Power Flow Equations: Complexity, Approximations and Relaxations

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Australian Research Council

NICTA Funding and Supporting Members and Partners

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Credits

- Power systems
  - PSCC'11, PES'12 (2), PSCC'14 (5)
  - IJEPES (to appear)
- Operations research
  - IJOC 2014, Mathematical Programming 2015
- Submissions
  - complexity results (under submission)
- People
  - *C. Coffrin*, A. Grastien, H. Hijazi, *K. Lehmann, T. Mak*
  - *R. Bent (LANL)*
  - D. Hill and I. Hiskens

Outline

- Motivation
- Power Flow Formulations
- Complexity
- The LPAC Approximation
- Case Studies for LPAC
- Convex Relaxations
- Case Studies for Convex Relaxations
- Beyond Steady States

Power Systems

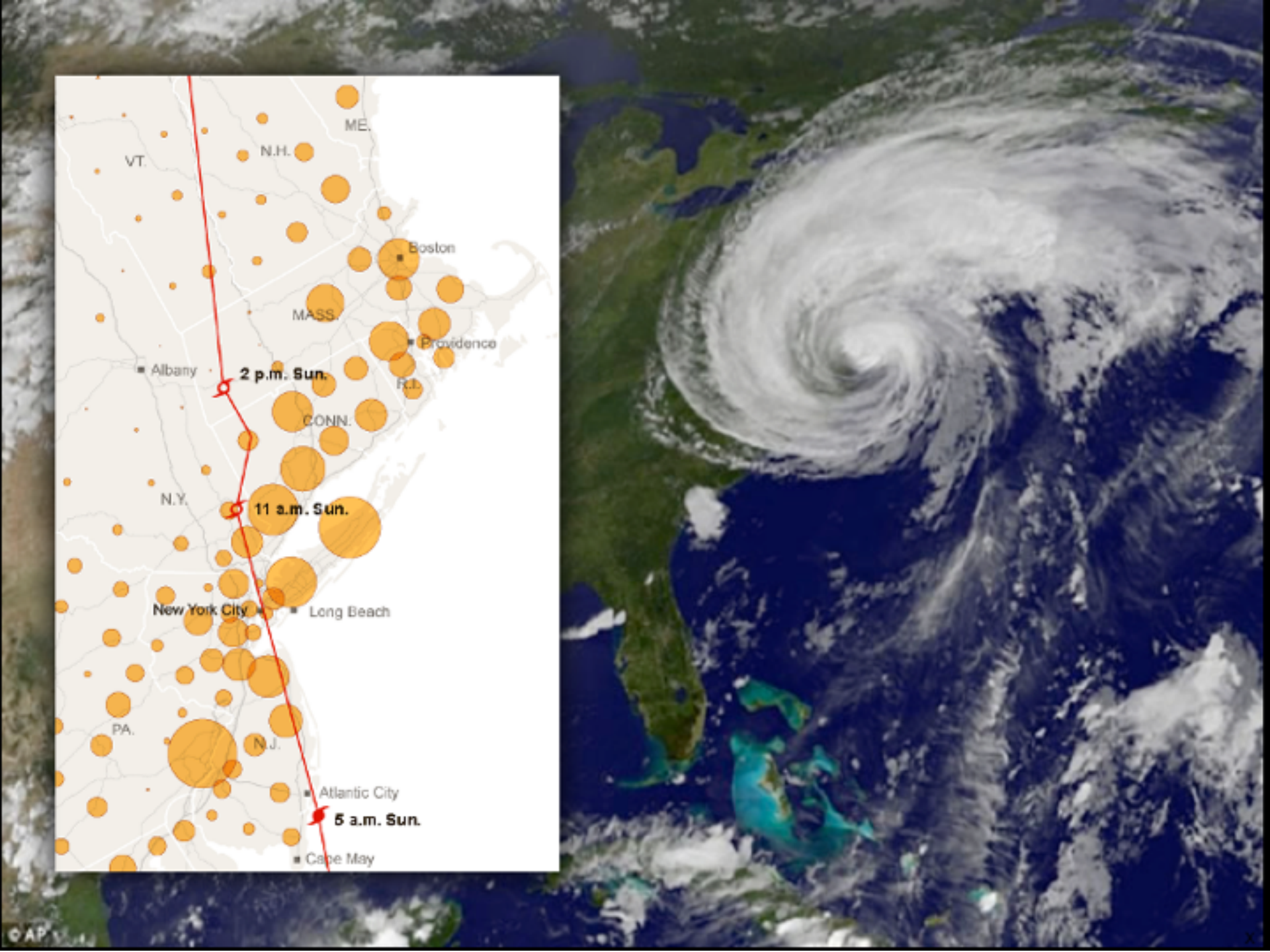
- Biggest machine on earth
  - 400 billions of electricity

Why fix it?

- New challenges
  - challenging existing assumptions
- New applications
  - requiring new technology
- New enabling technologies
  - enabling new functionalities

Motivation.

- Category 3
  - August 21-28, 2011
- Fatalities
  - 49 direct (+ 7)
- Damages
  - ~ $15 billions


ME.
N.H.
VT.
Boston
MASS.
Providence
Albany
2 p.m. Sun.
R.I.
CONN.
N.Y.
11 a.m. Sun.
New York City
Long Beach
PA.
N.J.
Atlantic City
5 a.m. Sun.
Cape May
© AP

San Diego Blackout

- Causes
  - Tripping of a line between Arizona and California
  - Cascading effect (not supposed to happen)
- Effects
  - >4 millions people without power, Sept. 8-9, 2011
- Economic Losses over US$ 100 million
  - Opportunity losses: $70 million
  - Overtime workers: $20 million
  - Spoiled food: $18 million

Joint Repair and Power Restoration

- The challenge
  - Schedule a fleet of repair crews to repair the grid and minimize the overall size of the blackout after a disaster
- Two fundamental aspects
  - Scheduling the repairs
  - Scheduling the power restoration
  - Both are challenging in their own right
- Assumptions for Last-Mile Restoration
  - Steady state behavior of the power grid
  - Ability to dispatch load and generation continuously

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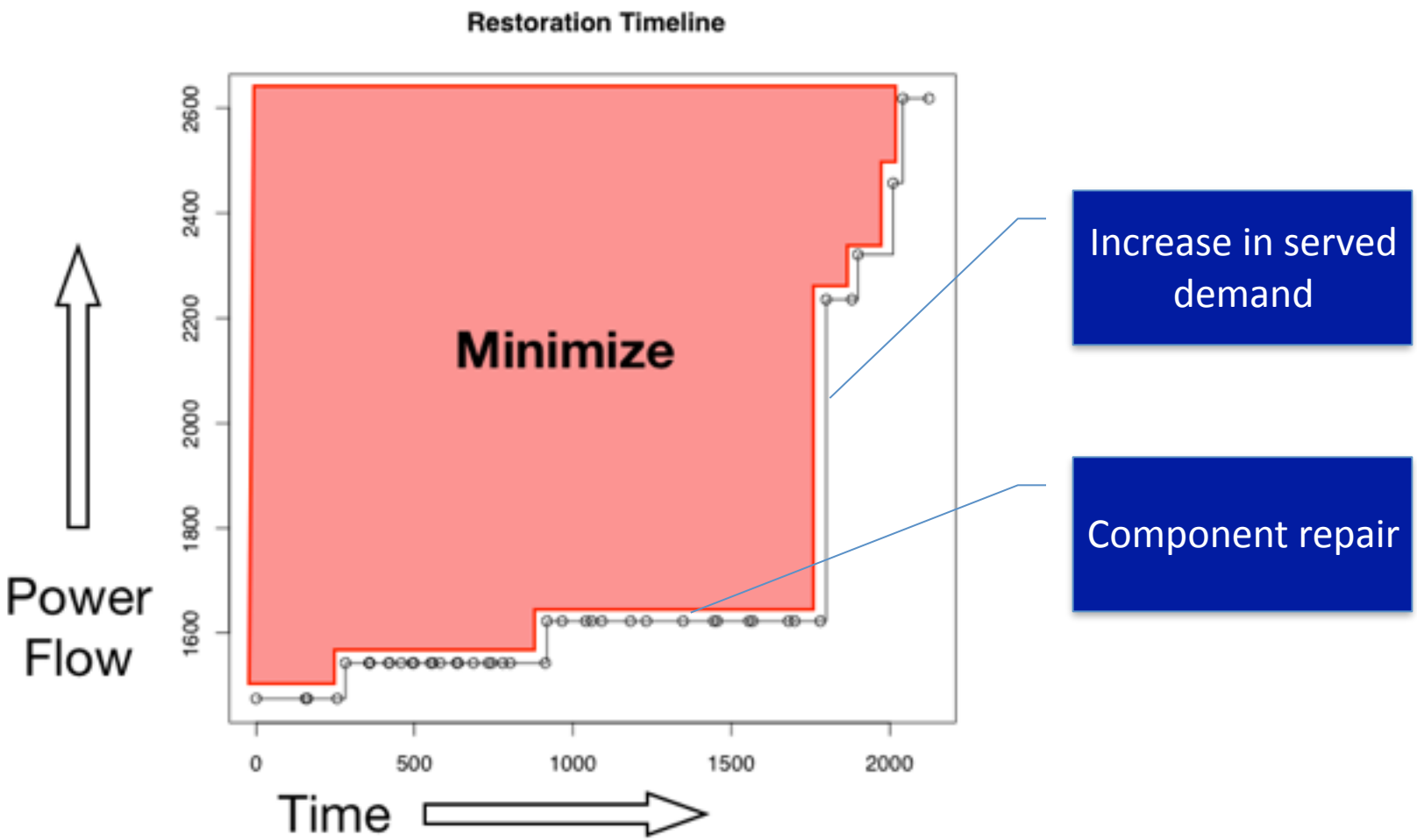
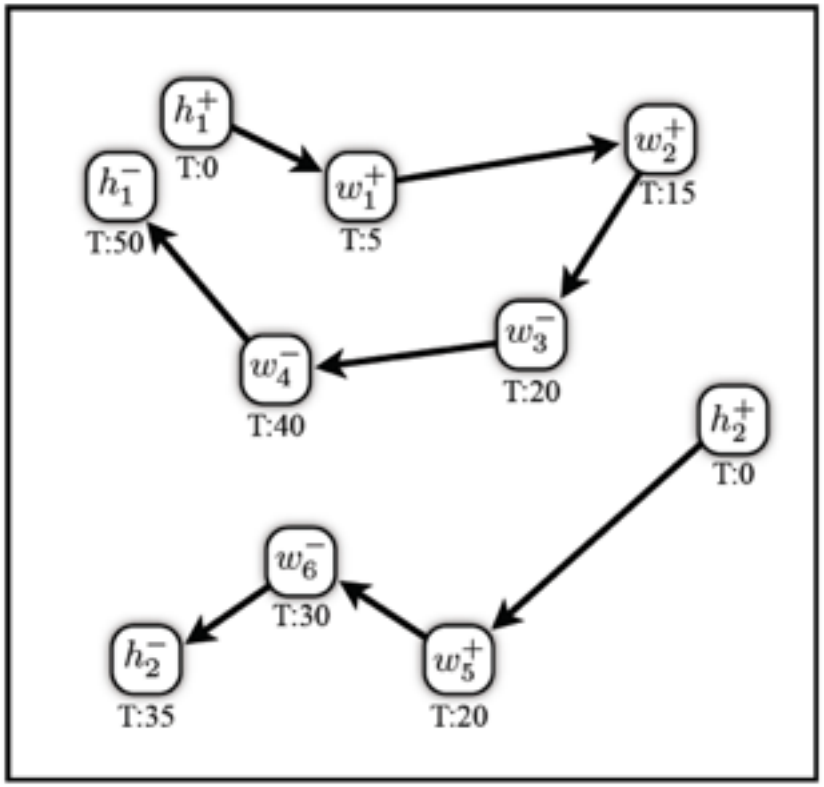
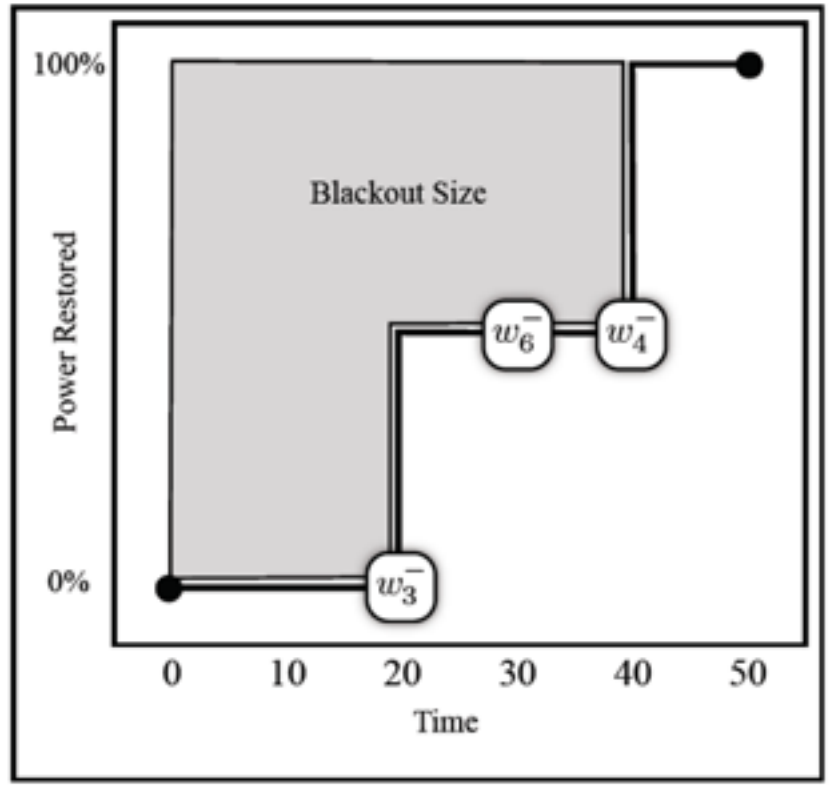
Joint Repair and Power Restoration



Joint Repair and Power Restoration



Routing Aspect

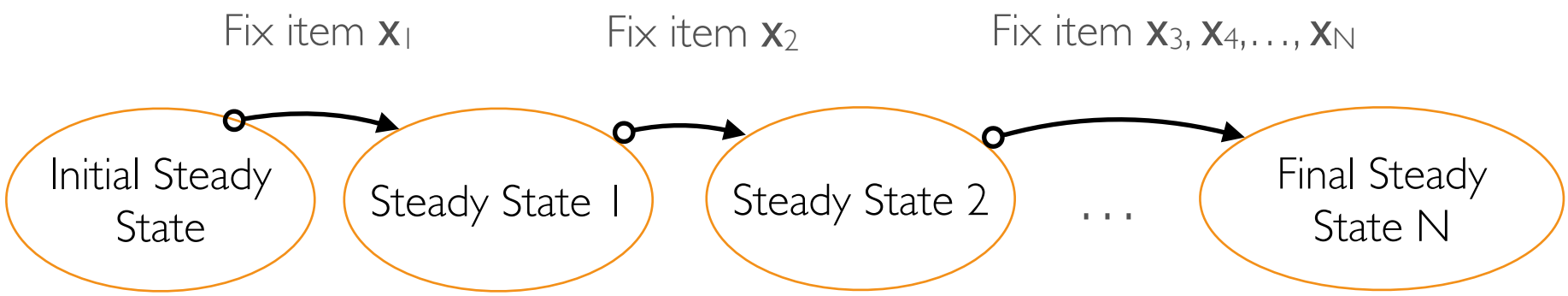


Power Flow Aspect

Joint Repair and Power Restoration

- 2-Step Approach (PSCC'11)
  - Restoration Ordering Problem (ROP)
  - Pickup and Delivery Routing with Precedences
    - Randomized Adaptive Decomposition over LNS over CP

Restoration Ordering Problem (ROP)



- Find the best sequence of restoration

  $[x_1, x_2, x_3, \ldots, x_n]$

  so that the size of the blackout is minimised

The ROP Problem

- Extremely challenging computationally
  - generalizes transmission switching

Computational Challenge

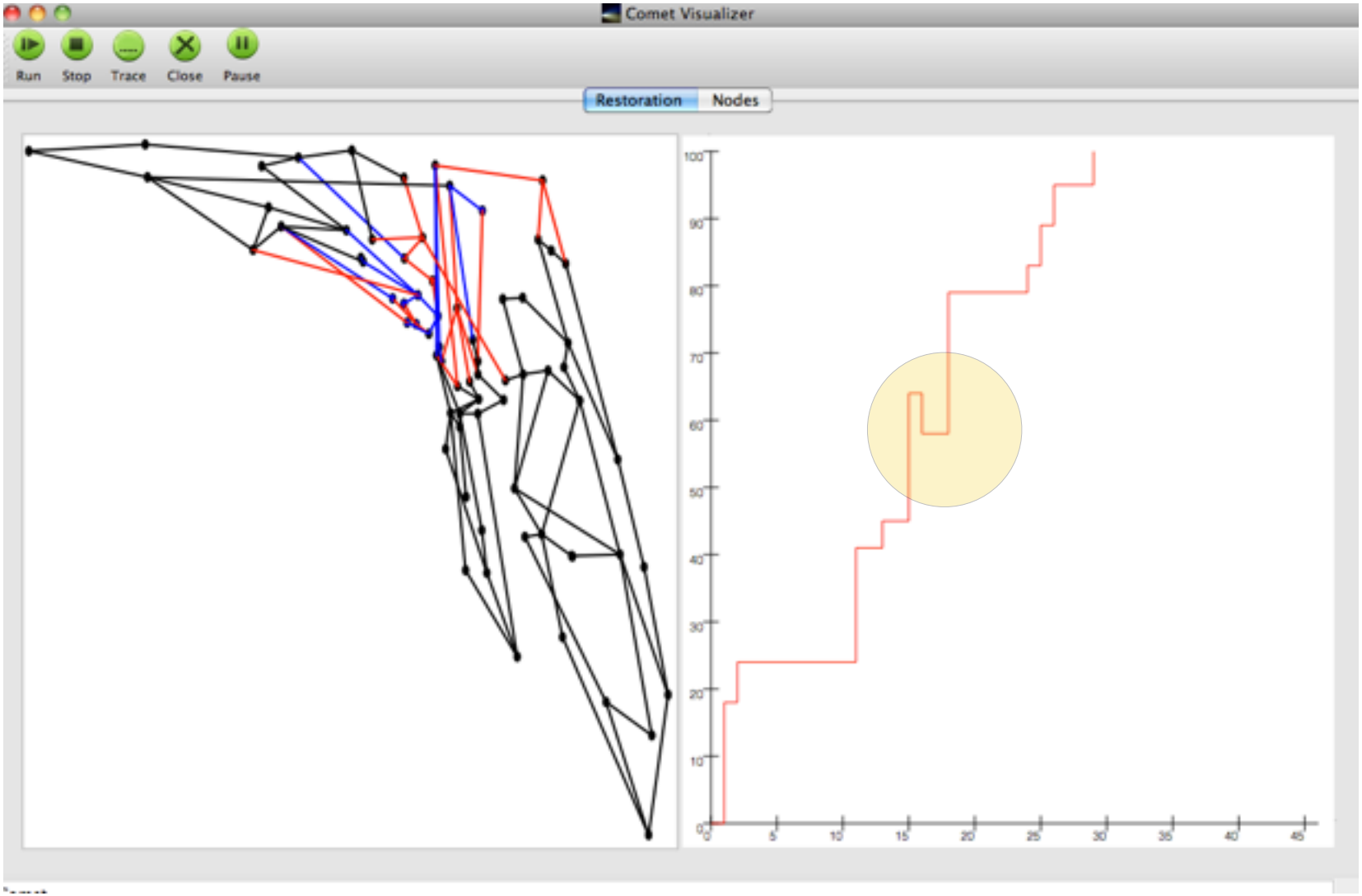
presolved problem has 1960 variables and 2186 constraints
510 constraints of type <varbound>
1475 constraints of type <linear>
201 constraints of type <logicor>
Presolving Time: 0.52

time	node	dualbound	primalbound	gap
1.0s	1	1.602999e+02	-0.000000e+00	100.00%
14.3s	1	1.602999e+02	1.213604e+02	24.29%
779s	12012	1.602999e+02	1.228179e+02	23.38%
812s	12658	1.602999e+02	1.238971e+02	22.71%
827s	12858	1.602999e+02	1.263328e+02	21.19%
1019s	18895	1.602999e+02	1.324728e+02	17.36%
1147s	24867	1.602999e+02	1.353018e+02	15.59%
1276s	33091	1.602999e+02	1.362924e+02	14.98%
1294s	34851	1.602999e+02	1.419186e+02	11.47%
1958s	91300	1.602999e+02	1.447137e+02	9.72%
5882s	435468	1.602999e+02	1.460944e+02	8.86%
202m	1038k	1.602999e+02	1.464802e+02	8.62%
253m	1344k	1.602999e+02	1.464802e+02	8.62%

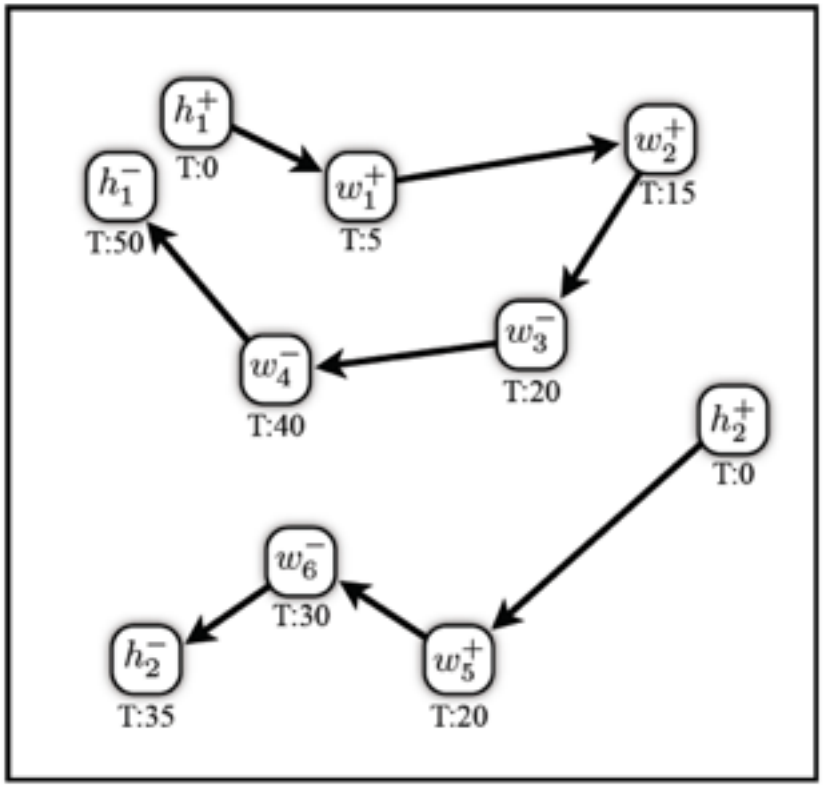
The ROP Problem

- Extremely challenging computationally
  - generalizes transmission switching
- Braess paradox
  - restoring a line may decrease the network flow

