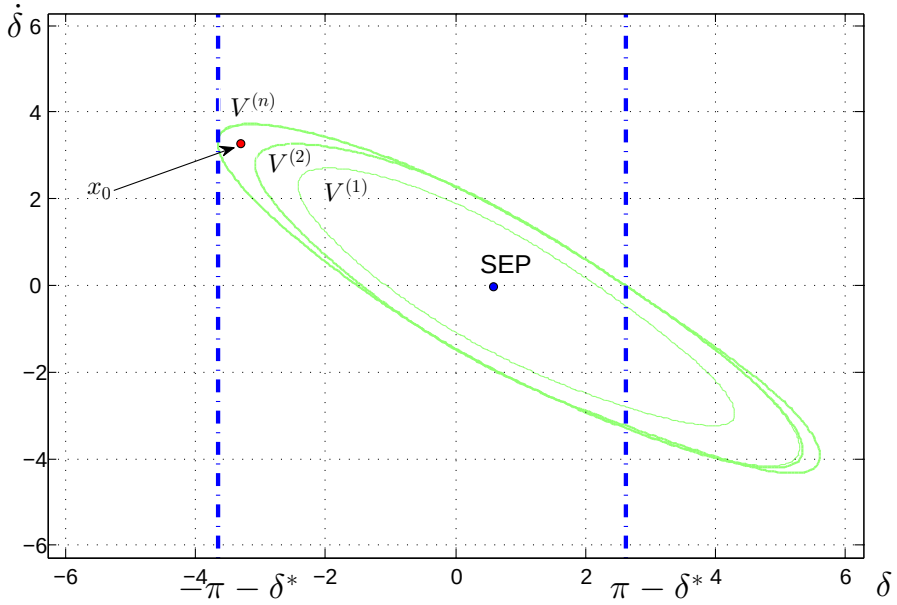


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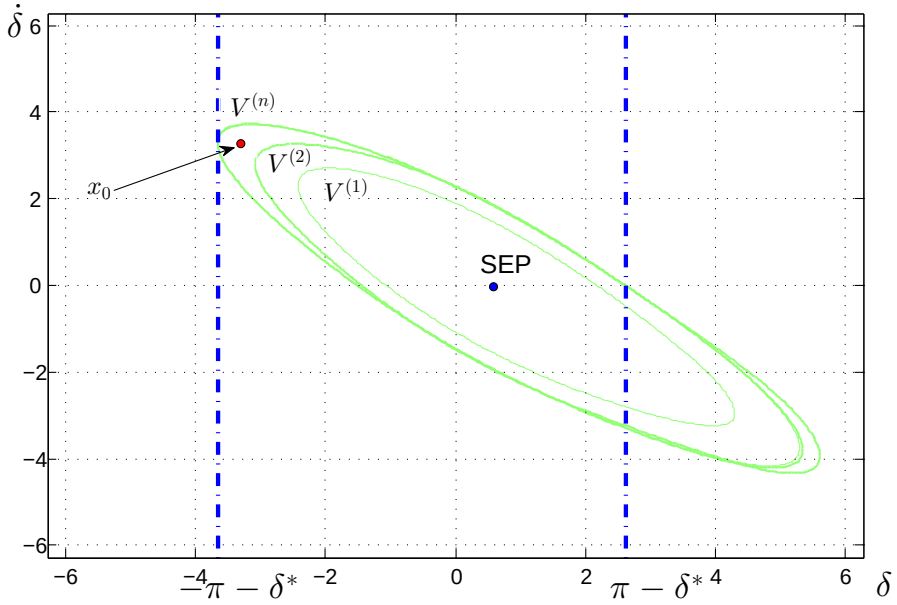
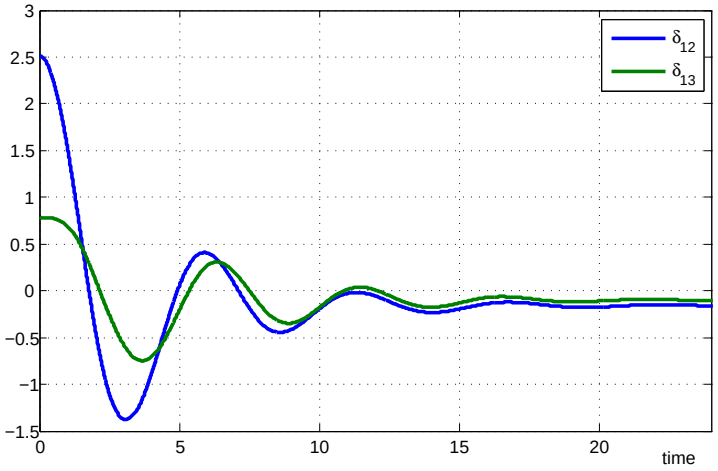
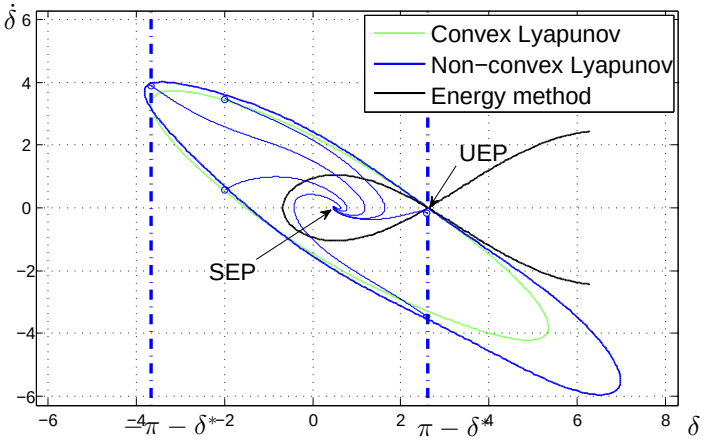