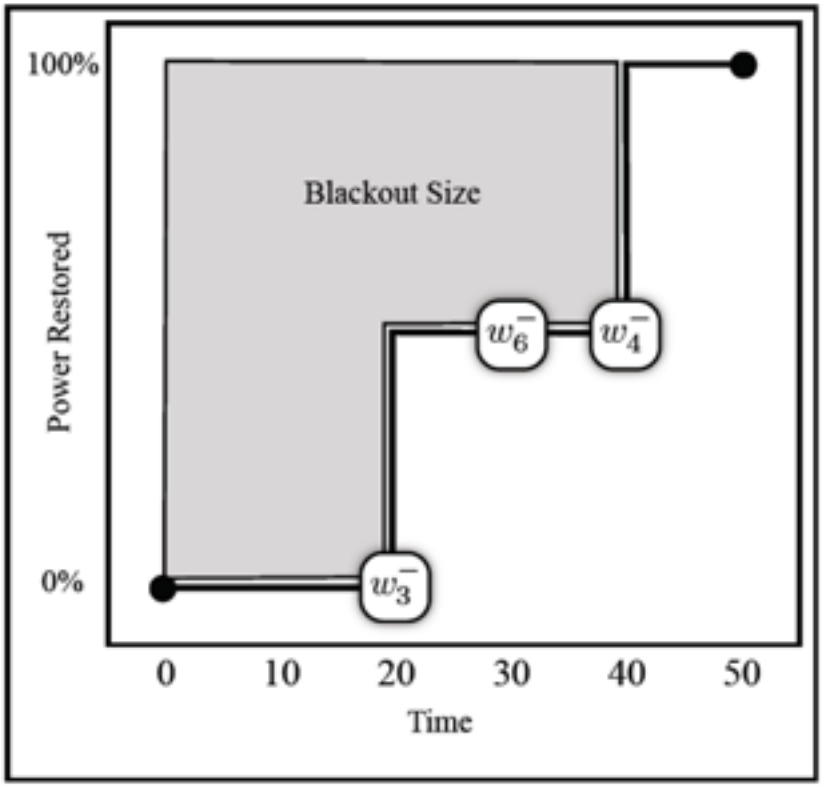
Braess Paradox



Joint Repair and Power Restoration



Routing Aspect



Power Flow Aspect

The ROP Problem

- Modeling the power system
  - Line capacities (thermal constraints)
  - Constraints on real and reactive power injections
  - Power Flow equations

Power Flow Equations

- Ohm's law for each line (i,j)

$$p_{ij} = \mathbf{g}_{ij} v_i^2 - \mathbf{g}_{ij} v_i v_j \cos(\theta_i - \theta_j) - \mathbf{b}_{ij} v_i v_j \sin(\theta_i - \theta_j)$$

$$q_{ij} = -\mathbf{b}_{ij} v_i^2 + \mathbf{b}_{ij} v_i v_j \cos(\theta_i - \theta_j) - \mathbf{g}_{ij} v_i v_j \sin(\theta_i - \theta_j)$$

- Kirchhoff's current law for each bus i

$$p_i = \sum_{(i,j) \in E} p_{ij}$$

$$q_i = \sum_{(i,j) \in E} q_{ij}$$

The Core ROP Subproblem

- Push as much load as possible in the network while satisfying the power flow and side constraints
  - nonlinear
  - nonconvex
  - discrete

The Linear DC Model

conductance is much smaller than susceptance

sin(x) close to x when x is small

$$p_{ij} = \qquad - \mathbf{b}_{ij} \qquad (\theta_i - \theta_j)$$

Ignore reactive power

Voltage magnitudes are close to 1.0

The Linear DC Model

- Ohm's law for each line (i,j)

$$p_{ij} = -\mathbf{b}_{ij}(\theta_i - \theta_j)$$

- Kirchhoff's current law for each bus i

$$p_i = \sum_{(i,j)\in E} p_{ij}$$

In the ROP

- The power equation becomes

$$\mathrm{p}_{ij} = -\mathbf{b}_{ij} z_{ij} (\theta_i - \theta_j)$$

- It is nonlinear: can be linearized since z_i is a 0/1 variable

$$p_{ij} \leqslant \mathbf{b}_{ij}(\theta_i - \theta_j) + \mathbf{M}(1 - z_{ij})$$
$$p_{ij} \geqslant \mathbf{b}_{ij}(\theta_i - \theta_j) - \mathbf{M}(1 - z_{ij})$$

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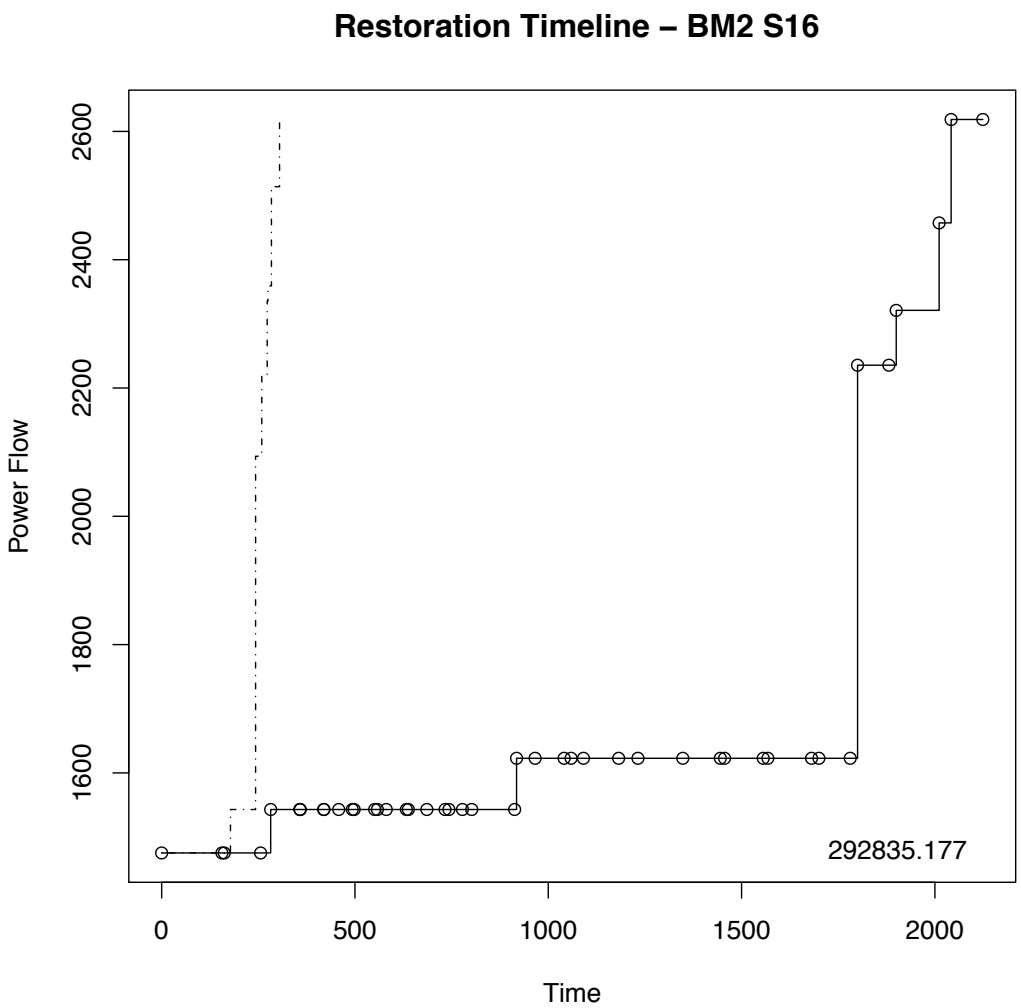
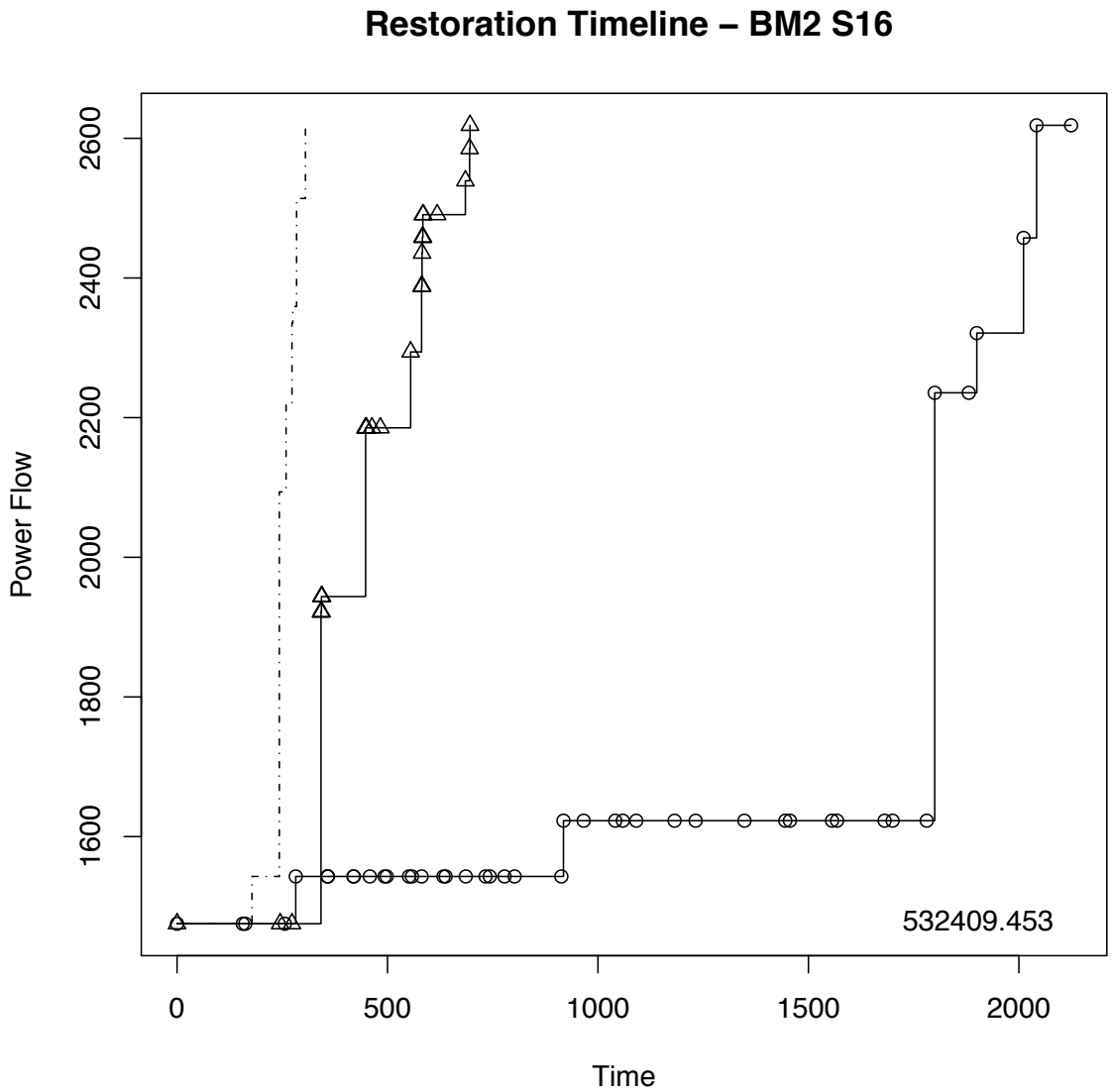
Case Studies

- 15 large disaster scenarios
  - with damage size in 50-1000
- Generated using,
  - US Transmission and Transportation Infrastructure
  - State-of-the-art disaster simulation tools (NHC, FEMA)
- Study quality over 0.5 - 8 hours
  - Average of 10 runs
- Comparing with best practices in the field
  - utilisation heuristics + greedy routing
- Lower bound
  - assuming infinitely many crews

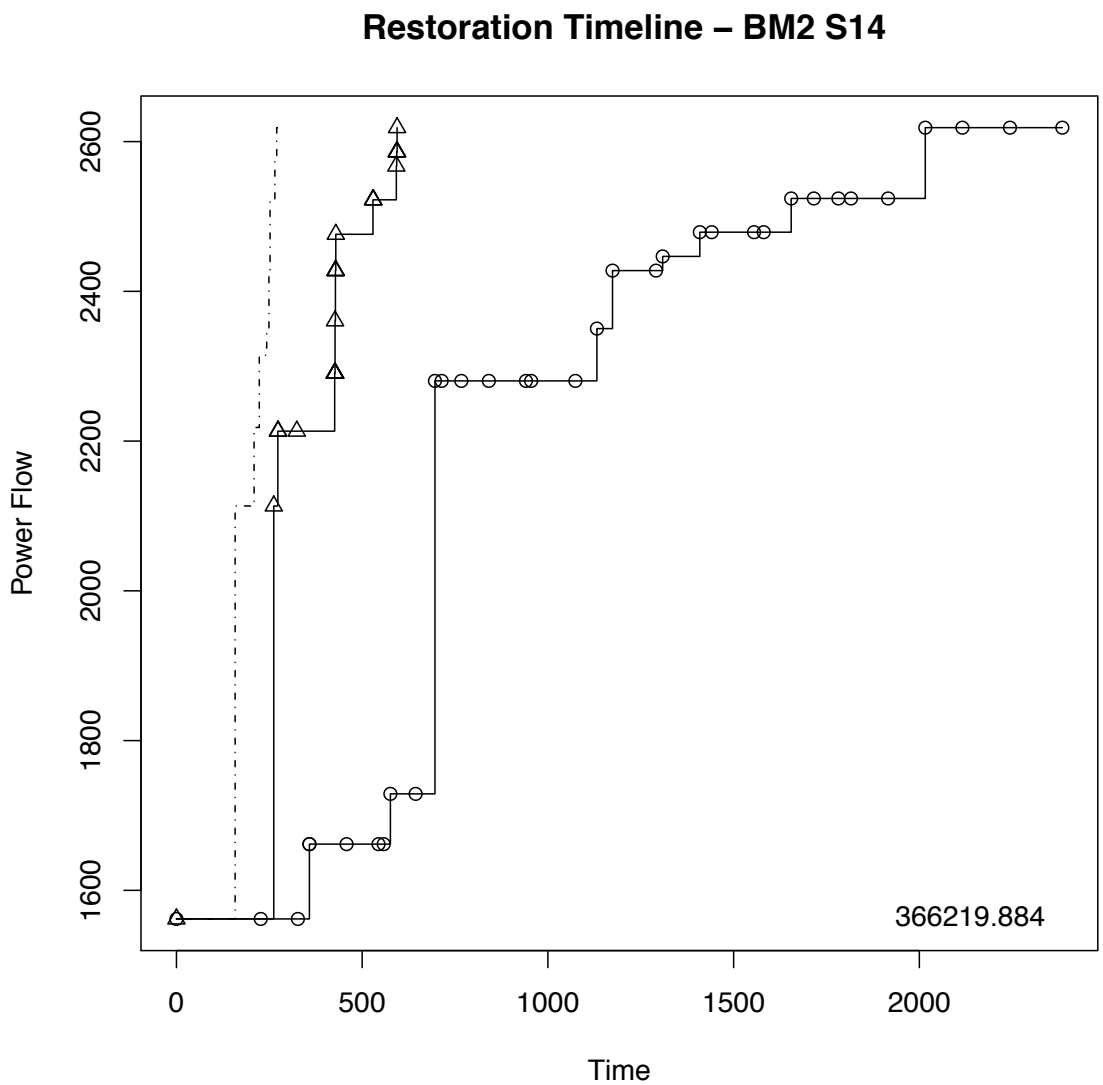
Case Studies



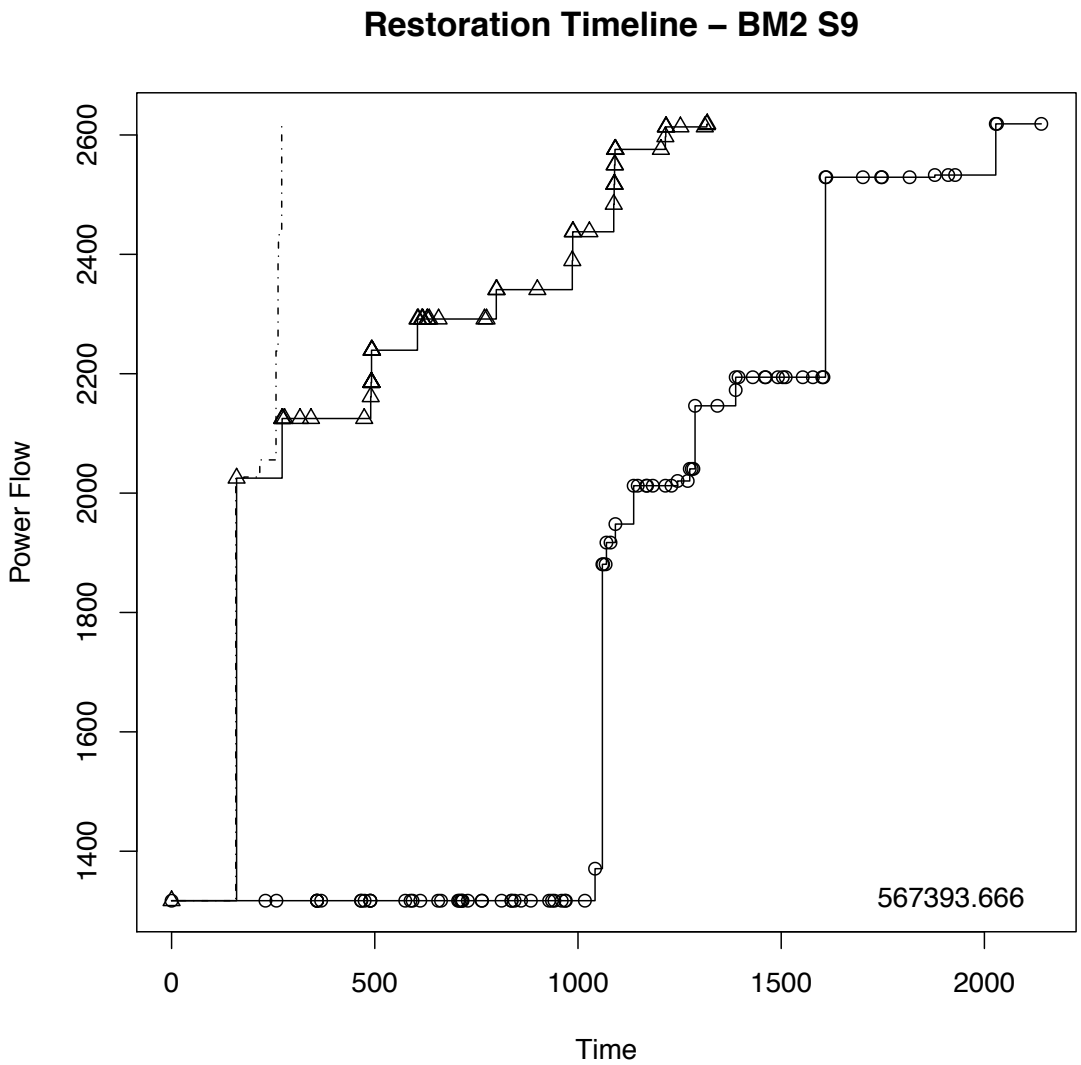
Case Studies



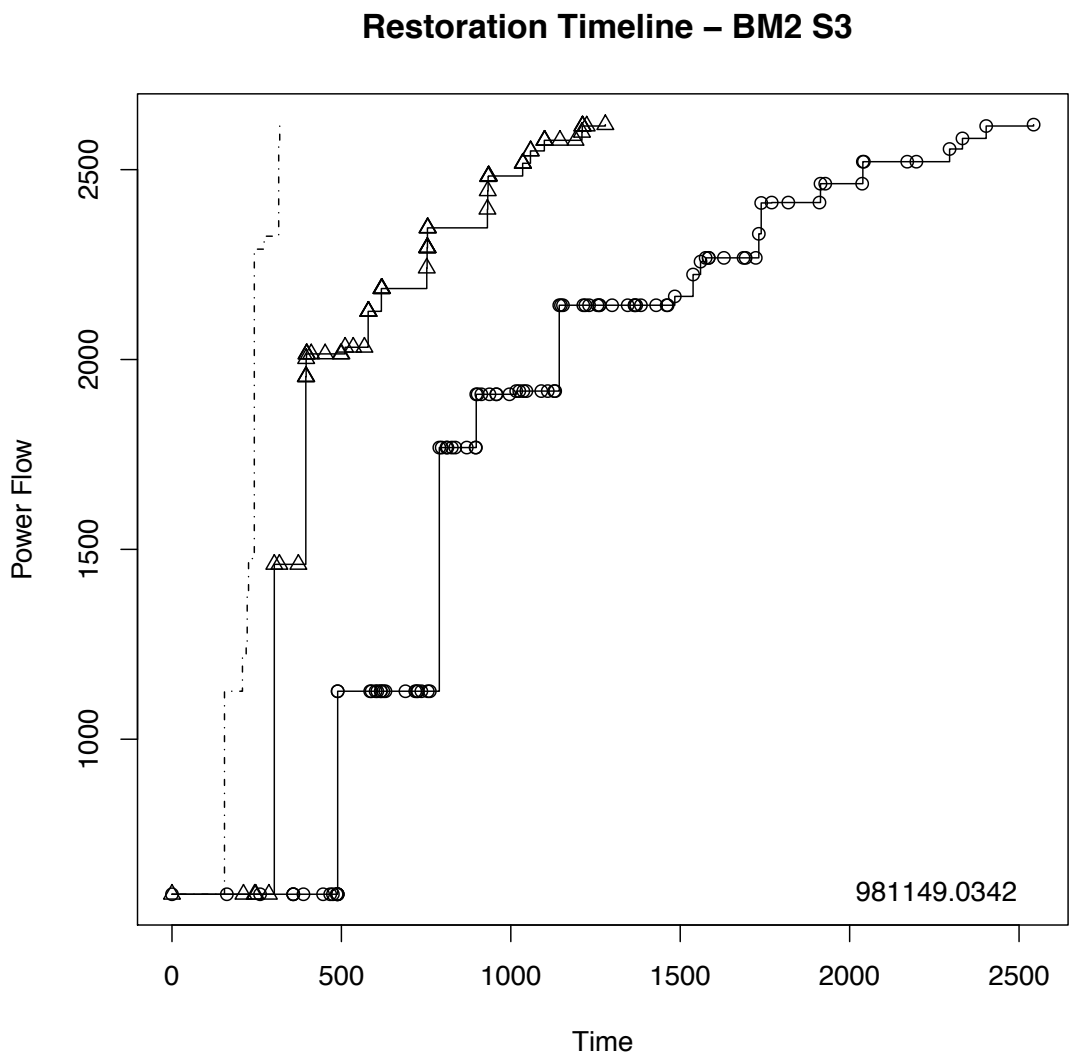
Case Studies



Case Studies



Case Studies



Case Studies

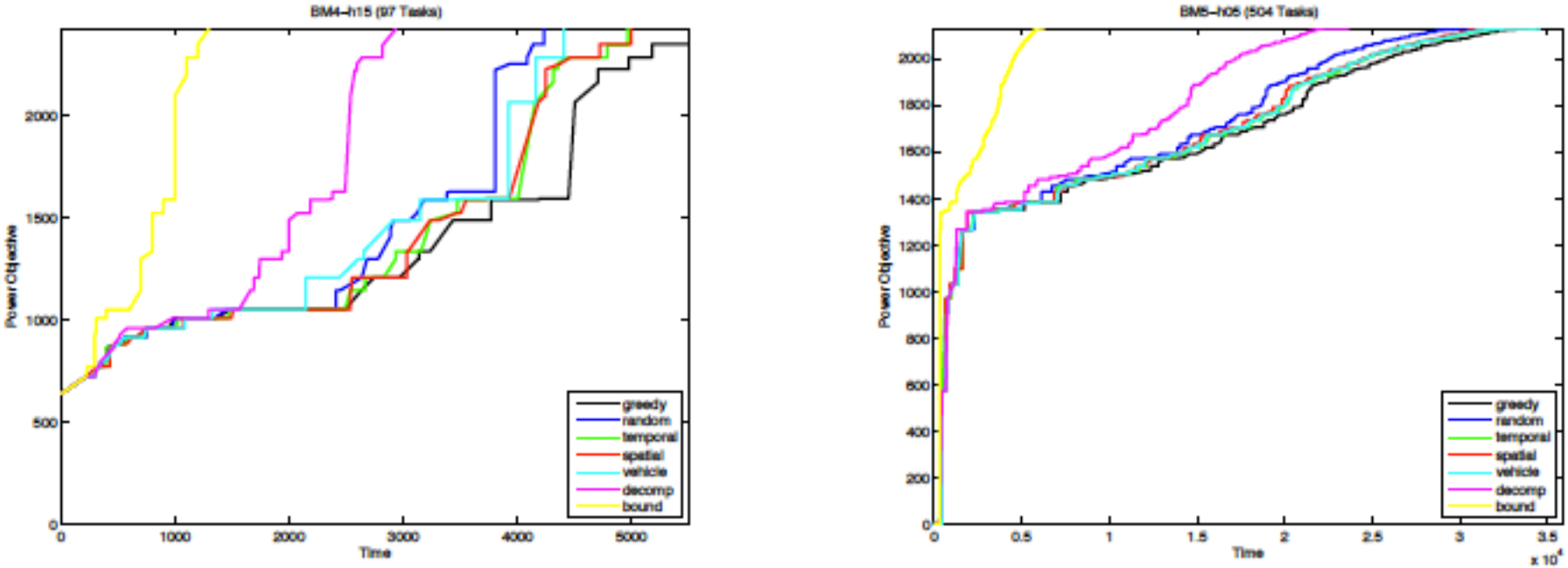


Fig. 4. Size of the Blackouts: 97 and 504 Tasks.

Power Restoration

Power Restoration

- A fundamental open question
  - Is this “optimal” restoration plan “feasible” operationally?

Power Restoration

- A fundamental open question
  - Is this "optimal" restoration plan "feasible" operationally?

- These are not "normal operating" conditions

  -"Maddeningly difficult" to find an AC solution in cold start contexts [Overbye et al, 2004]
- The ROP is stressing the network

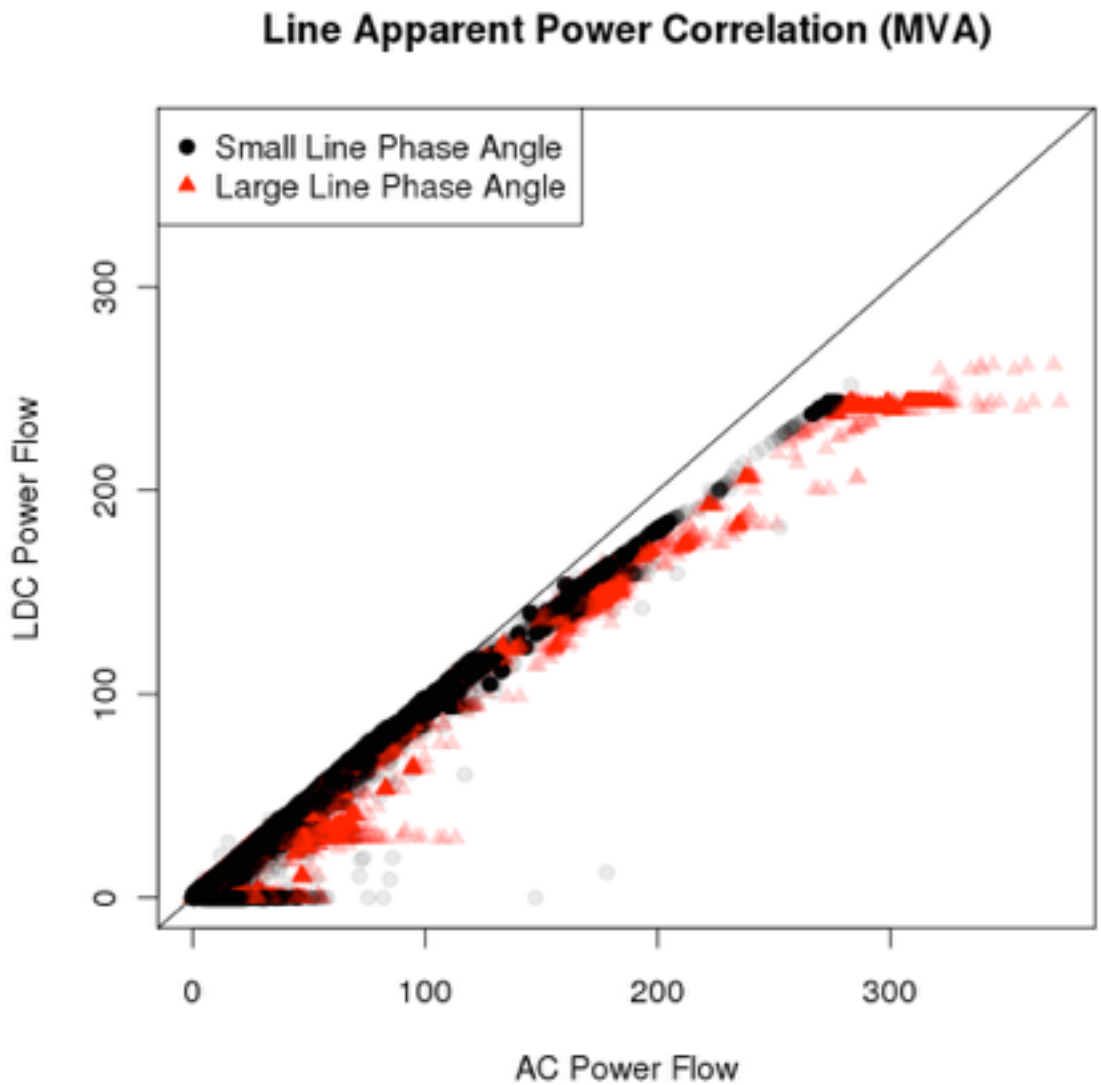
AC Power/Load Flow

- Seed a power flow study
  - with the optimization results
- Different information for different buses

	P	Q	V	δ
P-Q bus	known	known	unknown	unknown
P-V bus	known	unknown	known	unknown
Slack bus	unknown	unknown	known	known

Power Restoration

N-3 Contingencies (IEEE-30)



IEEE PES'12

N-10 Contingencies (IEEE-30)

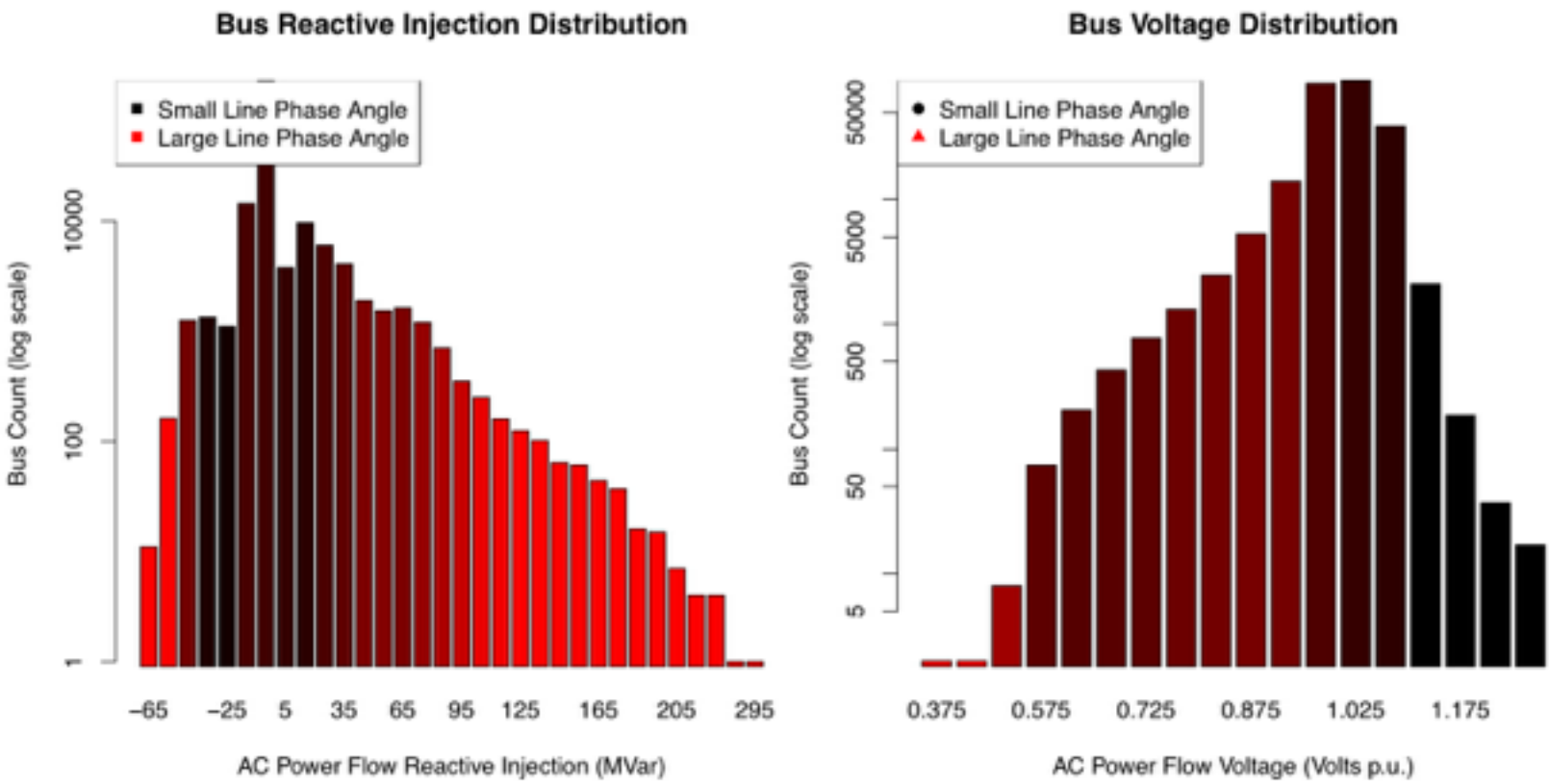


Fig. 3. Histograms of Reactive Injection (left) and Voltage Magnitudes (right) of N–10 Contingencies using the DC-GDP Algorithm.

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Expansion Planning

- Simplest problem formulation
  - starting point
- Network design problem
  - add lines to meet the increased load
- Under
  - voltage constraints
  - thermal limits
  - constraints on real and reactive power injection

Expansion Planning

	HAC-TNEP	DC-TNEP
Case	Cost	
6	160 (6)	
24	2310 (43)	
46	569810 (47)	
9	3 (3)	
14	15 (15)	
30	13 (13)	
39	47 (47)	
57	49 (49)	
118	37 (37)	

New Challenges

- Power applications that
  - are **mixed** nonlinear optimisation problems
  - require **accurate** (coupled?) models of the power system
    - congestion
- Observe that
  - the discrete nature precludes some technology
    - integrating discrete optimization and homotopy methods is not easy
  - need for relaxations!
    - performance guarantees

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Power Flows

- Complex Number Formulation

$$S_i^g - \boldsymbol{S}_i^d = \sum_{(i,j)\in E\cup E^R} S_{ij} \quad \forall i \in N$$

$$S_{ij} = \boldsymbol{Y}_{ij}^* V_i V_i^* - \boldsymbol{Y}_{ij}^* V_i V_j^* \quad (i,j) \in E \cup E^R$$

Power Flows

- Rectangular formulation

$$p_i^g - \boldsymbol{p}_i^d = \sum_{(i,j)\in E\cup E^R} p_{ij} \quad \forall i \in N$$

$$q_i^g - \boldsymbol{q}_i^d = \sum_{(i,j)\in E\cup E^R} q_{ij} \quad \forall i \in N$$

$$p_{ij} = \boldsymbol{g}_{ij}\left((v_i^R)^2 + (v_i^I)^2\right) - \boldsymbol{g}_{ij}\left(v_i^R v_j^R + v_i^I v_j^I\right) - \boldsymbol{b}_{ij}\left(v_i^I v_j^R - v_i^R v_j^I\right)$$

$$q_{ij} = -\boldsymbol{b}_{ij}\left((v_i^R)^2 + (v_i^I)^2\right) + \boldsymbol{b}_{ij}\left(v_i^R v_j^R + v_i^I v_j^I\right) - \boldsymbol{g}_{ij}\left(v_i^I v_j^R - v_i^R v_j^I\right)$$

Power Flows

- Hybrid formulation

$$p_i^g - \boldsymbol{p}_i^d = \sum_{(i,j)\in E\cup E^R} p_{ij} \quad \forall i \in N$$

$$q_i^g - \boldsymbol{q}_i^d = \sum_{(i,j)\in E\cup E^R} q_{ij} \quad \forall i \in N$$

$$p_{ij} = \boldsymbol{g}_{ij} v_i^2 - \boldsymbol{g}_{ij} v_i v_j \cos(\theta_i - \theta_j) - \boldsymbol{b}_{ij} v_i v_j \sin(\theta_i - \theta_j)$$

$$q_{ij} = -\boldsymbol{b}_{ij} v_i^2 + \boldsymbol{b}_{ij} v_i v_j \cos(\theta_i - \theta_j) - \boldsymbol{g}_{ij} v_i v_j \sin(\theta_i - \theta_j)$$

Power Flows

- Side constraints

$$\sqrt{p_{ij}^2 + q_{ij}^2} \leqslant \boldsymbol{s}_{ij}^{\boldsymbol{u}} \quad \forall (i,j) \in E \cup E^R$$

$$\boldsymbol{v}_i^{\boldsymbol{l}} \leqslant v_i \leqslant \boldsymbol{v}_i^{\boldsymbol{u}} \quad i \in N$$

$$-\boldsymbol{\theta}_{ij}^{\boldsymbol{\Delta}} \leqslant \theta_i - \theta_j \leqslant \boldsymbol{\theta}_{ij}^{\boldsymbol{\Delta}} \quad \forall (i,j) \in E$$

W-Formulation

- Define

$$W_{ij} = V_i V_j^*$$

- Power Flow becomes

$$S_i^g - \boldsymbol{S}_i^d = \sum_{(i,j)\in E\cup E^R} S_{ij} \quad \forall i \in N$$

$$S_{ij} = \boldsymbol{Y}_{ij}^* W_{ii} - \boldsymbol{Y}_{ij}^* W_{ij} \quad (i,j) \in E \cup E^R$$

$$W \succeq 0$$

$$\mathrm{rank}(W) = 1$$

W-Formulation

$$p_i^g - \boldsymbol{p}_i^d = \sum_{(i,j)\in E\cup E^R} p_{ij} \quad \forall i \in N$$

$$q_i^g - \boldsymbol{q}_i^d = \sum_{(i,j)\in E\cup E^R} q_{ij} \quad \forall i \in N$$

$$\begin{aligned} p_{ij} &= \mathbf{g}_{ij} w_{ii}^R - \mathbf{g}_{ij} w_{ij}^R - \mathbf{b}_{ij} w_{ij}^I \\ q_{ij} &= -\mathbf{b}_{ij} w_{ii}^R + \mathbf{b}_{ij} w_{ij}^R - \mathbf{g}_{ij} w_{ij}^I \end{aligned}$$

$$W \succeq 0$$

$$\mathrm{rank}(W) = 1$$

Dist-Flow Relaxation

- M. E. Baran and F. F. Wu, “Network reconfiguration in distribution systems for loss reduction and load balancing,” 1989.

$$p_{ij} = \sum_{k:(j,k)\in E} p_{jk} + \boldsymbol{r}_{ij} \frac{p_{ij}^2 + q_{ij}^2}{v_i^2}$$

$$q_{ij} = \sum_{k:(j,k)\in E} q_{jk} + \boldsymbol{x}_{ij} \frac{p_{ij}^2 + q_{ij}^2}{v_i^2}$$

$$v_j^2 = v_i^2 - 2(\boldsymbol{r}_{ij} p_{ij} + \boldsymbol{x}_{ij} q_{ij}) + (\boldsymbol{r}_{ij}^2 + \boldsymbol{x}_{ij}^2) \frac{p_{ij}^2 + q_{ij}^2}{v_i^2}$$

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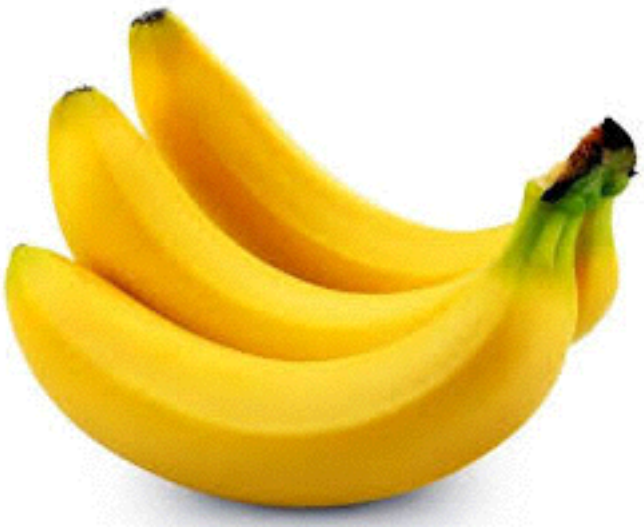
Complexity of AC-Feasibility

- AC-Feasibility is NP-Hard
  - D. Bienstock and A. Verma (2006)



- What are bananas?
  - small circuits that create discontinuities

Complexity of AC-Feasibility

- AC-Feasibility is NP-Hard
  - D. Bienstock and A. Verma (2006)
  - Introducing discontinuities

$$\max \sum_i p_i^d$$

subject to

$$p_i^g - p_i^d = \sum_{(i,j)} p_{ij}$$

$$p_{ij} = -\mathbf{b}_{ij} sin(\theta_i - \theta_j)$$

$$|\theta_i - \theta_j| \leq \theta^u$$

$$\mathbf{p}_i^{gl} \leq p_i^g \leq \mathbf{p}_i^{gu}$$

$$\mathbf{p}_i^{dl} \leq p_i^g \leq \mathbf{p}_i^{du}$$

Only extension to LDC needed to be NP-hard

Complexity of AC-Feasibility

- AC-Feasibility on Acyclic Networks is NP-Hard
  - K. Lehmann, A. Grastien, and P. Van Hentenryck

solve

$$\mathbf{p}_i^g - \mathbf{p}_i^d = \sum_{(i,j)} p_{ij}$$

$$\mathbf{q}_i^g - \mathbf{q}_i^d = \sum_{(i,j)} q_{ij}$$

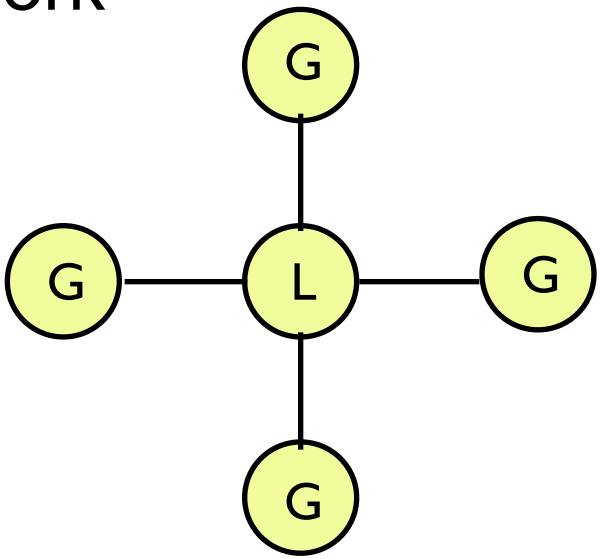
$$p_{ij} = \mathbf{g}_{ij}(1 - \cos(\theta_i - \theta_j)) - \mathbf{b}_{ij}\sin(\theta_i - \theta_j)$$

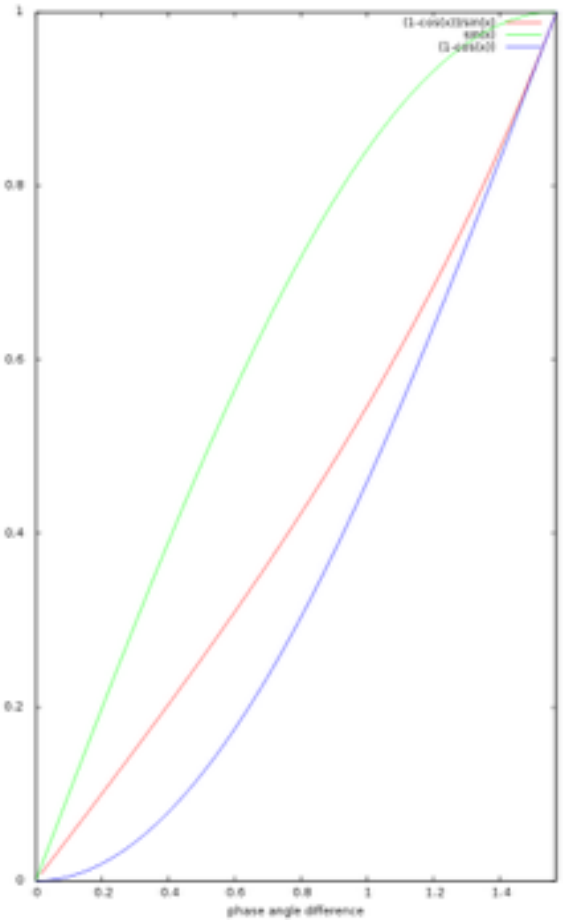
$$q_{ij} = -\mathbf{b}_{ij}(1 - \cos(\theta_i - \theta_j)) - \mathbf{g}_{ij}\sin(\theta_i - \theta_j)$$

$$|\theta_i - \theta_j| \leq \theta^u$$

AC-Feasibility on Acyclic Networks

- Star Network





- Key ideas

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The AC/DC Conundrum

- Find an approximation of AC power flows that
  - is more accurate than the LDC model
  - is useful outside normal operating conditions
  - reasons about voltage magnitudes and reactive power
  - can be embedded in discrete optimization solvers
    - mixed integer programming solvers

Trigonometric Functions

$$
\begin{aligned}
p_{ij} &= \mathbf{g}_{ij} v_i^2 - \mathbf{g}_{ij} v_i v_j \widehat{\cos}(\theta_i - \theta_j)) - \mathbf{b}_{ij} v_i v_j (\theta_i - \theta_j) \\
q_{ij} &= -\mathbf{b}_{ij} v_i^2 + \mathbf{b}_{ij} v_i v_j \widehat{\cos}(\theta_i - \theta_j)) - \mathbf{g}_{ij} v_i v_j (\theta_i - \theta_j)
\end{aligned}
$$

- Two approximations
  - sin(x) is replaced by x
  - cosine is replaced by its piecewise linear relaxation

Trigonometric Functions

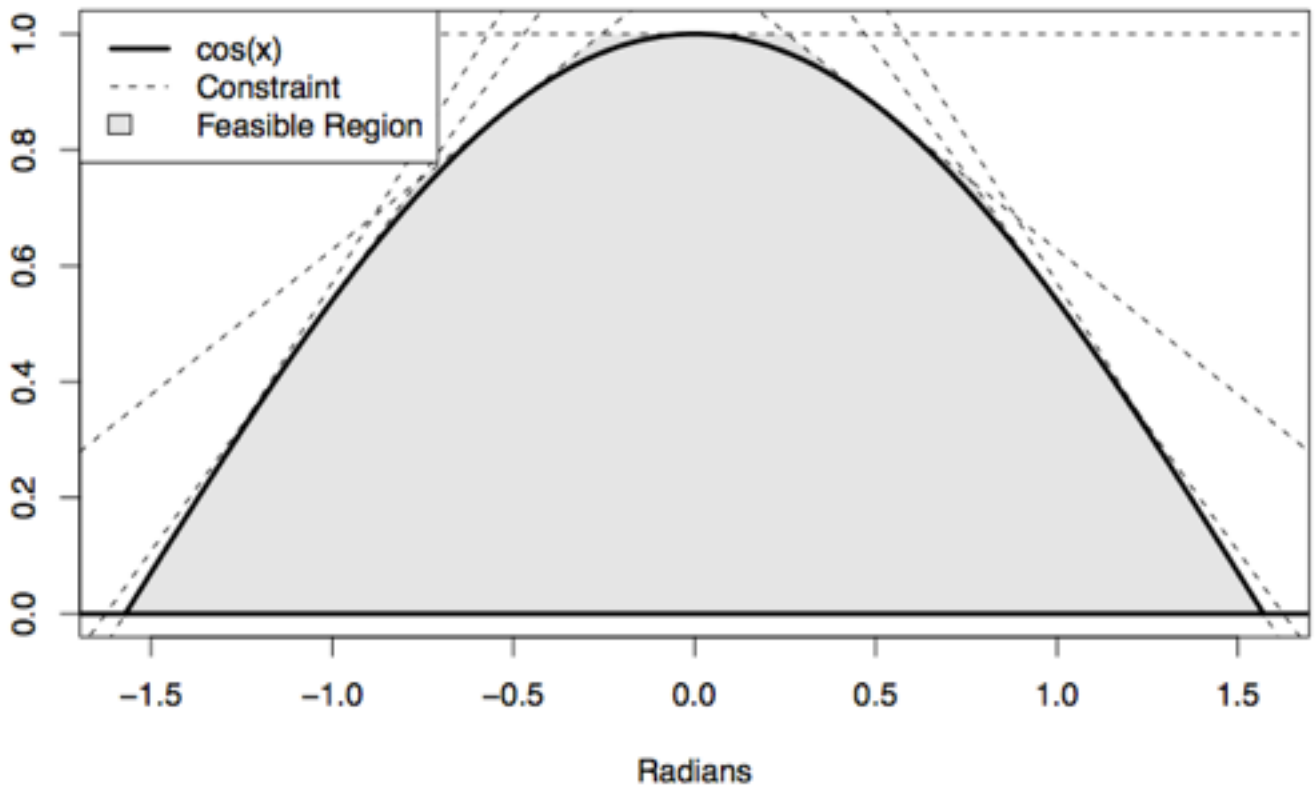


Fig. 1. A Piecewise-Linear Approximation of Cosine using 7 Inequalities.