- Lyapunov functions can be adapted to a given initial condition
- If a Lyapunov function exists it will be found in finite time

# Numerical simulations



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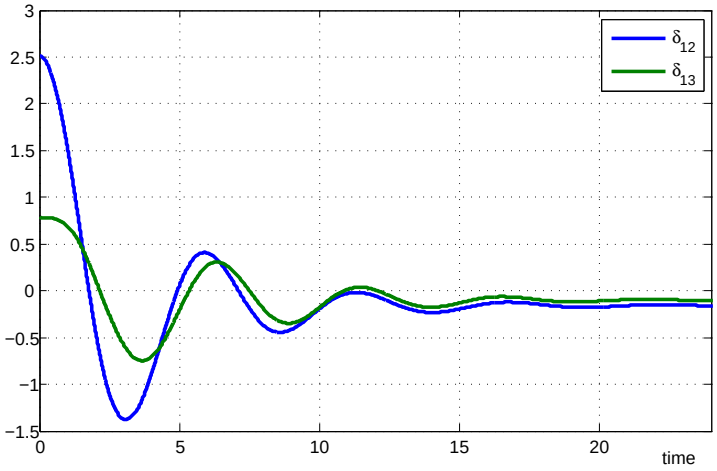


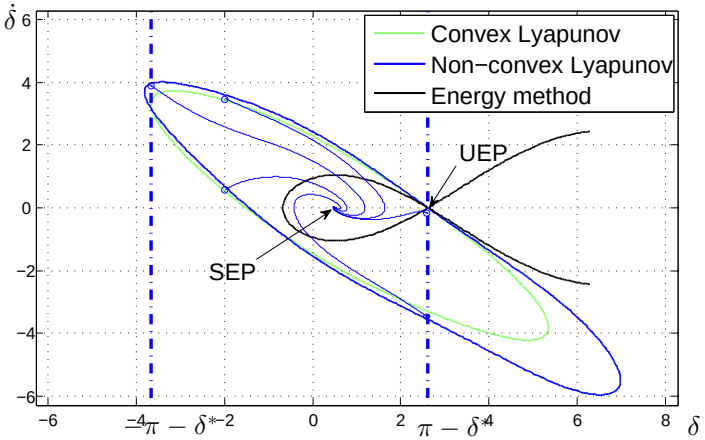


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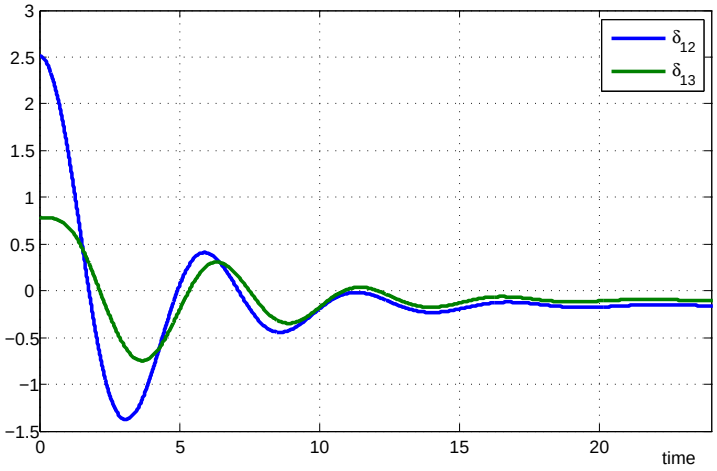


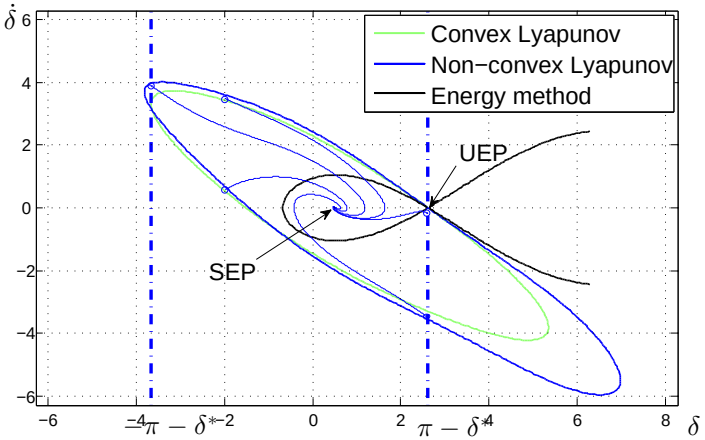