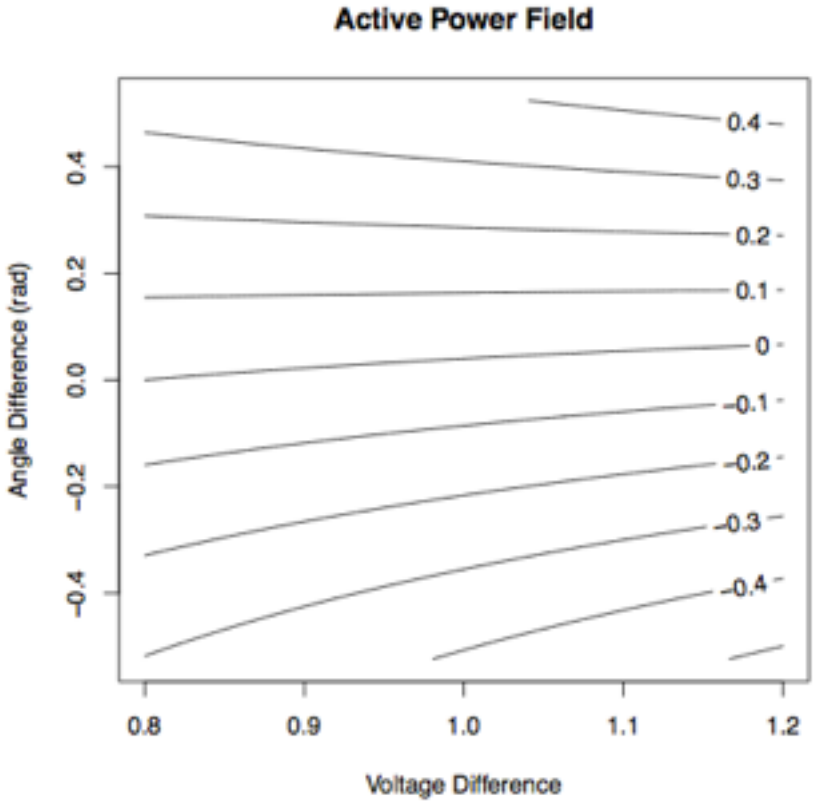
Voltage Magnitudes

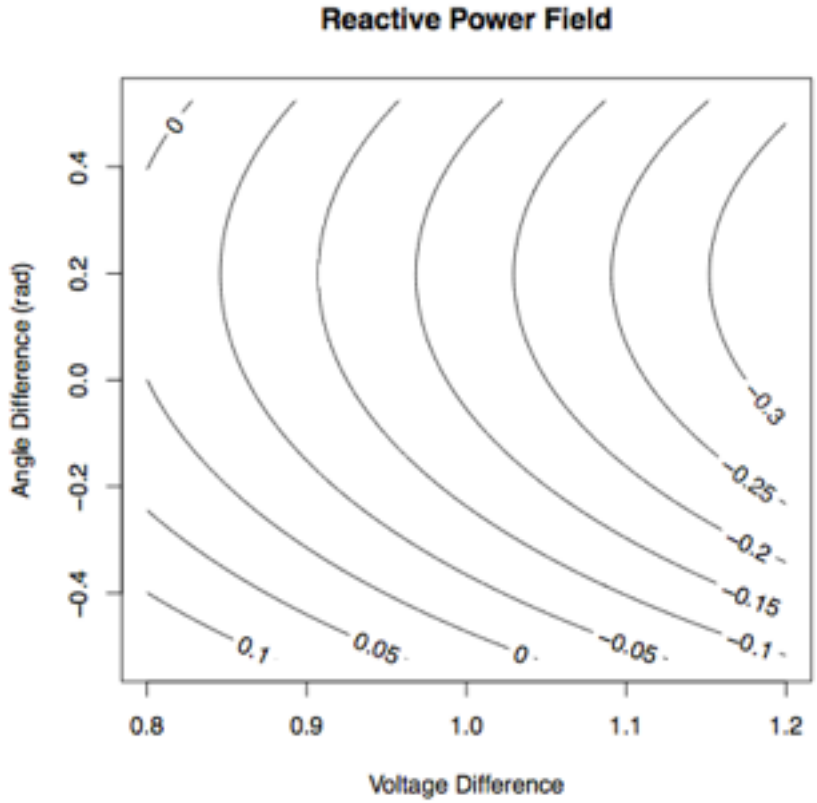
- Understanding power flows [Grainger, 94]
  - Phase angle differences determine active power
  - Voltage magnitude differences determine reactive power

- Experiments
  - Per unit system
  - Look at how the equations behave when
    - g = 0.2 and b = 1.0

$$v_i = 1.0, v_j \in (0.8, 1.2), (\theta_i - \theta_j) \in (-\pi/6, \pi/6)$$

Voltage Magnitudes





Voltage Magnitudes

- Key ideas
  - Substitute $v = \hat{v} + \phi$ into the power flow equations
  - First-order Taylor expansion to remove quadratic terms

$$p_{nm} = g_{nm} - g_{nm}\cos_{nm} - b_{nm}(\theta_n - \theta_m)$$
$$q_{nm} = -b_{nm} + b_{nm}\cos_{nm} - g_{nm}(\theta_n - \theta_m) - b_{nm}(\phi_n - \phi_m).$$

$$\cos_{nm} \in \widehat{\cos}(\theta_n - \theta_m).$$

Outline

- Motivation
- Power Flow Formulations
- Complexity
- The LPAC Approximation
- Case Studies for LPAC
- Convex Relaxations
- Case Studies for Convex Relaxations
- Beyond Steady States

Experimental Settings

- Bonmin (Bonami 2008)
  - heuristics for solving MINLPs.
  - outer approximation method for convex MINLPs
- QP and SOCP
  - CPLEX 12.8 or Gurobi
- Ipopt (Waechter and Biegler 2006)
  - NLP

Experimental Results

- Wide variety of IEEE and MATPOWER Benchmarks
  - ieee14,mp24,ieee30,mp30,mp39,ieee57,ieee118,ieedd17,mp300
  - Small benchmarks are easy in general
  - IEEE 118 is also easy
  - All LPAC models solved almost instantly (LPs)
- Comparison with an AC Solver
  - LDC and LPAC solutions versus an AC solution

Line Active Power

Table 1: Active Power Flow Accuracy Comparison

Benchmark	Active Power (MW)							
	Corr	$\mu(\Delta)$	$\max(\Delta)$	$\delta(\arg\max(\Delta))$	$\mu(\delta)$	$\max(\delta)$	$\Delta(\arg\max(\delta))$	approx(%)
The LDC Model								
ieee14	0.9994	1.392	10.64	6.783	6.052	24.33	0.3927	65
mp24	0.9989	5.659	19.7	23.65	6.447	29.89	6.656	47.06
ieee30	0.9993	1.046	13.1	7.562	6.406	31.23	0.5646	80.49
mp30	0.9993	0.2964	2.108	19.36	3.086	19.36	2.108	82.93
mp39	0.9995	7.341	43.64	6.527	9.566	52.18	12.86	76.09
ieee57	0.9989	1.494	8.216	8.055	105.8	4193	0.9607	52.56
ieee118	0.9963	3.984	56.3	44.74	29.2	445.9	6.526	51.96
ieeedd17	0.9972	4.933	201.3	13.84	15.2	215	0.5265	50.71
ieeedd17m	0.9975	4.779	191.1	13.23	14.56	231.3	3.066	51.43
mp300	0.991	11.09	418.5	90.02	29.35	2859	46.14	67.73
The LPAC-Cold Model								
ieee14	0.9989	1.636	5.787	13.13	11.52	35.67	2.623	40
mp24	0.9999	1.884	6.159	2.933	3.871	17.23	3.837	41.18
ieee30	0.9998	0.5475	2.213	2.523	5.751	31.33	0.5666	75.61
mp30	0.9995	0.2396	1.641	15.07	2.402	15.07	1.641	78.05
mp39	1	2.142	8.043	3.288	4.357	24.78	6.106	43.48
ieee57	0.9995	0.9235	4.674	9.728	110.1	4500	1.031	46.15
ieee118	1	0.622	3.708	2.038	5.318	99.61	0.5519	55.31
ieeedd17	0.9999	1.827	30.38	2.088	10.92	420.2	1.029	55.36
ieeedd17m	0.9999	1.475	20.21	1.399	7.766	144.5	2.547	56.79
mp300	0.9998	2.455	18	8.675	7.104	337.2	6.95	57.21

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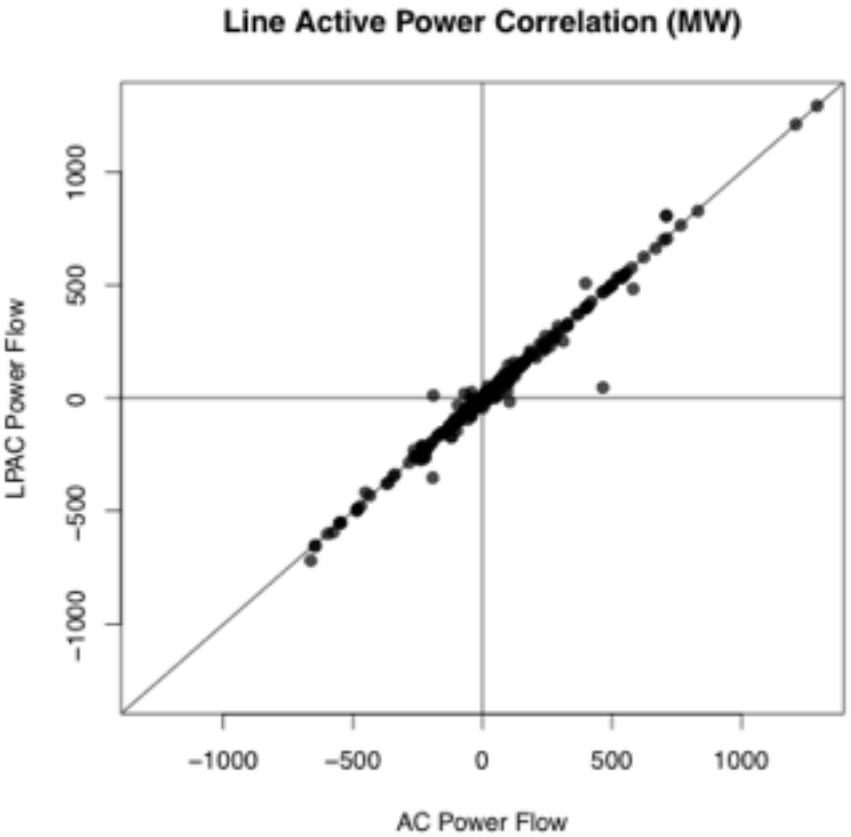
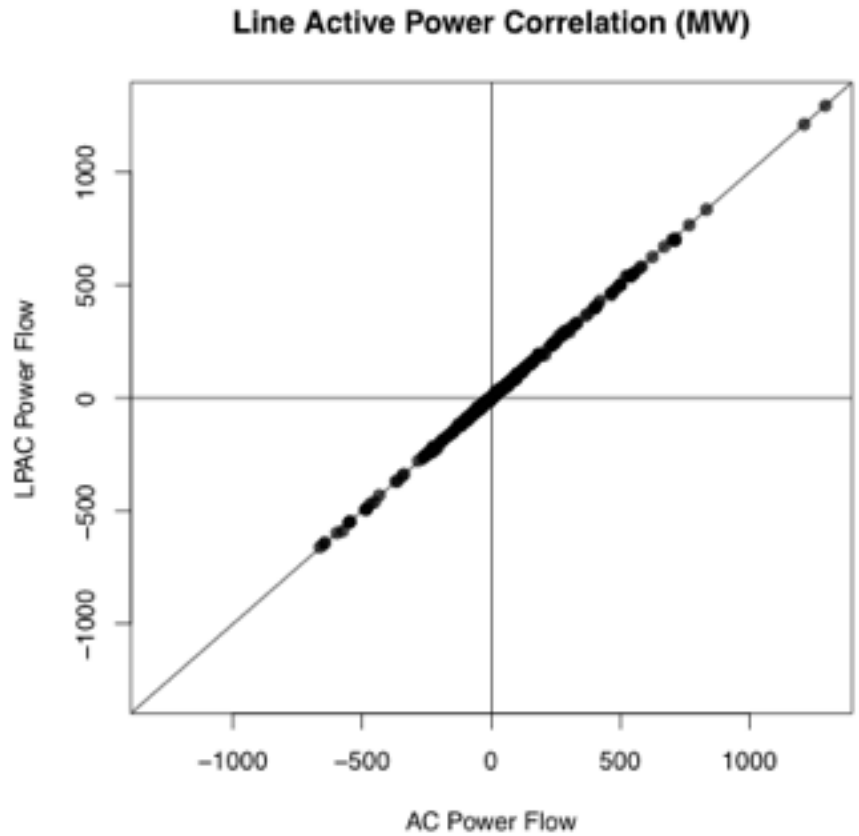
Line Active Power

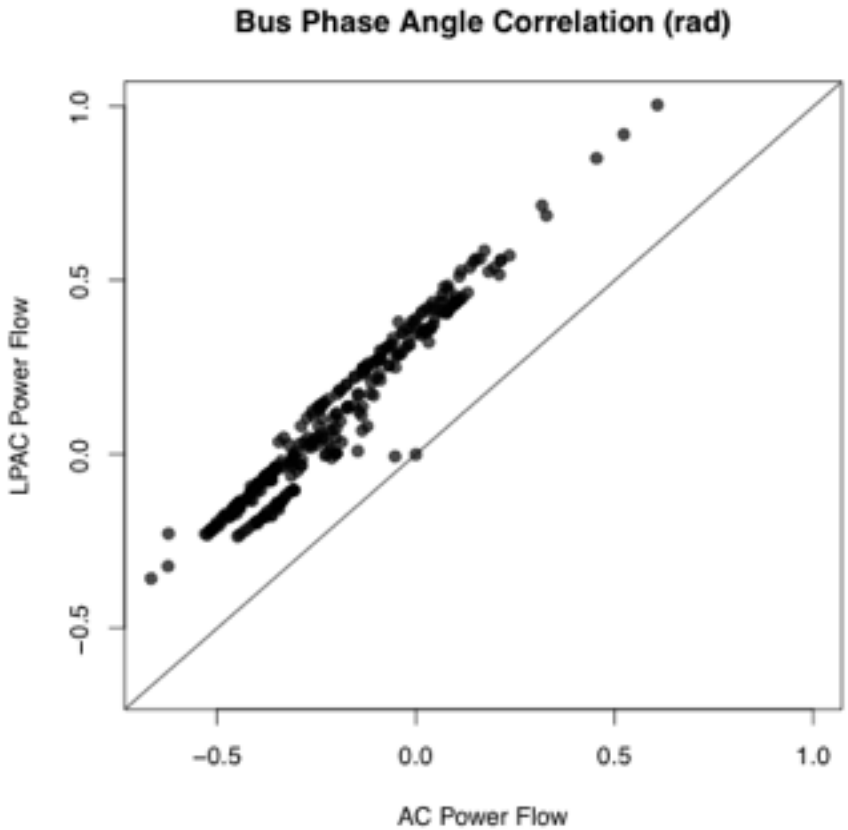


DC Model

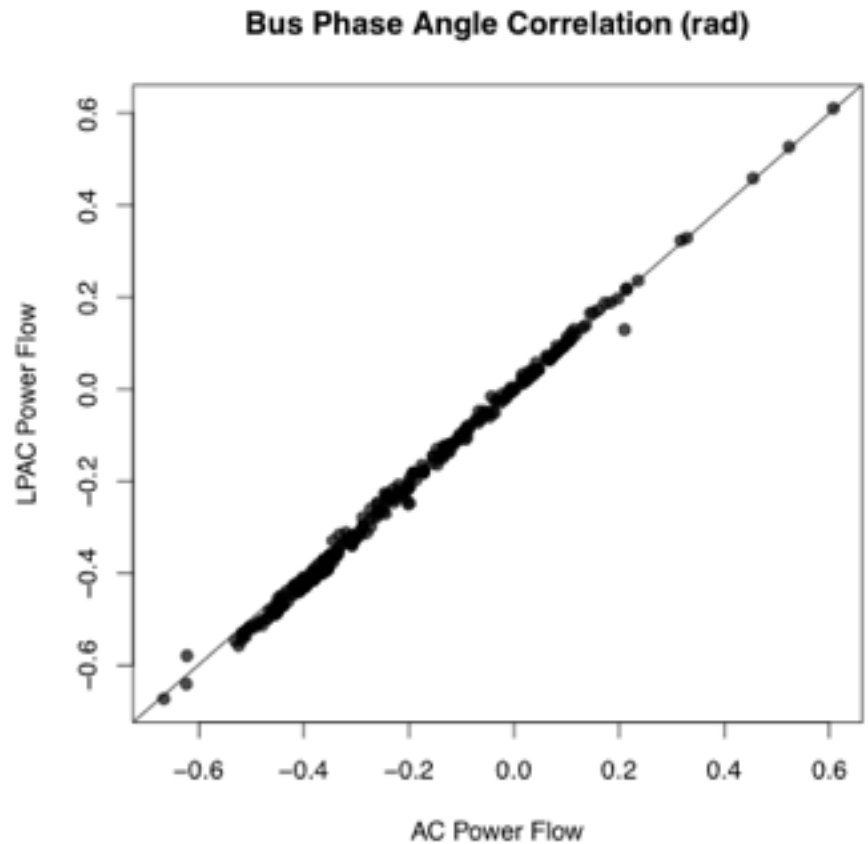


LPAC Model

Bus Angles



DC Model



LPAC Model

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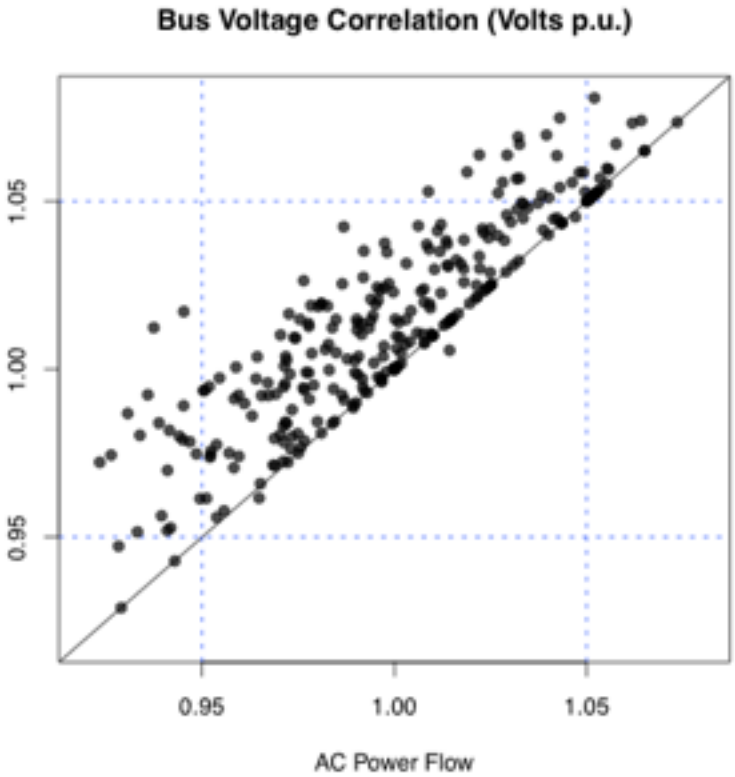
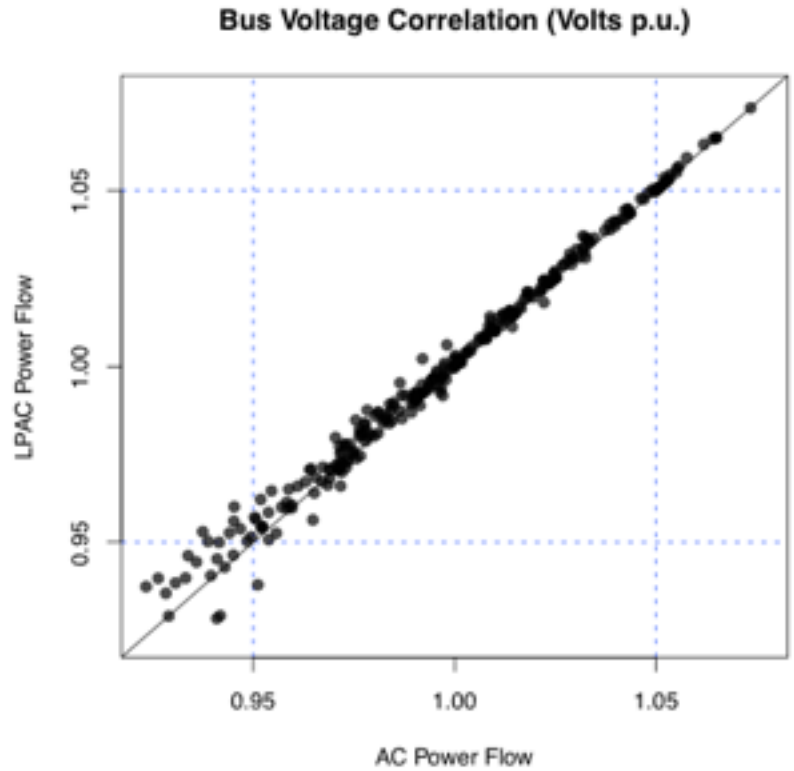
Importance of cos: Reactive Power





Cold-Start LPAC Model (cos=1)

Cold-Start LPAC Model

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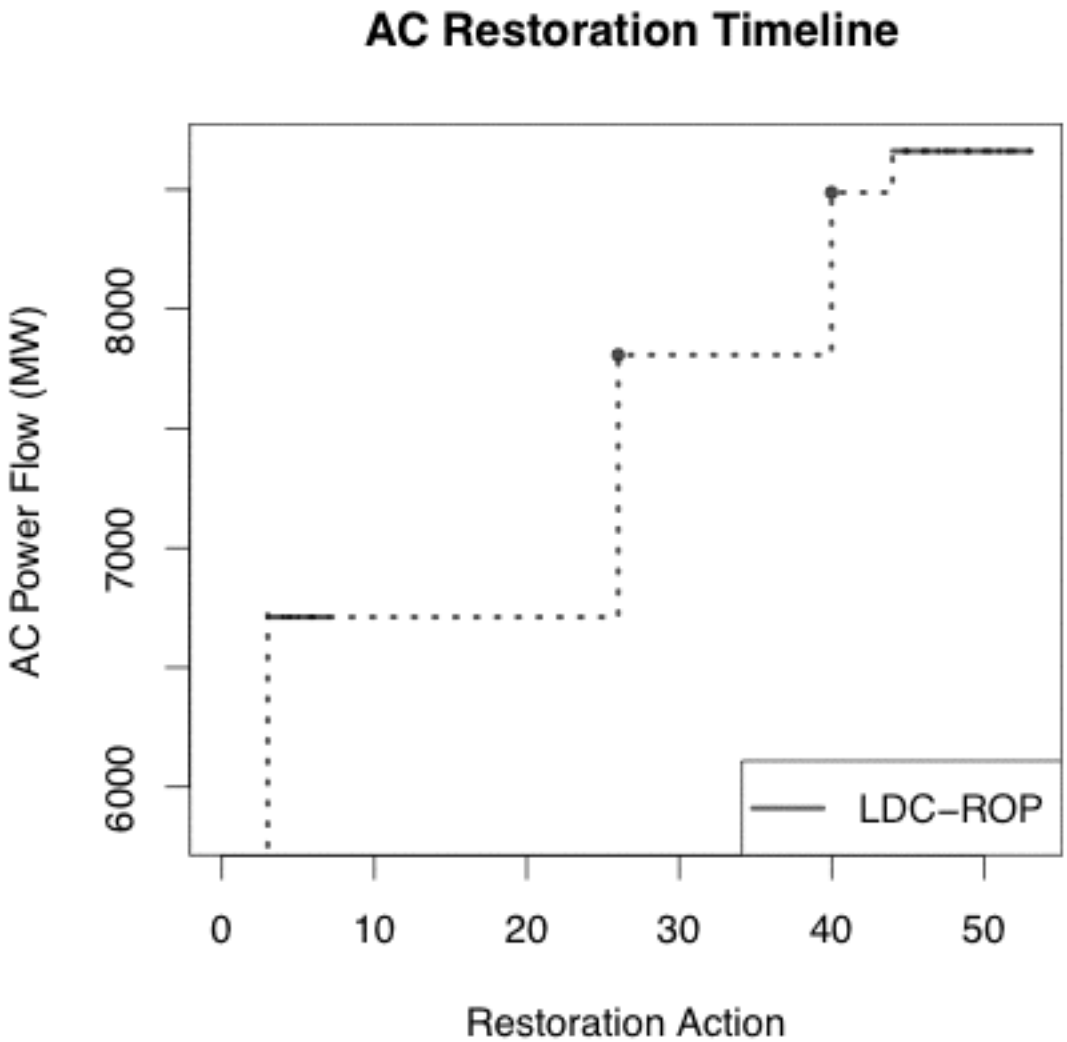
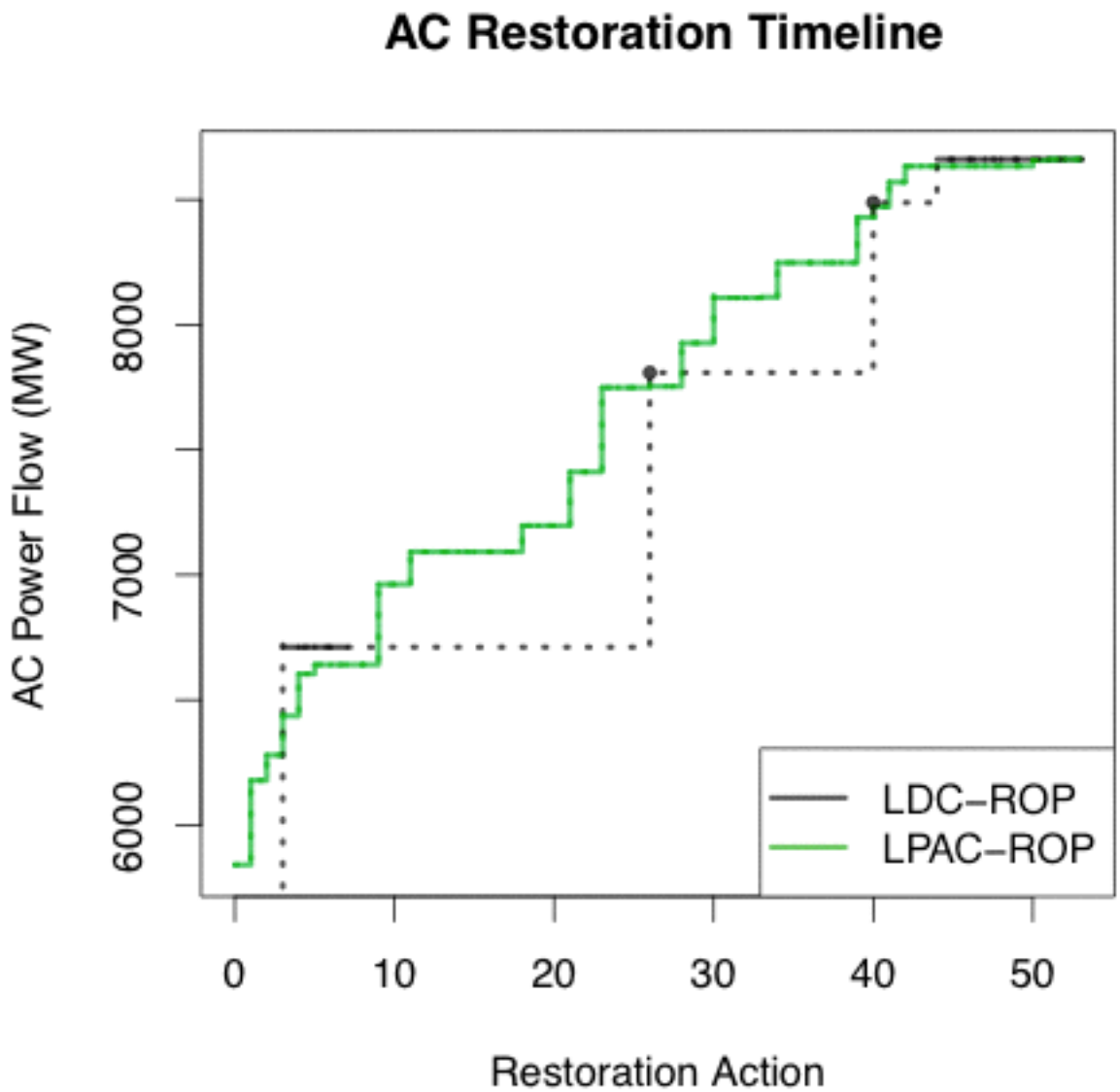
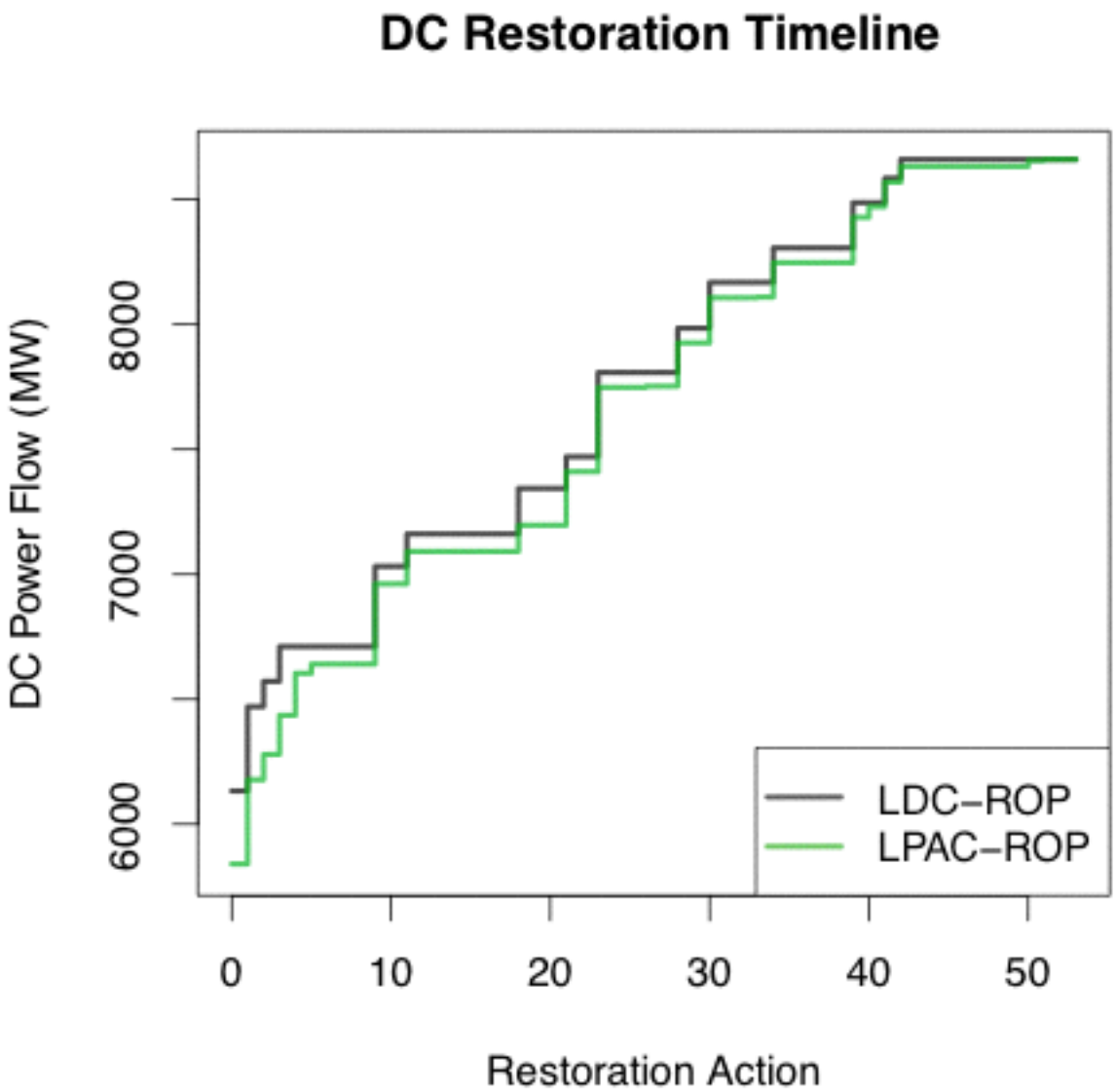
Power Restoration



Power Restoration



Power Restoration



DC versus LPAC in Restoration

Table 2 Quality of the DC-ROP and LPAC-ROP with a Fixed Restoration Order.

Model	Inst. Solved	Models Solved	Violations		
			MLCV	MRIV	MVV
All Instances					
DC	13/18	333/404	301.2	360.6	0.06073
LPAC	18/18	404/404	1.656	13.81	0
Group A - All E-DC-LPP dispatches are AC-feasible.					
DC	13/13	196/196	1	330.8	0.0005223
LPAC	13/13	196/196	1.261	12.41	0
Group B - Most DC-LPP dispatches are AC-feasible.					
DC	0/3	75/105	203.3	435.1	0.08339
LPAC	3/3	105/105	1.508	14.29	0
Group C - Few DC-LPP dispatches are AC-feasible.					
DC	0/2	62/103	331.3	341.4	0.1522
LPAC	2/2	103/103	2.561	15.98	0

DC versus LPAC in Restoration

How good is the DC ordering?

Table 3 Comparison of Blackout Sizes Produced by the DC-ROP and LPAC-ROP.

ID	$\lvert\mathcal{R}\rvert$	DC-ROP		LPAC-ROP	Relative Difference
		DC-LPP	LPAC-LPP		
1	61	35,320	73,807	47,687	64.61%
16	53	33,104	60,655	53,138	87.61%
3	41	12,906	31,581	22,963	72.71%
13	40	14,655	35,468	31,267	88.15%
12	36	16,954	41,442	41,415	99.94%
2	32	6,508	20,515	13,459	65.60%
4	24	2,245	11,956	5,230	43.75%

Expansion Planning

- Simplest problem formulation
  - starting point
- Network design problem
  - add lines to meet the increased load
- Under
  - voltage constraints
  - thermal limits
  - constraints on real and reactive power injection

Expansion Planning

Minimize:

$$\sum_{\langle n,m\rangle \in L} c_{nm}(z_{nm} - 1) + \sum_{\langle n,m\rangle \in L^+} c_{nm} z_{nm} \qquad \text{(M1.1)}$$

Subject To:

$$\Im(\widetilde{V}_s) = 0 \qquad \text{(M1.2)}$$

$$\underline{|\widetilde{V}_n|} \leq |\widetilde{V}_n| \leq \overline{|\widetilde{V}_n|};\ \forall n \in N \qquad \text{(M1.3)}$$

$$p_n^g - p_n^l = \sum_{m \in L(n)} z_{nm} p_{nm} \quad \forall n \in N \qquad \text{(M1.4)}$$

$$q_n^g - q_n^l = \sum_{m \in L(n)} z_{nm} q_{nm} \quad \forall n \in N \qquad \text{(M1.5)}$$

$$1 \leq z_{nm} \quad \forall \langle n,m\rangle \in L \qquad \text{(M1.6)}$$

$$\forall \langle n,m\rangle \in L \cup L^r \cup L^+ \cup L^{+r}$$

$$p_{nm} = g_{nm}|\widetilde{V}_n|^2 - g_{nm}\Re(\widetilde{V}_n\widetilde{V}_m^*) - b_{nm}\Im(\widetilde{V}_n\widetilde{V}_m^*) \qquad \text{(M1.7)}$$

$$q_{nm} = -b_{nm}|\widetilde{V}_n|^2 + b_{nm}\Re(\widetilde{V}_n\widetilde{V}_m^*) - g_{nm}\Im(\widetilde{V}_n\widetilde{V}_m^*) \qquad \text{(M1.8)}$$

$$p_{nm}^2 + q_{nm}^2 \leq \overline{|\widetilde{S}_{nm}|}^2 \qquad \text{(M1.9)}$$

Heuristic AC Method (HAC)

- Many proposals

R. Bent, G. Toole, and A. Berscheid, "Transmission network expansion planning with complex power flow models," *IEEE Transactions on Power Systems*, vol. 27, no. 2, pp. 904–912, 2012.

- Destructive heuristics
  - start with all the possible lines (feasible solution)
  - consider each line l in turn for removal
    - if feasible in AC model, remove the line l
  - order the line by increasing relative load
- Outperforms the state of the art
  - constructive methods

Case Studies

- Traditional benchmarks

L. L. Garver, "Transmission network estimation using linear programming," *Power Apparatus and Systems, IEEE Transactions on*, no. 7, pp. 1688–1697, 1970.

S. Haffner, A. Monticelli, A. Garcia, J. Mantovani, and R. Romero, "Branch and bound algorithm for transmission system expansion planning using a transportation model," *IEE Proceedings-Generation, Transmission and Distribution*, vol. 147, no. 3, pp. 149–156, 2000.

- New benchmarks
  - MathPower benchmarks
  - load and generation scaled by a factor of 3
  - reaction injection is half of the real injection
  - cost is 1

Measuring Accuracy

- Thermal limits

$$\max\left(\frac{\sqrt{p_{nm}^2 + q_{nm}^2}}{S_{nm}^u}, \frac{\sqrt{p_{mn}^2 + q_{mn}^2}}{S_{mn}^u}\right)$$

- Voltage magnitudes

$$\max(0, \mathbf{v}_n^u - v_n, v_n - \mathbf{v}_n^l)$$

Core Results

Case	HAC-TNEP Cost	DC-TNEP Cost	Capacity vio. Max (MVA)	Capacity vio. Avg. (MVA)	Vol. vio. Max
6	160 (6)	110 (4)	137.32	(5) 121.90	0.13188
24	2310 (43)	152 (5)	AC-PF did not converge		
46	569810 (47)	89889 (9)	AC-PF did not converge		
9	3 (3)	2 (2)	136	(5) 124.07	0.18994
14	15 (15)	1 (1)	120.44	(2) 116.17	0.11955
30	13 (13)	5 (5)	192.86	(16) 128.21	0.06847
39	47 (47)	26 (26)	113.46	(8) 108.21	0.08713
57	49 (49)	1 (1)	AC-PF did not converge		
118	37 (37)	5 (5)	172.03	(17) 113.36	0.08198

Case	LPAC-TNEP Cost	Capacity vio. Max (MVA)	Capacity vio. Avg. (MVA)	Vol. vio. Max
6	130 (5)	0	(0) 0.00	0.00347
24	689 (17)	101.02	(1) 101.02	0.00435
46	310688 (29)	102.96	(1) 102.96	0
9	2 (2)	104.45	(1) 104.45	0.01587
14	4 (4)	0	(0) 0.00	0
30	10 (10)	110.84	(5) 103.55	0.00303
39	28 (28)	103.01	(10) 100.80	0.01368
57	39 (39)	111.15	(2) 106.98	0.01785
118	7 (7)	116.97	(13) 105.96	0.00259

Constraint Tightening (10%)

Case	HAC-TNEP Cost	DC-TNEP Cost	Capacity vio. Max (MVA)	Capacity vio. Avg. (MVA)	Vol. vio. Max
6	160 (6)	130 (5)	105.91	(3) 105.91	0.05778
24	2378 (44)	266 (8)	AC-PF did not converge		
46	569810 (47)	130110 (13)	AC-PF did not converge		
9	3 (3)	2 (2)	126.24	(3) 121.07	0.17817
14	15 (15)	1 (1)	0	(0) 0.00	0.11628
30	21 (21)	8 (8)	158.68	(13) 119.80	0.07081
39	48 (48)	32 (32)	108.32	(1) 108.32	0.07016
57	50 (50)	4 (4)	AC-PF did not converge		
118	39 (39)	7 (7)	158.42	(6) 120.16	0.07591

Case	LPAC-TNEP Cost	Capacity vio. Max (MVA)	Capacity vio. Avg. (MVA)	Vol. vio. Max
6	130 (5)	0	(0) 0.00	0.00498
24	681 (15)	0	(0) 0.00	0.00484
46	316551 (30)	0	(0) 0.00	0.00097
9	3 (3)	0	(0) 0.00	0.01219
14	4 (4)	0	(0) 0.00	0
30	12 (12)	106.62	(2) 105.03	0.0057
39	34 (34)	0	(0) 0.00	0.00696
57	42 (42)	0	(0) 0.00	0.01268
118	11 (11)	112.99	(2) 107.70	0.00294

VAr Compensation

E. Kahn and R. Baldick, "Reactive power is a cheap constraint," *The Energy Journal*, no. 4, pp. 191–202, 1994.

- VAr compensation is cheaper
  - can be used to meet voltage bounds
- Case study: Perfect Voltage Profile (PVP)
  - unlimited Var compensation at each bus
  - bus becomes synchronous condenser
    - unlimited reactive power injection, voltage set-point at 1.0
  - DC model
    - VAr compensation used in second step (cross-over)
  - LPAC model
    - a single, integrated model

VAr Compensation

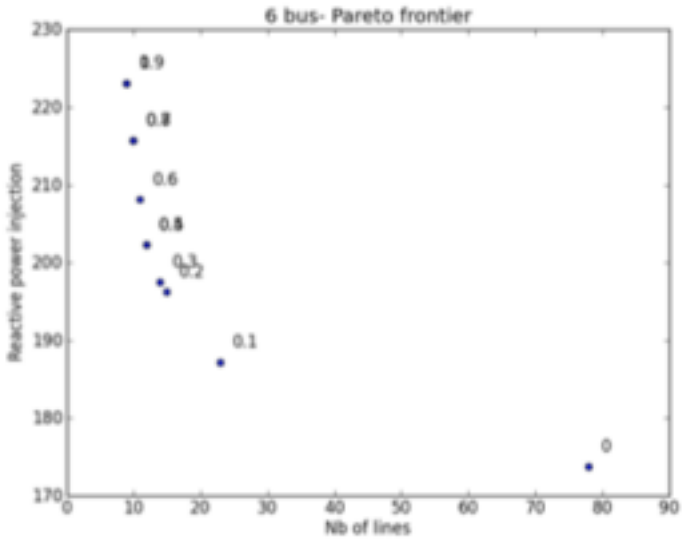
Case	HAC-PVP-TNEP Cost	DC-PVP-TNEP Cost	Capacity vio. Max (MVA)	Capacity vio. Avg. (MVA)	Vol. vio. Max
6	130 (5)	110 (4)	104.65	(4) 104.52	0
24	573 (10)	152 (5)	111.97	(9) 104.79	0
46	277592 (22)	89889 (9)	129.86	(10) 108.26	0
9	2 (2)	2 (2)	105.86	(4) 102.36	0
14	2 (2)	1 (1)	113.41	(2) 108.38	0
30	8 (8)	5 (5)	119.61	(9) 109.88	0
39	24 (24)	20 (20)	100.41	(7) 100.31	0
57	2 (2)	0 (0)	122.95	(3) 112.11	0
118	2 (2)	1 (1)	160.31	(17) 113.99	0

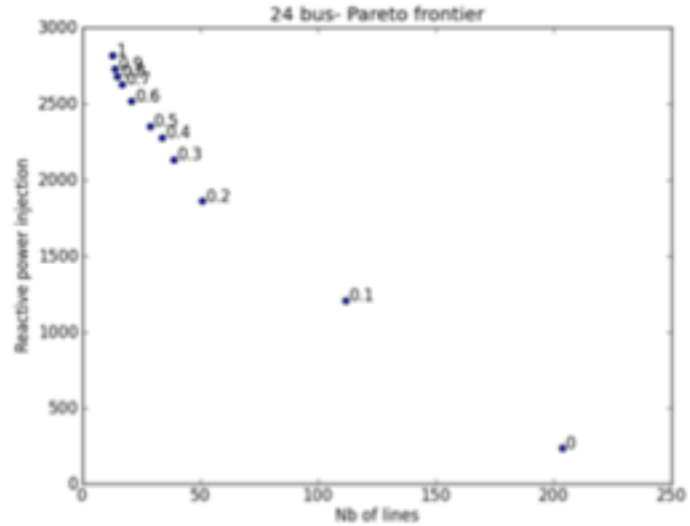
Case	LPAC-PVP-TNEP Cost	Capacity vio. Max (MVA)	Capacity vio. Avg. (MVA)	Vol. vio. Max
6	130 (5)	0	(0) 0.00	0
24	218 (6)	0	(0) 0.00	0
46	128948 (13)	100.41	(1) 100.41	0
9	2 (2)	0	(0) 0.00	0
14	1 (1)	0	(0) 0.00	0
30	7 (7)	100.49	(2) 100.49	0
39	22 (22)	0	(0) 0.00	0
57	2 (2)	0	(0) 0.00	0
118	2 (2)	0	(0) 0.00	0