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- About 2 seconds for certificate construction in CVX for 39 bus system on a regular laptop

# Ongoing work

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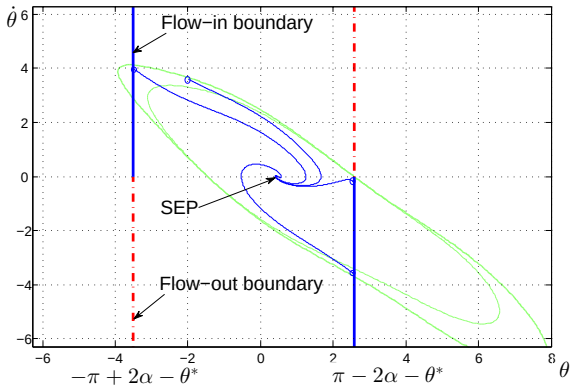
- Generalization to lossy grids (ACC)



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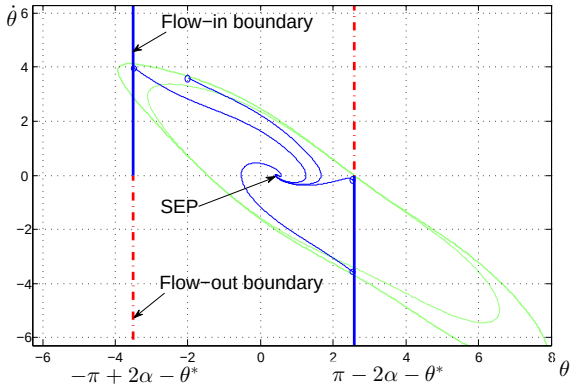