VAr Compensation

Case	HAC-PVP-TNEP		DC-PVP-TNEP		LPAC-PVP-TNEP	
	P (MW)	Q (MVAr)	P	Q	P	Q
6	772.79	279.9	781.9	370.98	772.46	276.58
24	8819.93	4140.34	8802.53	3979.02	8758.64	3824.76
46	7313.82	3618.02	7806.4	5740.78	7549.26	4443.6
9	977.22	659.86	973.11	635.66	965.52	626.66
14	826.62	630.82	821.3	510.91	785.68	414.99
30	586.74	671.41	595.57	695.46	586.61	623.85
39	18932.56	8398.51	18937.74	8249.37	18937.61	8518.99
57	3890.33	1985.7	3926.14	2113.43	3900.46	1986.25
118	13170.39	7615.2	13249.42	8023.94	13047.29	7067.46

VAr Compensation





	LPAC-PVP-TNEP				
		Q injection	Capacity vio. (MVA)		Vol. vio.
λ	Cost	(MVar)	Max	Avg.	Max
0	10834 (204)	231.82	0	(0) 0.00	0
0.1	5609 (112)	1201.39	0	(0) 0.00	0
0.2	2310 (51)	1856.36	0	(0) 0.00	0
0.3	1642 (39)	2126.24	0	(0) 0.00	0
0.4	1376 (34)	2270.73	0	(0) 0.00	0
0.5	1214 (29)	2344.62	0	(0) 0.00	0
0.6	910 (21)	2511.46	0	(0) 0.00	0
0.7	745 (17)	2620.82	0	(0) 0.00	0
0.8	645 (15)	2672.72	0	(0) 0.00	0
0.9	573 (14)	2722.19	0	(0) 0.00	0
1	509 (13)	2810.23	0	(0) 0.00	0

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- Case Studies for Convex Relaxations
- Beyond Steady States

SDP Relaxation

- Lavaei, J., Low, S.: Zero duality gap in optimal power flow problem. (2012)

$$S_i^g - \boldsymbol{S}_i^d = \sum_{(i,j)\in E\cup E^R} S_{ij} \quad \forall i \in N$$

$$S_{ij} = \boldsymbol{Y}_{ij}^* W_{ii} - \boldsymbol{Y}_{ij}^* W_{ij} \quad (i,j) \in E \cup E^R$$

$$W \succeq 0$$

SOCP Relaxation

- Jabr, R.: Radial distribution load flow using conic programming. (2006)
- Sojoudi, S., Lavaei, J.: Network Topologies Guaranteeing Zero Duality Gap for Optimal Power Flow Problem.(2012)

$$\begin{bmatrix} w_{ii} & w_{ij} \\ w_{ji} & w_{jj} \end{bmatrix} \succeq 0$$

SOCP Relaxation

- Jabr, R.: Radial distribution load flow using conic programming. (2006)
- Sojoudi, S., Lavaei, J.: Network Topologies Guaranteeing Zero Duality Gap for Optimal Power Flow Problem.(2012)

$$
\begin{aligned}
w_{ii} &\geq 0 \\
w_{ii} w_{jj} &\geq (w_{ij}^R)^2 + (w_{ij}^I)^2
\end{aligned}
$$

Dist-Flow Relaxation

$$p_{ij} = \sum_{k:(j,k)\in E} p_{jk} + \boldsymbol{r}_{ij}\frac{p_{ij}^2 + q_{ij}^2}{v_i^2}$$

$$q_{ij} = \sum_{k:(j,k)\in E} q_{jk} + \boldsymbol{x}_{ij}\frac{p_{ij}^2 + q_{ij}^2}{v_i^2}$$

only appear squared

$$v_j^2 = v_i^2 - 2(\boldsymbol{r}_{ij}p_{ij} + \boldsymbol{x}_{ij}q_{ij}) + (\boldsymbol{r}_{ij}^2 + \boldsymbol{x}_{ij}^2)\frac{p_{ij}^2 + q_{ij}^2}{v_i^2}$$

- Define

$$l_{ij} = \frac{p_{ij}^2 + q_{ij}^2}{vv_i}$$

Dist-Flow Relaxation

- M. Farivar et al: Inverter var control for distribution systems with renewables, 2011

$$p_{ij} = \sum_{k:(j,k)\in E} p_{jk} + \boldsymbol{r}_{ij} l_{ij}$$

$$q_{ij} = \sum_{k:(j,k)\in E} q_{jk} + \boldsymbol{x}_{ij} l_{ij}$$

$$vv_j = vv_i - 2(\boldsymbol{r}_{ij} p_{ij} + \boldsymbol{x}_{ij} q_{ij}) + (\boldsymbol{r}_{ij}^2 + \boldsymbol{x}_{ij}^2) l_{ij}$$

$$p_{ij}^2 + q_{ij}^2 \leqslant l_{ij} vv_i$$

Transcendental Relaxation

- Key ideas
  - compositional, structural relaxation
  - relax trigonometric functions
  - exploit the narrow bounds in power systems

$$\boldsymbol{\theta}^u \leqslant \pi/2.$$

  - dual modelling
- Resulting optimization model
  - quadratic and convex

Power Flow Equations

- Ohm's law for each line (i,j)

$$p_{ij} = \mathbf{g}_{ij} v_i^2 - \mathbf{g}_{ij} v_i v_j \cos(\theta_i - \theta_j) - \mathbf{b}_{ij} v_i v_j \sin(\theta_i - \theta_j)$$

$$q_{ij} = -\mathbf{b}_{ij} v_i^2 + \mathbf{b}_{ij} v_i v_j \cos(\theta_i - \theta_j) - \mathbf{g}_{ij} v_i v_j \sin(\theta_i - \theta_j)$$

- Kirchhoff's current law for each bus i

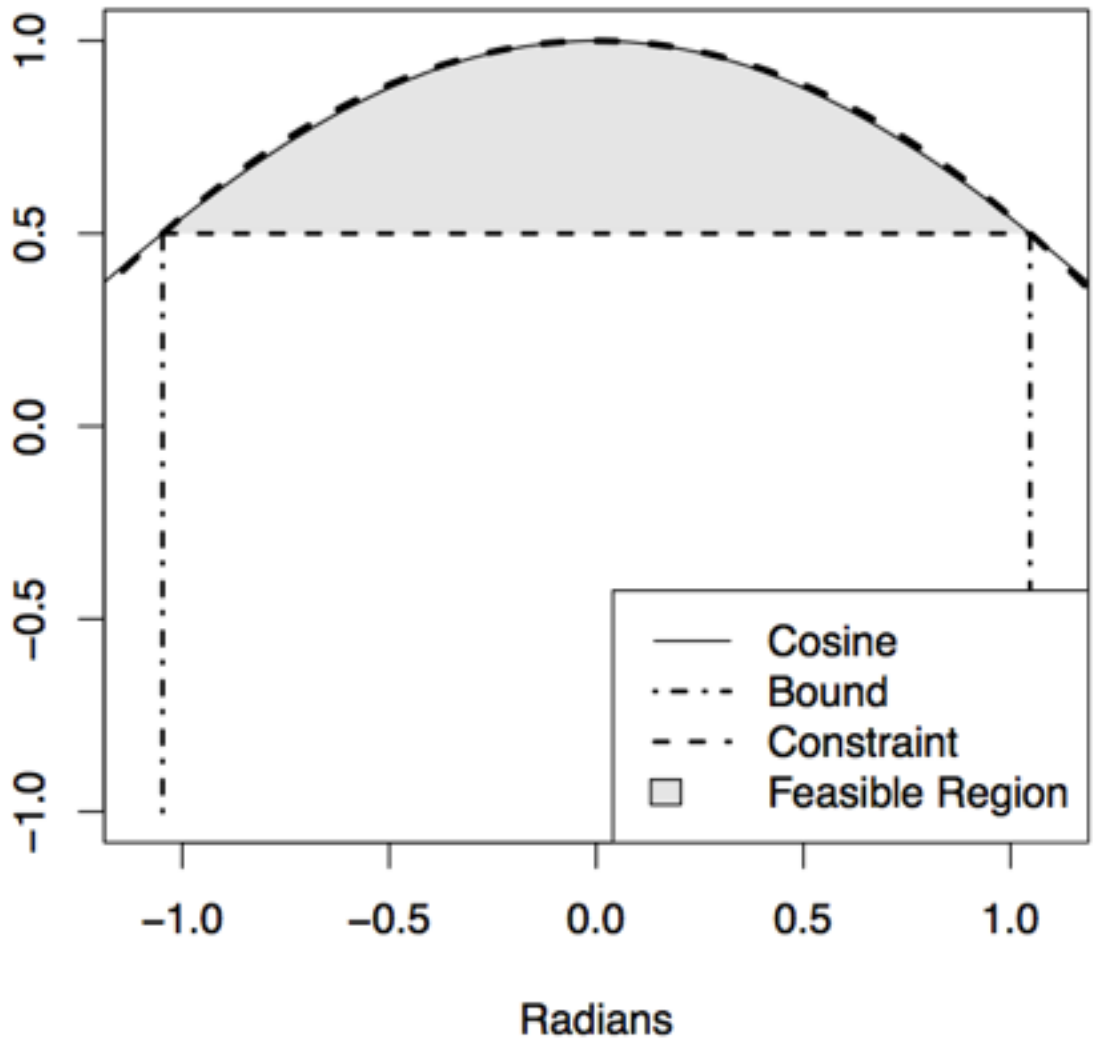
$$p_i = \sum_{(i,j) \in E} p_{ij}$$

$$q_i = \sum_{(i,j) \in E} q_{ij}$$

Convex Quadratic Relaxation

- Quadratic convex relaxation of
  - cosine function
  - square function
- Polyhedral relaxation of
  - sine function
- MacCormick relaxation of
  - multi-linear terms

QP Relaxation of Cosine



QP Relaxation of Cosine

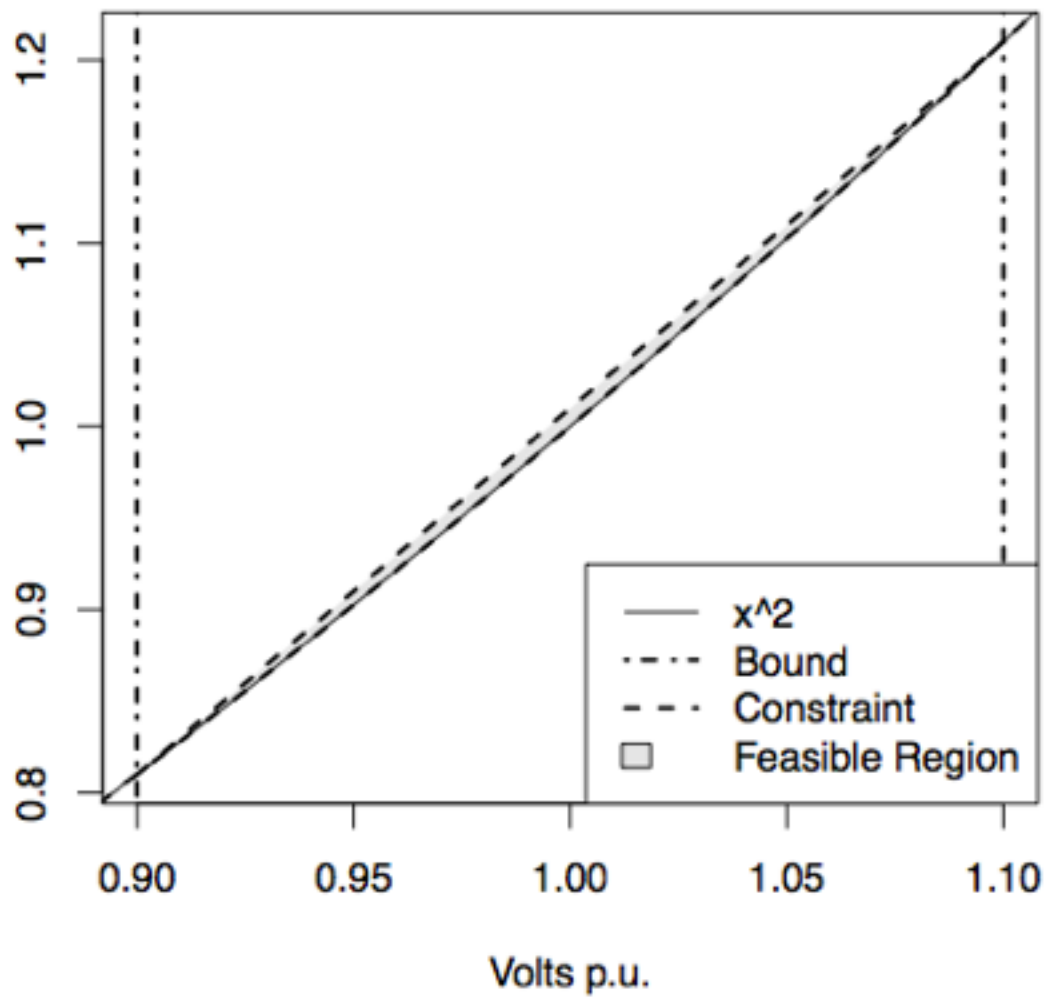
- Convex quadratic relaxation of cosine

$$\widetilde{\cos}(\theta) = 1 - \left(\frac{1 - \cos(\theta^u)}{(\theta^u)^2} \right) \theta^2.$$

- Proposition:

$$\forall \theta \in [-\theta^u,\ \theta^u] :\ \widetilde{\cos}(\theta) \geqslant \cos(\theta).$$

QP Relaxation of Square



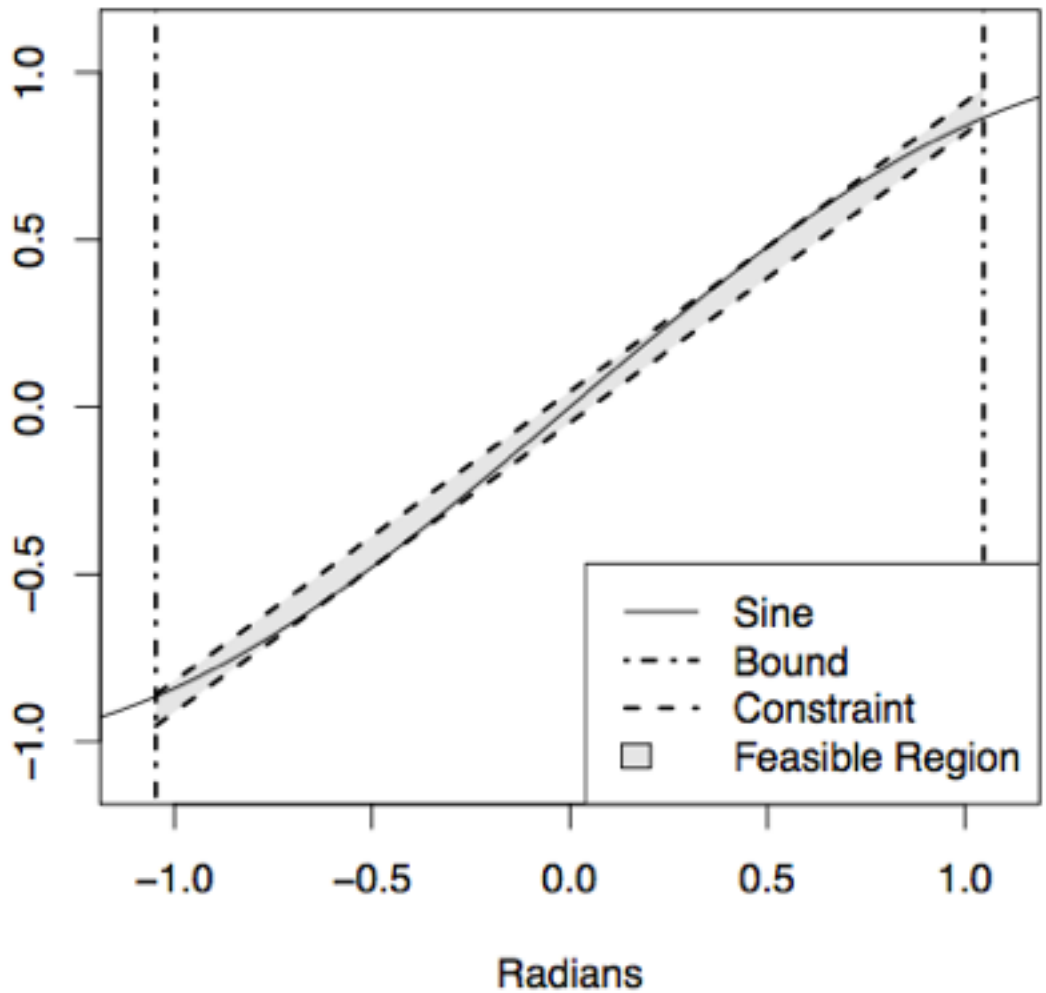
QP Relaxation of Square

- Convex quadratic Relaxation

$$\breve{v^2} \geqslant v^2$$

$$\breve{v^2} \leqslant \left(v^u + v^l\right) v - v^u v^l$$

QP Relaxation of Sine



Polynomial Relaxation of Sine

- Polynomial relaxation

$$\widetilde{\sin}(\theta) \leqslant \cos\left(\frac{\theta^u}{2}\right)\left(\theta - \frac{\theta^u}{2}\right) + \sin\left(\frac{\theta^u}{2}\right)$$

$$\widetilde{\sin}(\theta) \geqslant \cos\left(\frac{\theta^u}{2}\right)\left(\theta + \frac{\theta^u}{2}\right) - \sin\left(\frac{\theta^u}{2}\right)$$

- Proposition:

$$\forall \theta \in [-\theta^u,\ \theta^u]: \ \cos\left(\tfrac{\theta^u}{2}\right)\left(\theta - \tfrac{\theta^u}{2}\right) + \sin\left(\tfrac{\theta^u}{2}\right) \geqslant \sin(\theta).$$

$$\sin(\theta) \geqslant \cos\left(\tfrac{\theta^u}{2}\right)\left(\theta + \tfrac{\theta^u}{2}\right) - \sin\left(\tfrac{\theta^u}{2}\right)$$

MacCormick Relaxations

- Sequential Bilinear Relaxations

$$\widetilde{v_i v_j} \geqslant v_i^l v_j + v_j^l v_i - v_i^l v_j^l$$
$$\widetilde{v_i v_j} \geqslant v_i^u v_j + v_j^u v_i - v_i^u v_j^u$$
$$\widetilde{v_i v_j} \leqslant v_i^l v_j + v_j^u v_i - v_i^l v_j^u$$
$$\widetilde{v_i v_j} \leqslant v_i^u v_j + v_j^l v_i - v_i^u v_j^l$$

The Initial QC Model

$$
(\text{QC-Init}) = \begin{cases}
p_{ij} = \boldsymbol{g}_{ij}\breve{v}_i - \boldsymbol{g}_{ij}\widetilde{wc}_{ij} - \boldsymbol{b}_{ij}\widetilde{ws}_{ij} & (10) \\
q_{ij} = -\boldsymbol{b}_{ij}\breve{v}_i + \boldsymbol{b}_{ij}\widetilde{wc}_{ij} - \boldsymbol{g}_{ij}\widetilde{ws}_{ij} & (11) \\
\breve{cs}_{ij} \in \langle \cos(\theta_i - \theta_j) \rangle^R & (12) \\
\breve{s}_{ij} \in \langle \sin(\theta_i - \theta_j) \rangle^R & (13) \\
\breve{v}_i \in \langle v^2 \rangle^R & (14) \\
\breve{w}_{ij} \in \langle v_i, v_j \rangle^M & (15) \\
\widetilde{wc}_{ij} \in \langle \breve{w}_{ij}, \breve{cs}_{ij} \rangle^M & (16) \\
\widetilde{ws}_{ij} \in \langle \breve{w}_{ij}, \breve{s}_{ij} \rangle^M & (17)
\end{cases}
$$

Dual Modelling

- Key idea
  - Use several models of the optimisation problem
  - The solution set is the intersection
- Heavily used in optimisation for decades
  - constraint programming (early 1990s)
    - improving propagation
  - mathematical programming
- In nonlinear optimisation
  - Formalized by [Liberty, 04] to strengthen the relaxation
  - Generalized reduction constraints [Ruiz & Grossmann, 11]
    - intersecting several formulations based on physical interpretation of the problem

Redundancy in MINLP

$$\begin{aligned} \min \;\; & f(\mathbf{x},\mathbf{y}) \\ \text{s.t.} \;\; & z_i = g_i(\mathbf{x},\mathbf{y}), \;\; \forall i \in \{1,\ldots,m\}, \\ & \mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{Z}^m. \end{aligned}$$

$$\downarrow$$

$$\begin{aligned} \min \;\; & o \\ \text{s.t.} \;\; & o \in f(\mathbf{x},\mathbf{y})^R \\ & z_i \in g_i(\mathbf{x},\mathbf{y})^R, \;\; \forall i \in \{1,\ldots,m\}, \\ & \mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{Z}^m. \end{aligned}$$

Redundancy in MINLP

$$
\begin{aligned}
\min\ & f(\mathbf{x},\mathbf{y}) \\
\text{s.t. } & z_i = g_i(\mathbf{x},\mathbf{y}),\ \forall i \in \{1,\ldots,m\}, \\
& z_i + z_j = g_i(\mathbf{x},\mathbf{y}) + g_j(\mathbf{x},\mathbf{y}),\ \forall (i,j) \in \{1,\ldots,m\}^2, \\
& \mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{Z}^m.
\end{aligned}
$$

$$\downarrow$$

$$
\begin{aligned}
\min\ & o \\
\text{s.t. } & o \in f(\mathbf{x},\mathbf{y})^R \\
& z_i \in g_i(\mathbf{x},\mathbf{y})^R,\ \forall i \in \{1,\ldots,m\}, \\
& z_i + z_j \in (g_i(\mathbf{x},\mathbf{y}) + g_j(\mathbf{x},\mathbf{y}))^R,\ \forall (i,j) \in \{1,\ldots,m\}^2, \\
& \mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{Z}^m.
\end{aligned}
$$

Power Loss Formulation

- Power loss on line (i,j)

Current magnitude squared

$$p_{ij} + p_{ji} = \mathbf{r}_{ij} \frac{p_{ij}^2 + q_{ij}^2}{v_i^2}$$

$$q_{ij} + q_{ji} = \mathbf{x}_{ij} \frac{p_{ij}^2 + q_{ij}^2}{v_i^2}$$

The QC-NLP Model

- Power loss on line (i,j)

$$\left\{ (10) - (17) \right.$$

(QC-NLP (18) (19) (20)

How to prove this?

Use the perspective z f(x/z,y/z)
where $f(x,y) = x^2 + y^2$

- Propositi

$$\textit{The function } f(\mathbb{R}^2 \times \mathbb{R}^{*+} \rightarrow \mathbb{R}) : f(x,y,z) = \frac{x^2+y^2}{z} \textit{ is convex}$$

The QC-SOCP Model

- Second-Order Cone Formulation
  - [Farivar, M., Clarke, C., Low, S., Chandy, K. 2011]

$$l_{ij}\breve{v}_i \geqslant p_{ij}^2 + q_{ij}^2 \tag{20a}$$

$$(\text{QC-SOCP}) = \begin{cases} (10) - (19) \\ (20a) \end{cases}$$

One/Off Constraints

$$
\begin{aligned}
\min \;\; & f(\mathbf{x}, \mathbf{z}) \\
\text{s.t. } & \mathbf{h}(\mathbf{x}, \mathbf{z}) \leqslant \mathbf{0}, \\
& g_k(\mathbf{x}) \leqslant 0 \text{ if } z_k \;=\; 1, \;\; \forall k \in K \\
& \mathbf{x} \in \mathbb{R}^n, \mathbf{z} \in \mathbb{Z}^m.
\end{aligned}
$$

- Can we avoid big-M transformations?