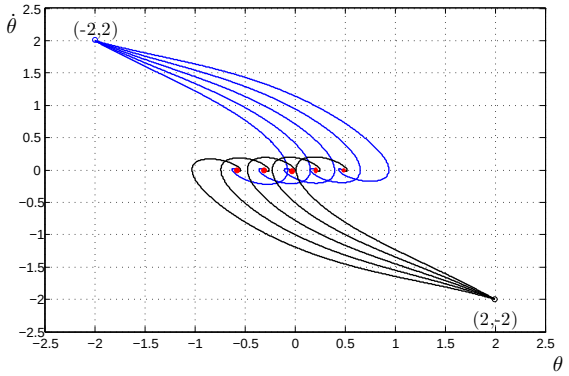

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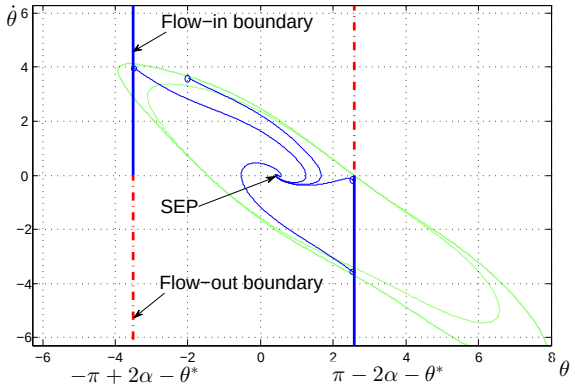
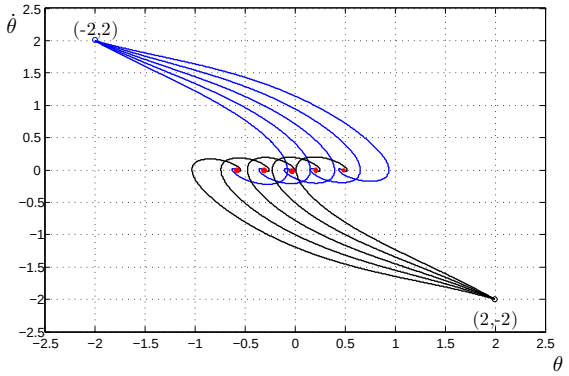
- Robustness with respect to operating point (ACC)
- Higer-order generator models - ongoing



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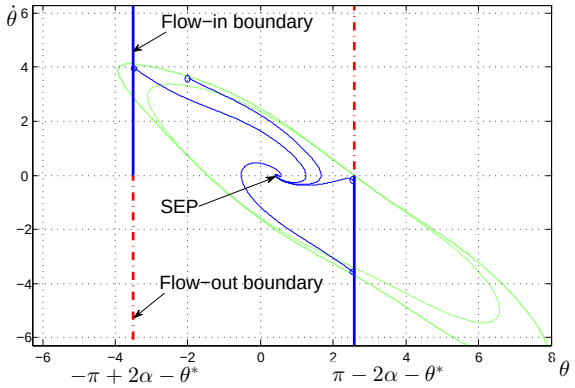
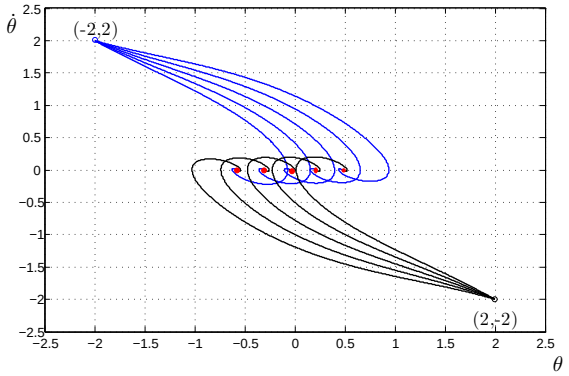
- Robustness with respect to operating point (ACC)
- Higer-order generator models - ongoing
- Topology change - planned