On/Off Constraints

$$
\begin{aligned}
\min \;\; & f(\mathbf{x}, \mathbf{z}) \\
\text{s.t. } & \mathbf{h}(\mathbf{x}, \mathbf{z}) \leqslant \mathbf{0}, \\
& (\mathbf{x}, z_k) \in \Gamma_0^k \cup \Gamma_1^k, \;\; \forall k \in K \\
& \Gamma_0^k = \{(\mathbf{x}, z_k) : z_k = 0, \;\; \mathbf{l}^0 \leqslant \mathbf{x} \leqslant \mathbf{u}^0\} \\
& \Gamma_1^k = \{(\mathbf{x}, z_k) : z_k = 1, \;\; g_k(\mathbf{x}) \leqslant 0, \;\; \mathbf{l}^1 \leqslant \mathbf{x} \leqslant \mathbf{u}^1\}. \\
& \mathbf{x} \in \mathbb{R}^n, \mathbf{z} \in \mathbb{Z}^m.
\end{aligned}
$$

On/Off Constraints

$$
\begin{aligned}
\min \;\; & h(\mathbf{x}, z) \\
\text{s.t.} \;\; & g(\mathbf{x}, \mathbf{z}) \leq 0, \\
& (\mathbf{x}, z_k) \in conv(\Gamma_0^k \cup \Gamma_1^k), \qquad \forall k \in \{1,\ 2, ..., K\}, \\
& \mathbf{x} \in \mathbb{R}^n,\ \mathbf{z} \in \{0,1\}^K.
\end{aligned}
$$

- Can I represent the convex hull in the original space of variables?

On/Off Constraints: Point Case

- When $\Gamma_0^k = \{(\mathbf{x}, z_k) : z_k = 0,\ \mathbf{l}^0 \leqslant \mathbf{x} \leqslant \mathbf{u}^0\}$ is a point, the convex hull becomes

$$\Gamma^* = \left\{ \begin{array}{l} (\mathbf{x}, z) \in \mathbb{R}^{m+1} : \\ zg(\mathbf{x}/z) \leqslant 0, \\ z\mathbf{l}^1 \leqslant \mathbf{x} \leqslant z\mathbf{u}^1,\ 0 \leqslant z < 1 \end{array} \right\}$$

On/Off Constraints: quadratic

- Consider the function

$$g(\mathbf{x}) = a_1 x_1^2 + a_2 x_2^2 - a_3 x_3$$

- The convex hull becomes

$$\Gamma^* = \left\{ \begin{array}{l} (\mathbf{x}, z) \in \mathbb{R}^4 : \\ a_1 x_1^2 + a_2 x_2^2 \leqslant z(a_3 x_3), \\ z\mathbf{l}^1 \leqslant \mathbf{x} \leqslant z\mathbf{u}^1, \ 0 \leqslant z \leqslant 1 \end{array} \right\}$$

On/Off Constraints

- The idea can be generalised for intervals and monotone functions: e.g. linear constraints

$$g(\mathbf{x}) = \boldsymbol{a}^T\mathbf{x} - b,$$

On/Off Constraints

- The idea can be generalised for intervals and monotone functions: e.g. linear constraints

$$\Gamma^* = \left\{ \begin{array}{l} (\mathbf{x}, z) \in \mathbb{R}^{n+1} : \\ \sum_{i \notin S} \boldsymbol{a}_i x_i \leqslant z \left(b - \sum_{\substack{i \in S, \\ \boldsymbol{a}_i < 0}} \boldsymbol{a}_i u_i^1 - \sum_{\substack{i \in S, \\ \boldsymbol{a}_i > 0}} \boldsymbol{a}_i l_i^1 \right) \\ +(1-z) \left(\sum_{\substack{i \notin S, \\ \boldsymbol{a}_i < 0}} \boldsymbol{a}_i l_i^0 + \sum_{\substack{i \notin S, \\ \boldsymbol{a}_i > 0}} \boldsymbol{a}_i u_i^0 \right), \ \forall S \subset \{1, \ldots, m\} \\ z l_i^1 + (1-z) l_i^0 \leqslant x_i \leqslant z u_i^1 + (1-z) u_i^0, \ \forall i \in \{1, \ldots, m\}, \\ 0 \leqslant z \leqslant 1 \end{array} \right\}$$

On/Off Constraints in Power Systems

- When a line is switched off, the phase angle difference must increase.
- We can then apply the above results to each constraint

On/Off Constraints in Power Systems

- Sine constraint

$$\Gamma_s^0 = \left\{ (\breve{s}, \theta, z) \in \mathbb{R}^3 : -1 \leqslant \breve{s} \leqslant 1, \ \ -|E|\boldsymbol{\theta}^u \leqslant \theta \leqslant |E|\boldsymbol{\theta}^u, \ z = 0 \right\}$$

$$\Gamma_s^1 = \left\{ \begin{array}{l} (\breve{s}, \theta, z) \in \mathbb{R}^3 : \\ \breve{s} - \cos(\boldsymbol{\theta}^u/2)\theta \leqslant \sin(\boldsymbol{\theta}^u/2) - \cos(\boldsymbol{\theta}^u/2)\boldsymbol{\theta}^u/2, \\ -\breve{s} + \cos(\boldsymbol{\theta}^u/2)\theta \leqslant \sin(\boldsymbol{\theta}^u/2) - \cos(\boldsymbol{\theta}^u/2)\boldsymbol{\theta}^u/2, \\ \sin(-\boldsymbol{\theta}^u) \leqslant \breve{s} \leqslant \sin(\boldsymbol{\theta}^u), \ \ -\boldsymbol{\theta}^u \leqslant \theta \leqslant \boldsymbol{\theta}^u, \ z = 1 \end{array} \right\}$$

On/Off Constraints in Power Systems

$$
\Gamma_s^* = \left\{ \begin{array}{l}
(\breve{s}, \theta, z) \in \mathbb{R}^3 : \\
\breve{s} - \cos(\boldsymbol{\theta}^u/2)\theta \leqslant z(\sin(\boldsymbol{\theta}^u/2) - \cos(\boldsymbol{\theta}^u/2)\boldsymbol{\theta}^u/2) \\
+(1-z)(\cos(\boldsymbol{\theta}^u/2)|E|\boldsymbol{\theta}^u + 1), \\
-\breve{s} + \cos(\boldsymbol{\theta}^u/2)\theta \leqslant z(\sin(\boldsymbol{\theta}^u/2) - \cos(\boldsymbol{\theta}^u/2)\boldsymbol{\theta}^u/2) \\
+(1-z)(\cos(\boldsymbol{\theta}^u/2)|E|\boldsymbol{\theta}^u + 1), \\
|\breve{s}| \leqslant z(\sin(\boldsymbol{\theta}^u/2) + \cos(\boldsymbol{\theta}^u/2)\boldsymbol{\theta}^u/2) + (1-z), \\
\cos(\boldsymbol{\theta}^u/2)|\theta| \leqslant z(\sin(\boldsymbol{\theta}^u/2) - \cos(\boldsymbol{\theta}^u/2)\boldsymbol{\theta}^u/2 + sin(\boldsymbol{\theta}^u)) \\
+(1-z)(\cos(\boldsymbol{\theta}^u/2)|E|\boldsymbol{\theta}^u), \\
z\sin(-\boldsymbol{\theta}^u) - (1-z) \leqslant \breve{s} \leqslant z\sin(\boldsymbol{\theta}^u) + (1-z), \\
-z\boldsymbol{\theta}^u - (1-z)|E|\boldsymbol{\theta}^u \leqslant \theta \leqslant z\boldsymbol{\theta}^u + (1-z)|E|\boldsymbol{\theta}^u,\ 0 \leqslant z \leqslant 1
\end{array} \right\}
$$

Outline

- Motivation
- Power Flow Formulations
- Complexity
- The LPAC Approximation
- Case Studies for LPAC
- Convex Relaxations
- Case Studies for Convex Relaxations
- Beyond Steady States

Experimental Settings

- Instances: [Matpower 2011]

id	1	2	3	4	5	6	7	8	9	10	11	12	13
$\|N\|$	6	9	9	14	30	30	30	39	57	118	300	2383	2736
$\|E\|$	11	9	9	20	41	41	41	46	78	179	409	2886	3494

Table 1 Sizes of Power System Benchmarks.

- Congested Instances

id	c-1	c-2	c-3	c-4	c-5	c-6	c-7	c-8
$\|N\|$	3	9	14	30	39	57	118	300
$\|E\|$	3	9	20	41	46	78	179	409

Optimal Power Flows

- The problem
  - Minimize the cost of generation for a given load
  - Intensively studied
  - Purest

Optimal Power Flows

$$
\begin{aligned}
\min\ & \textstyle\sum_{i=1}^{m} \mathbf{c}_i'' p_i^2 + \mathbf{c}_i' p_i + \mathbf{c}_i \\
\text{s.t.}\ & p_i = \textstyle\sum_{j=1}^{m} p_{ij},\ \forall i \in \{1,\ldots,m\}, \\
& q_i = \textstyle\sum_{j=1}^{m} q_{ij},\ \forall i \in \{1,\ldots,m\}, \\
& (1)-(2) \\
& p_{ij}^2 + q_{ij}^2 \leqslant \mathbf{s}_{ij}^2,\ \forall (i,j) \in \{1,\ldots,m\}^2, \\
& \mathbf{p}_i^l \leqslant p_i \leqslant \mathbf{p}_i^u,\ \forall i \in \{1,\ldots,m\}, \\
& \mathbf{q}_i^l \leqslant q_i \leqslant \mathbf{q}_i^u,\ \forall i \in \{1,\ldots,m\}, \\
& \mathbf{v}_i^l \leqslant v_i \leqslant \mathbf{v}_i^u,\ \forall i \in \{1,\ldots,m\}, \\
& -\theta_i^u \leqslant \theta_i - \theta_j \leqslant \theta_i^u,\ \forall (i,j) \in \{1,\ldots,m\}^2, \\
& \mathbf{v}, \boldsymbol{\theta}, \mathbf{p}, \mathbf{q} \in \mathbb{R}^n.
\end{aligned}
\qquad \text{(AC-OPF)}
$$

quadratic production cost

Optimal Power Flows

	Quality	Optimality Gap				
	AC	QC			[24]	[27]
Benchmark	NLP	Init.	NLP	SOCP	SOCP	SDP
1	3143	0.79%	0.47%	0.47%	0.63%	0.00%
2	5296	1.52%	0.00%	0.00%	0.78%	-0.01%
3	5296	1.52%	0.00%	0.00%	0.78%	-0.01%
4	8081	4.51%	0.08%	0.08%	0.08%	0.00%
5	576	2.03%	0.57%	0.57%	0.58%	0.00%
6	576	2.03%	0.57%	0.57%	0.58%	0.00%
7	8906	5.25%	0.04%	0.04%	0.04%	0.00%
8	41864	1.34%	0.02%	0.03%	0.06%	-0.06%
9	41737	1.75%	0.06%	0.06%	0.07%	0.00%
10	129658	2.48%	0.18%	0.18%	0.25%	0.00%
11	722847	1.65%	0.32%	0.32%	0.71%	0.39%
12	1898276	6.48%	2.49%	2.49%	inf.	1.66%
13	1308771	2.50%	0.45%	0.45%	inf.	err.

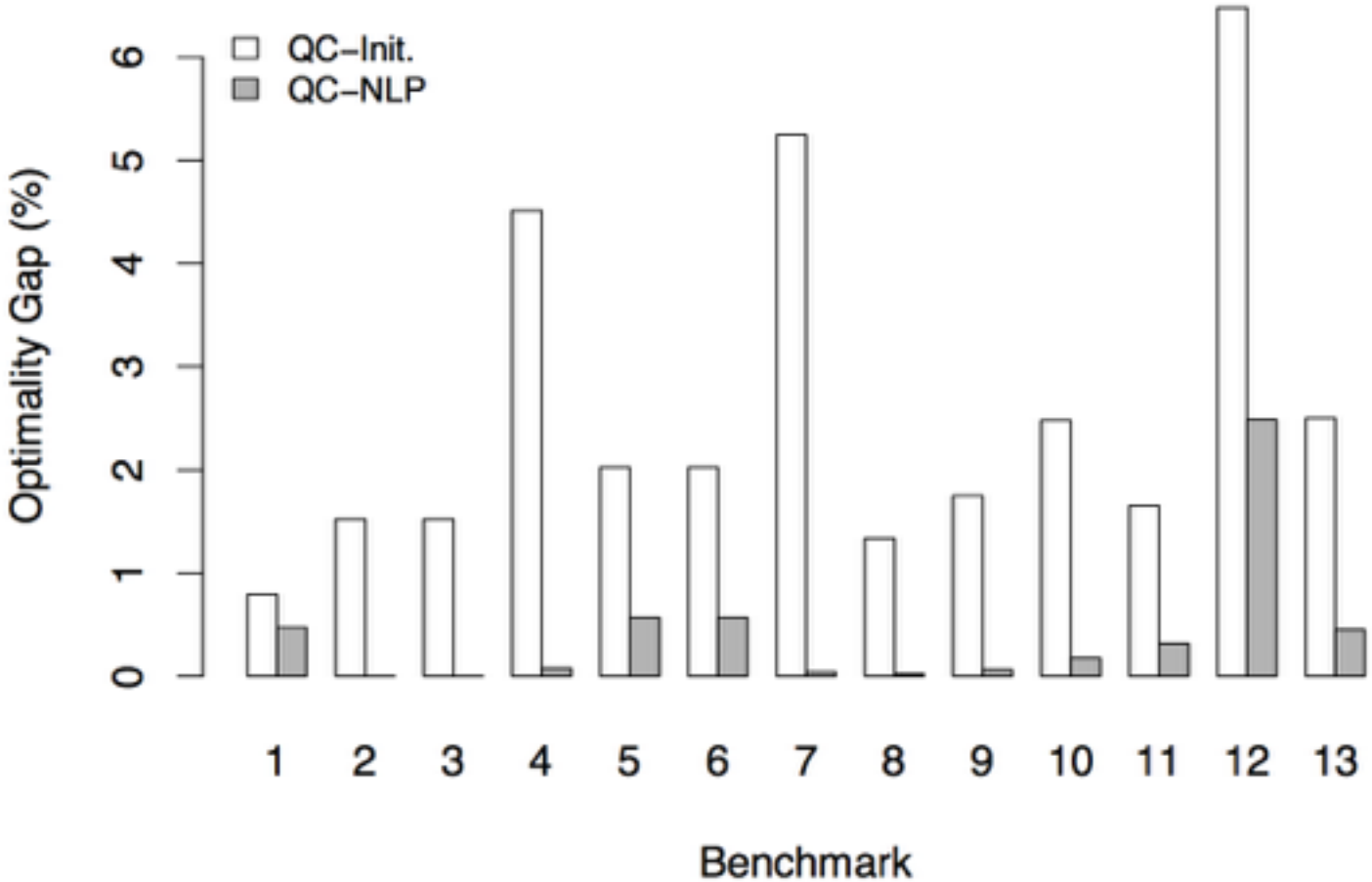
Table 3 Optimality Gap for Different Relaxations of the OPF Problem.

Optimal Power Flows

	Runtime (seconds)					
Benchmark	AC NLP	Init.	QC NLP	SOCP	[24] SOCP	[27] SDP
1	0.12	0.23	0.08	0.44	0.05	16.16
2	0.07	0.06	0.06	0.13	0.04	8.00
3	0.05	0.10	0.06	0.13	0.03	7.43
4	0.04	0.18	0.08	0.20	0.09	6.33
5	0.06	0.09	0.13	0.41	0.15	16.65
6	0.07	0.18	0.15	0.37	0.18	16.45
7	0.06	0.22	0.12	0.53	0.15	8.78
8	0.12	0.20	0.20	0.40	0.22	16.13
9	0.22	0.18	0.24	0.59	0.32	11.48
10	0.18	0.51	0.57	8.25	0.82	19.57
11	0.52	1.60	1.80	5.37	3.59	34.12
12	9.78	37.51	34.22	1468.19	131.97	2666.00
13	8.59	39.28	34.25	2707.78	32.48	T.L.

Table 4 Runtimes for Different Relaxations of the OPF Problem.

Optimal Power Flows



Optimal Power Flows

Congested	Quality	Optimality Gap			Runtime (seconds)			
	AC	QC		[24]	AC	QC		[24]
Benchmark	NLP	NLP	SOCP	SOCP	NLP	NLP	SOCP	SOCP
c-1	1499	0.00%	0.00%	0.00%	0.05	0.06	0.05	0.03
c-2	22081	0.00%	0.00%	0.01%	0.05	0.07	0.20	0.08
c-3	47259	2.04%	2.08%	Inf.	0.05	0.10	0.51	0.32
c-4	1979	14.57%	14.57%	14.59%	0.07	0.13	0.56	0.21
c-5	107536	0.23%	0.36%	Inf.	0.16	0.50	0.71	0.85
c-6	137116	1.54%	2.76%	Inf.	0.12	0.37	2.74	0.83
c-7	2530242	1.69%	2.61%	Inf.	0.61	4.61	38.08	4.49
c-8	14773480	1.26%	1.66%	Inf.	2.86	20.84	29.52	10.33

Optimal Power Flows: Summary

- Dual modelling is critical
- QC-NLP dominates the SOCP relaxation
  - especially on congested benchmarks
- QC-NLP is orders of magnitude faster than the SDP relaxation with minimal loss in accuracy
- QC-NLP is orders of magnitude faster than QC-SOCP

Outline

- Motivation
- Complexity
- The LPAC Approximation
  - The model
  - Power restoration
- The QC Relaxation
  - The model
  - Optimal power flow
  - Optimal line-switching power flow
  - Capacitor placement
- Conclusion and future work

Optimal Line-Switching Power Flows

- The Problem
  - same as OPF but allows to switch off lines
  - exploit Braess paradox in power systems
  - very hard computationally
- Prior work
  - [Fisher, O'Neil, Ferris, 07], [Bienstock, 12]
  - shows the cost benefits of switching lines off
  - based on the LDC model
- This work
  - First attempt at provable quality bounds

Optimal Line-Switching Power Flows

- The AC Model

$$\min \sum_{i \in G} \boldsymbol{c}_i^{''}(p_i^g)^2 + \boldsymbol{c}_i^{'}(p_i^g)$$

$$\text{s.t. } (3) - (7), (9) \qquad \text{(AC-MINLP)}$$

$$p_{ij} = z_{ij}(\mathbf{g}_{ij} v_i^2 - \mathbf{g}_{ij} v_i v_j \cos(\theta_i - \theta_j) - \mathbf{b}_{ij} v_i v_j \sin(\theta_i - \theta_j)),$$

$$q_{ij} = z_{ij}(-\mathbf{b}_{ij} v_i^2 + \mathbf{b}_{ij} v_i v_j \cos(\theta_i - \theta_j) - \mathbf{g}_{ij} v_i v_j \sin(\theta_i - \theta_j)),$$

$$-\boldsymbol{\theta}_i^u \leqslant z_{ij}(\theta_i - \theta_j) \leqslant \boldsymbol{\theta}_i^u,$$

$$z_{ij} \in \{0, 1\}.$$

Optimal Line-Switching Power Flows

- AC-MINLP
  - the AC version
- DC
  - the original version: Fisher, O'Neill, Ferris, 2008
- QC
  - QC-NLP
  - QC-SOP

Optimal Line-Switching Power Flows

Benchmark	Cost AC	Relative Cost DC	Relative Cost DC to AC
1	3128	-2.63%	—
2	5296	-1.52%	1.73%
3	5296	-1.52%	1.73%
4	8081	-5.43%	0.85%
5	573	-1.52%	—
6	573	-1.52%	—
7	8906	-6.32%	—
8	41857	-1.42%	0.23%
9	41725	-1.72%	—
10	129444	-2.69%	3.75%
11	721576	-1.88%	—

Optimal Line-Switching Power Flows

	Cost	Gap			Runtime (seconds)			
	AC	QC^r			AC	QC^r		
		Strong		Weak		Strong		Weak
Bench.	MINLP	NLP	SOCP	SOCP	MINLP	NLP	SOCP	SOCP
1	3128	0.16%	0.16%	0.18%	0.12	0.13	0.34	0.22
2	5296	0.01%	0.01%	0.09%	0.11	0.06	0.20	0.13
3	5296	0.01%	0.01%	0.09%	0.11	0.09	0.38	0.13
4	8081	0.12%	0.12%	0.20%	0.43	0.12	0.56	0.27
5	573	0.12%	0.12%	0.17%	0.89	0.22	0.56	1.43
6	573	0.12%	0.12%	0.17%	0.81	0.25	1.02	0.85
7	8906	0.12%	0.12%	0.23%	0.23	0.36	1.06	0.66
8	41857	0.02%	0.02%	0.06%	0.75	0.39	0.97	0.62
9	41725	0.17%	0.17%	0.21%	15.88	0.58	1.95	1.16
10	129444	0.11%	0.11%	0.18%	66.58	0.96	5.23	5.42
11	721576	0.47%	0.47%	0.55%	1111.06	3.64	20.21	22.99

Table 7 OTS root-node optimality gap for the QC models on standard benchmarks.

Optimal Line-Switching Power Flows

	Cost	Gap			Runtime (seconds)			
	AC	QC^r			AC	QC^r		
		Strong		Weak		Strong		Weak
Bench.	MINLP	NLP	SOCP	SOCP	MINLP	NLP	SOCP	SOCP
c-1	1500	0.05%	0.05%	0.05%	0.08	0.08	0.23	0.12
c-2	22081	0.36%	0.36%	0.55%	0.10	0.09	0.75	0.14
c-3	47259	8.06%	8.08%	14.58%	0.86	0.15	0.71	0.30
c-4	1695	0.76%	0.76%	1.02%	4.35	0.24	0.46	0.52
c-5	107536	1.21%	1.21%	1.62%	3.66	0.44	0.68	0.58
c-6	133473	10.29%	10.29%	11.03%	15.00	0.63	1.97	1.06
c-7	2478127	29.98%	29.98%	40.01%	73.37	2.47	3.86	3.02
c-8	14745788	13.47%	13.48%	20.37%	786.33	17.56	39.59	36.79

Table 8 OTS root-node optimality gap for the QC models on congested benchmarks.

Optimal Line-Switching Power Flows

	Gap	Gap			Runtime (seconds)		
	QC^r	QC			QC		
		Strong		Weak	Strong		Weak
Bench.	NLP	MINLP	SOCP	SOCP	MINLP	SOCP	SOCP
c-1	0.05%	0.05%	0.05%	0.05%	0.20	0.24	0.13
c-2	0.36%	0.00%	0.00%	0.00%	0.21	0.48	0.43
c-3	8.06%	3.35%	3.36%	3.51%	0.79	3.98	2.12
c-4	0.76%	0.44%	0.44%	0.43%	13.22	16.91	10.74
c-5	1.21%	0.43%	0.45%	0.50%	2.78	5.32	4.76
c-6	10.29%	3.35%	3.44%	5.61%	2367.89	294.74	911.95
c-7	29.98%	3.51%	3.58%	err.	T.L.	15992.07	436.61
c-8	13.47%	4.41%	4.94%	10.54%	T.L.	T.L.	T.L.

Table 9 OTS optimality gap for the QC models on congested benchmarks (10 hrs T.L.)