# Ongoing work

- Generalization to lossy grids (ACC)
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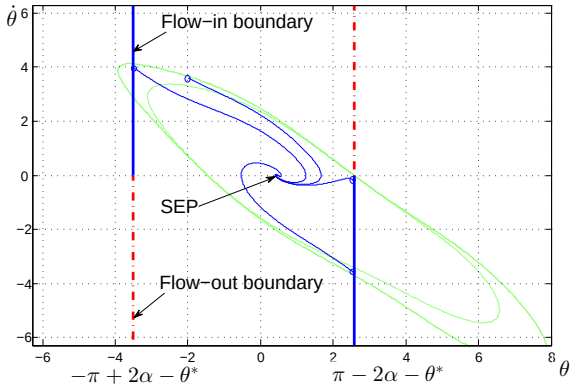
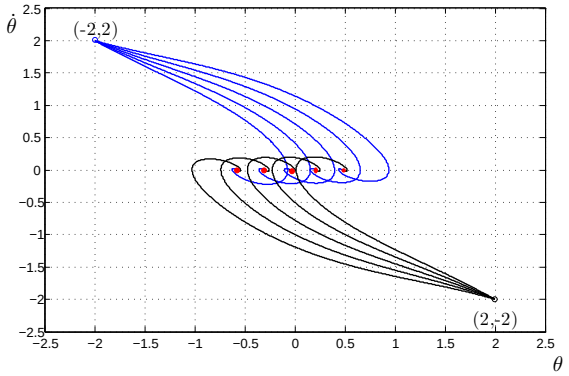
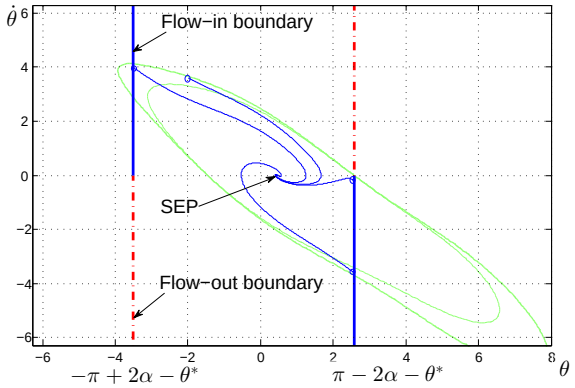
- Higer-order generator models - ongoing
- Topology change - planned
- Fault-on dynamics - planned

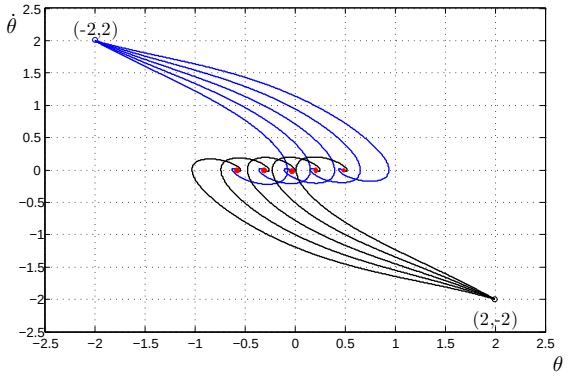


# Ongoing work

- Generalization to lossy grids (ACC)
- Robustness with respect to operating point (ACC)
- Higer-order generator models - ongoing
- Topology change - planned
- Fault-on dynamics - planned
- Compositional approaches - planned



# Key points

- Seconds time-scale dynamics critical for security
- System is inherently stochastic
- Simple models can be developed for fluctuations
- Ambient fluctuations may lead to collapse
- Frequency control is crucial for stability
- Enough energy to collapse in large systems
- Need to account for entropy growth
- Alternatives to Energy methods exist
- Questions ?