Optimal Line-Switching Power Flows

- The DC model is overly optimistic
  - infeasible configurations
  - suboptimal solutions when crossed-over
- QC bounds shows the quality of heuristic B&B
  - small gaps: 0.13% and 1.95% on traditional benchmarks at the root node
  - gaps can reached 40% on congested benchmarks at the root node
  - gaps can be reduced to 4% by solving the mixed integer version

Outline

- Motivation
- Complexity
- The LPAC Approximation
  - The model
  - Power restoration
- The QC Relaxation
  - The model
  - Optimal power flow
  - Optimal line-switching power flow
  - Capacitor placement
- Conclusion and future work

Capacitor Placement

- Capacitors inject reactive power
- The problem is to place capacitors in a network to improve the voltage profile
  - well-studied problems
  - almost always tackled by heuristics
    - need to reason about reactive power and voltage magnitudes
- The goal is to minimize the number of capacitors

Capacitor Placement

$$
\begin{aligned}
\min\ & \textstyle\sum_{i=1}^{m} z_i \\
\text{s.t.}\ & p_i = \textstyle\sum_{j=1}^{m} p_{ij},\ \forall i \in \{1,\ldots,m\}, \\
& q_i = \textstyle\sum_{j=1}^{m} q_{ij},\ \forall i \in \{1,\ldots,m\}, \\
& (1)-(2) \\
& p_{ij}^2 + q_{ij}^2 \leqslant \mathbf{s}_{ij}^2,\ \forall (i,j) \in \{1,\ldots,m\}^2, \\
& \mathbf{p}_i^l \leqslant p_i \leqslant \mathbf{p}_i^u,\ \forall i \in \{1,\ldots,m\}, \\
& \mathbf{q}_i^l \leqslant q_i \leqslant \mathbf{q}_i^u + \mathbf{q}^c z_i,\ \forall i \in \{1,\ldots,m\}, \\
& \mathbf{v}_i^l \leqslant v_i \leqslant \mathbf{v}_i^u,\ \forall i \in \{1,\ldots,m\}, \\
& -\theta_i^u \leqslant \theta_i - \theta_j \leqslant \theta_i^u,\ \forall (i,j) \in \{1,\ldots,m\}^2, \\
& \mathbf{v}, \boldsymbol{\theta}, \mathbf{p}, \mathbf{q} \in \mathbb{R}^n, \mathbf{z} \in \mathbb{N}^m.
\end{aligned}
\tag{AC-CP}
$$

Capacitor Placement

- Key Messages
  - MINLP heuristics give near optimal solutions
  - QC gives tight lower bounds
    - justifying the first message
  - The QC model scales well to medium-sized instances

id	C1	C2	C3	C4	C5	C6
$\|N\|$	14	30	30	57	118	300
$\|E\|$	20	41	41	78	179	409

Table 6 Sizes of Capacitor Placement Benchmarks.

Capacitor Placement

	Cost	Gap		Runtime (seconds)		
	AC	QC		AC	QC	
Benchmark	MINLP	MINLP	SOCP	MINLP	MINLP	SOCP
cpp-1	5	0.00%	0.00%	1.31	0.54	0.52
cpp-2	8	0.00%	0.00%	6.12	4.59	3.34
cpp-3	7	0.00%	0.00%	4.73	1.86	3.07
cpp-4	15	0.00%	0.00%	586.99	9.52	43.59
cpp-5	67	4.48%	5.97%	T.L.	T.L.	T.L.
cpp-6	241	5.39%	5.81%	T.L.	T.L.	T.L.

Table 11 CPP Runtime and Optimality Gaps using the QC Relaxations with $v^l = 1$.

Reconfiguration Problems

- Find out the network topology (by opening and closing switches) to
  - minimize losses
  - or maximize balances
- while satisfying operational constraints
  - thermal limits
  - phase angle and voltage magnitude constraints
  - …
- Relaxation
  - Dist-flow for radial networks, QC for meshed networks

Loss Minimization

	Obj. Value	Opt. Gap	LB. Run Time	UB. Run Time	Closed Switch.
Network	Radial				
32-bus	0.01395	0%	2.90	–	32/37
70-bus	0.03016	0%	7.45	–	68/76
135-bus	0.28013	0%	24.80	–	135/156
880-bus	0.45703	0%	2886.94	–	873/900
Network	Meshed				
32-bus	0.01232	0.55%	6.26	0.10	36/37
70-bus	0.02977	0.17%	35.42	0.21	75/76
135-bus	0.27079	3.00%	T.L.	3.42	149/156
880-bus	0.45175	8.72%	T.L.	T.L.	900/900

TABLE I. MINIMAL LOSS RESULTS

Loss Minimization

	present approach		QP approx. [7]		SOCP approx. [7]		MICP relax. [14]	
Network	Val	Time	Val	Time	Val	Time	Val	Time
32-bus	0.01395	2.90	0.01395	0.39	0.01395	23.83	0.01395	5.23
70-bus	0.03016	7.45	0.03016	0.98	0.03016	769.63	0.03016	52.48
135-bus	0.28013	24.80	0.28013	54.74	0.28661	T.L.	0.29171	T.L.
880-bus	0.45703	2886.94	0.45704	T.L.	–	T.L.	mem. err.	–

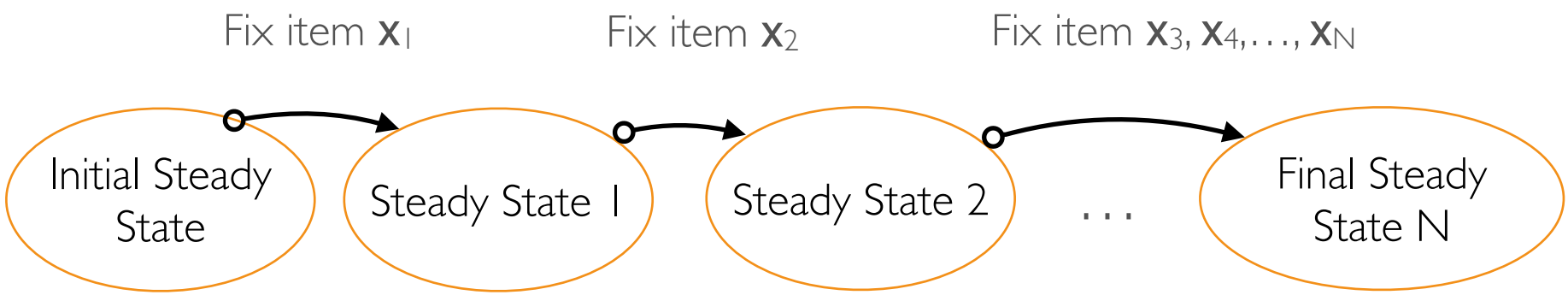
Reconfiguration Problems

- Key messages
  - Zero-gap relaxation on radial networks
  - Tight on meshed networks
    - loses accuracy on the largest one
  - Scalability

Outline

- Motivation
- Power Flow Formulations
- Complexity
- The LPAC Approximation
- Case Studies for LPAC
- Convex Relaxations
- Case Studies for Convex Relaxations
- Beyond Steady States

Restoration Ordering Problem (ROP)



- Find the best sequence of restoration

  $[x_1, x_2, x_3, \ldots, x_n]$

  so that the size of the blackout is minimised

Key Question

- Can I move from steady states to steady states?

- Capturing the dynamics

$$\dot{x} = f(x, y)$$

$$0 = g(x, y)$$

Dynamics

Network State

Dynamic Variables

State Variables

Transient Analysis

- Can you correct the steady states to ensure transient stability?
  - rotor angle stability
  - Swing equation

$$\dot{x} = f(x, y)$$

$$\delta_{it+1} - \delta_{it} - \frac{\Delta}{2}(\omega_{it+1} + \omega_{it}) = 0$$

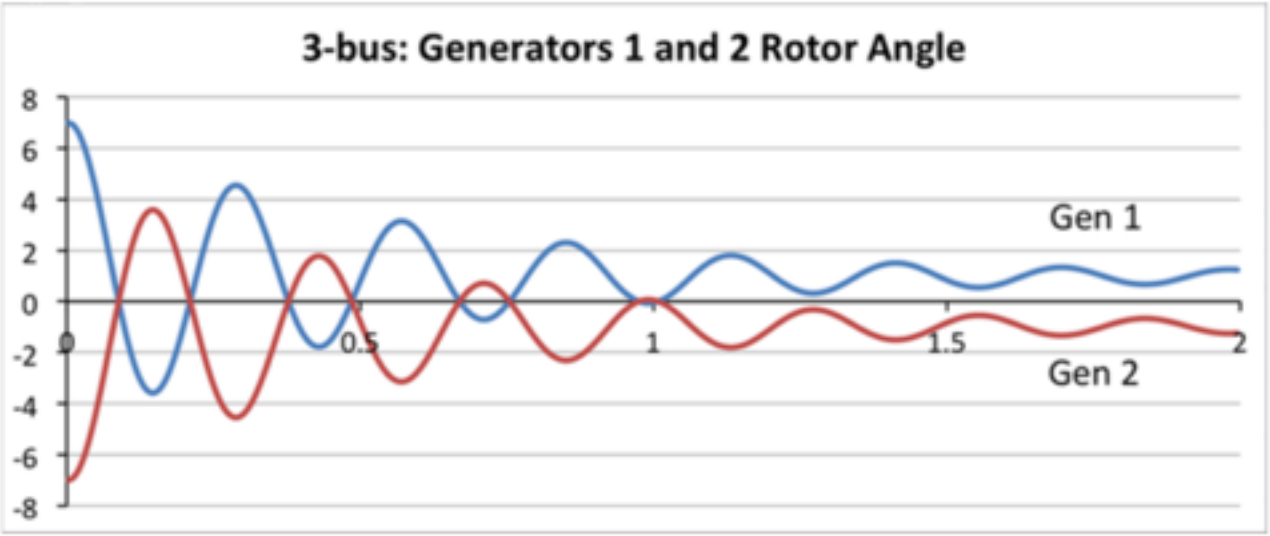
$$\omega_{it+1} - \omega_{it} - \frac{\Delta}{2}(a_{it+1} + a_{it}) = 0$$

$$a_{it} - \frac{\omega_0}{2H_i}(p^e_{i1} - p^e_{it} - D_i\omega_{it}) = 0$$

Transient Analysis

- Can you correct the steady states to ensure transient stability?
  - rotor angle stability
  - Swing equation

$$\dot{x} = f(x, y)$$



Transient Stability

- Angle for the center of inertia

$$\delta_t^r = \frac{\sum_{i \in G} H_i \delta_{it}}{\sum_{i \in G} H_i} \quad (1 \leq t \leq T).$$

The stability constraints then become:

$$\forall i \in G, 1 \leq t \leq T : -\overline{\delta} \leq \delta_{it} - \delta_t^r \leq \overline{\delta}$$

Transient Stability

- A generator is modeled as an internal bus

$$p_{it}^{e} = \frac{v_i^g v_{it}}{X_i} \sin(\theta_{it} - \delta_{it})$$

$$q_{i1}^{e} = \frac{v_{i1}^2}{X_i} - \frac{v_i^g v_{i1}}{X_i} \cos(\theta_{i1} - \delta_{i1})$$

Network State

- Power flow equations before closing

$$\sum_{j \in N} [v_{i1} v_{j1} (G^o_{ij} \cos(\theta_{ij1}) + B^o_{ij} \sin(\theta_{ij1}))] = \sum_{j \in G(i)} (p^e_{j1} - p^l_j l_j) \tag{9}$$

$$\sum_{j \in N} [v_{i1} v_{j1} (G^o_{ij} \sin(\theta_{ij1}) - B^o_{ij} \cos(\theta_{ij1}))] = \sum_{j \in G(i)} (q^e_{j1} - p^l_j l_j) \tag{10}$$

Network State

- Power flow equations after closing

$$(i \in N, 2 \leq t \leq T):$$

$$\sum_{j \in N} [v_{it} v_{jt} (G^c_{ij} \cos(\theta_{ijt}) + B^c_{ij} \sin(\theta_{ijt}))] = \sum_{j \in G(i)} (p^e_{jt} - p^l_j l_j) \tag{7}$$

$$\sum_{j \in N} [v_{it} v_{jt} (G^c_{ij} \sin(\theta_{ijt}) - B^c_{ij} \cos(\theta_{ijt}))] = \sum_{j \in G(i)} (q^e_{jt} - p^l_j l_j) \tag{8}$$

Network State

- Voltage Stability Constraints

$$v_{it} - \overline{v^{\Delta}} \leq v_{it+1} \leq v_{it} + \overline{v^{\Delta}}.$$

Minimal Transient-Stable Correction

Model 2 The One-Step Line-Closing Model

Inputs:

$\mathcal{PN}$	- Power network
Y^o, Y^c	- Reduced matrix before and after line closing
p_i^T, q_i^T	- i^{th} generator active/reactive target dispatch
X_i, H_i, D_i	- transient reactance, inertia, and damping constant
$f_q = \frac{\omega_0}{2\pi}, \Delta, T$	- Grid frequency, integration step, time horizon
v^{Δ}	- Maximum voltage fluctuations

Variables: $(\forall t : 1 \leq t \leq T)$

$\forall i \in N,$

θ_{it}	- Terminal bus angle
v_{it}	- Bus voltage

$\forall i \in G,$

v_i^g	- Internal bus voltage
p_{it}^e, q_{it}^e	- Generator active and reactive injection
l_i	- load percentage of load i
$\delta_{it}, \omega_{it}, a_{it}$	- Generator rotor angle, velocity, and acceleration

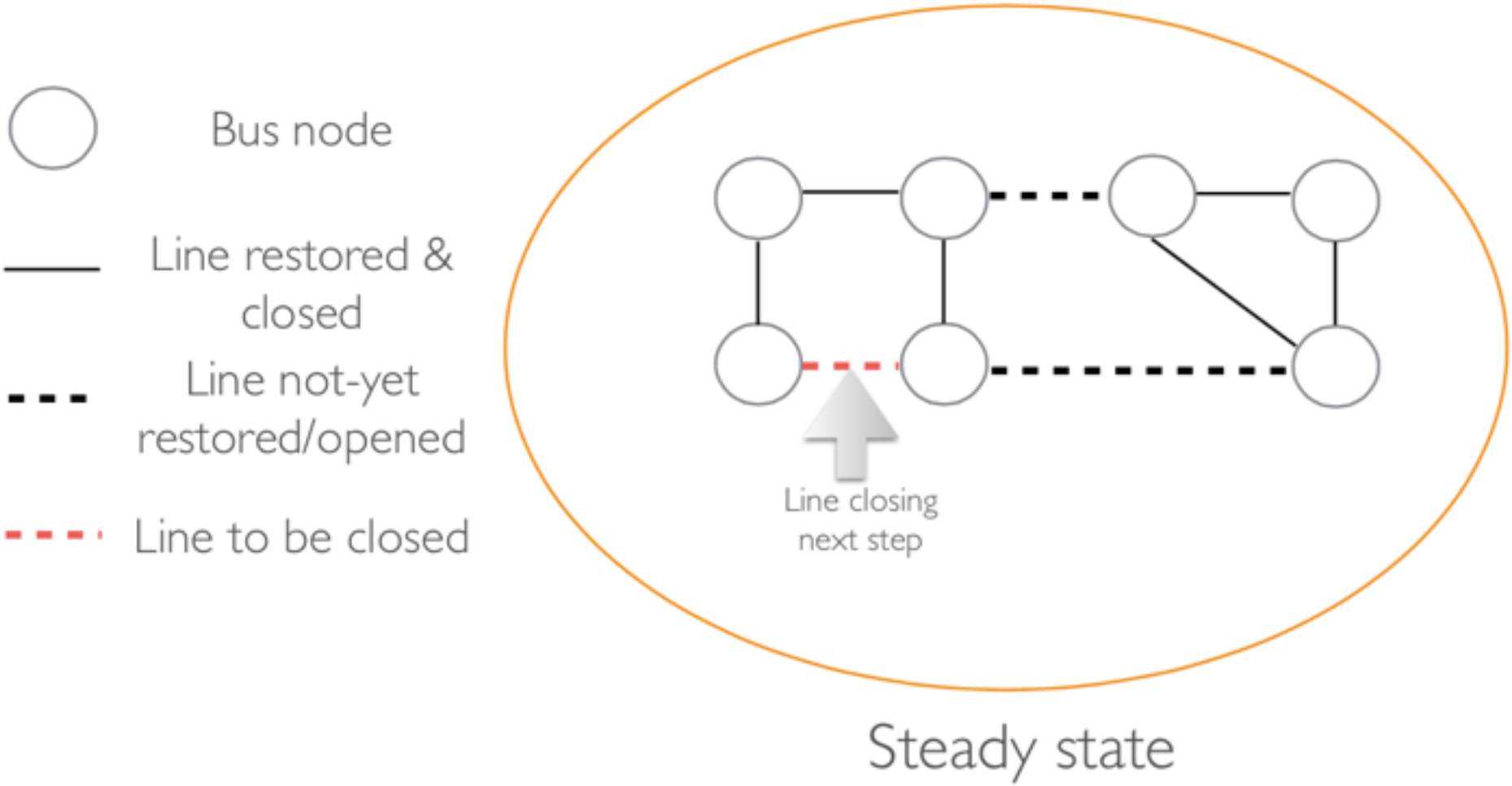
Minimize

$$\sum_{i \in G} (p_{i1}^e - p_i^T)^2 + (q_{i1}^e - q_i^T)^2$$

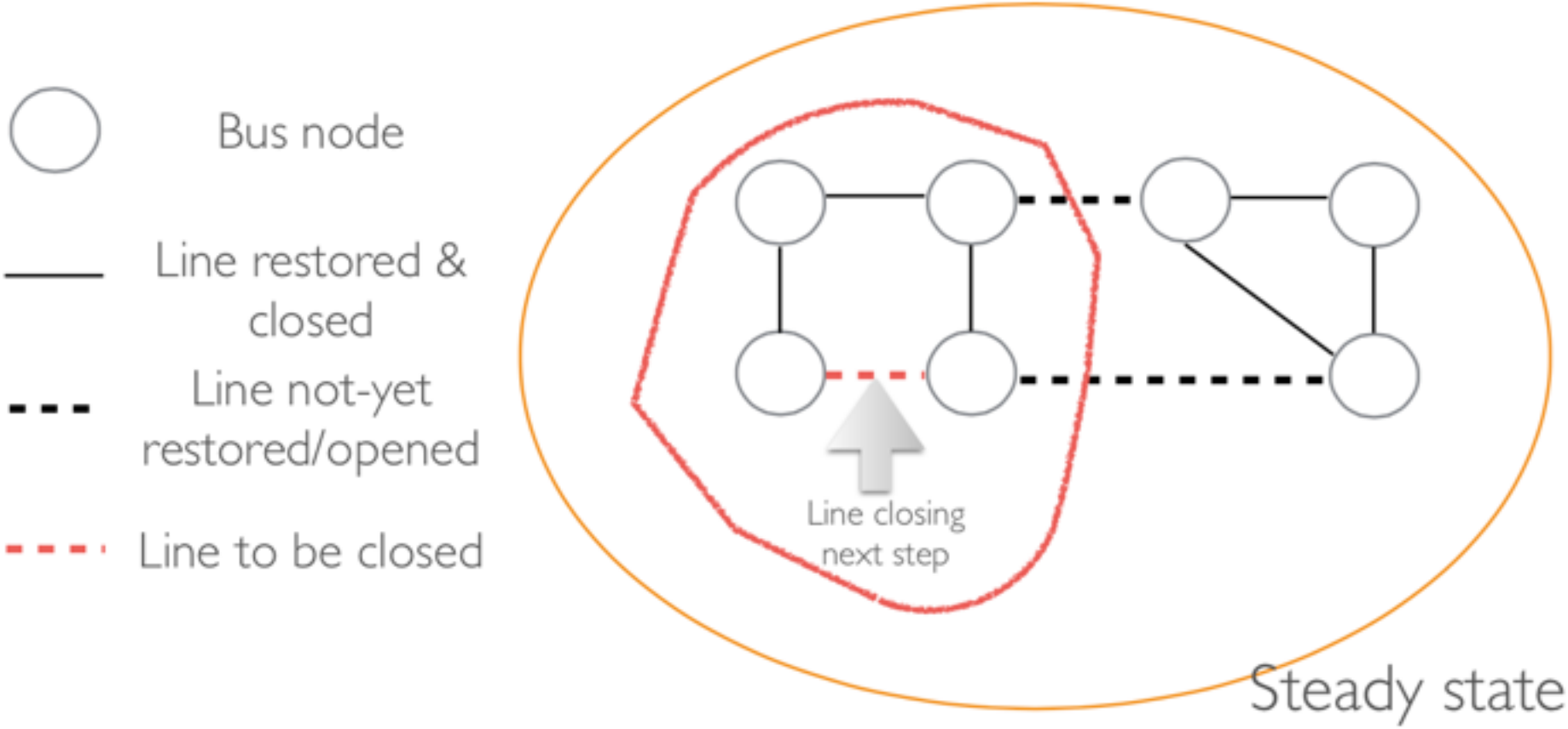
Subject to:

Equations (1) - (17)

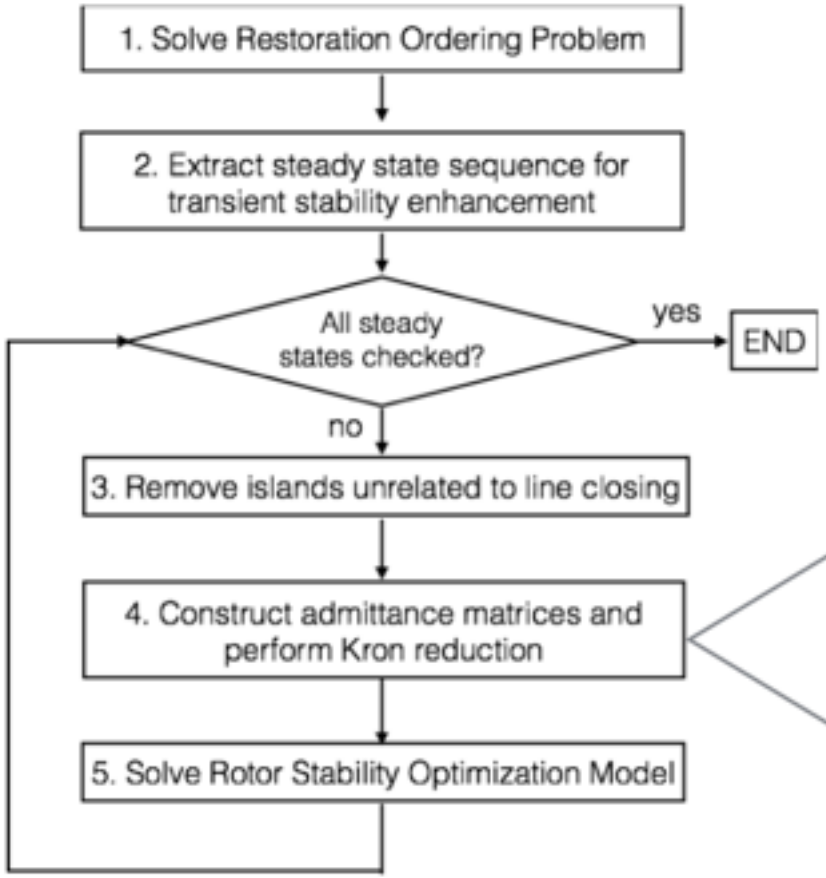
Overall Procedure



Overall Procedure



Overall Procedure



- Perform **Kron reduction** to further remove all "non-generator" buses

- Kron reduction:

1. Admittance preserving reduction
2. Widely used in transient stability *to reduce buses not under study*
3. Based on Gaussian Elimination and Kirchhoff's Current Law

Power Restoration with Transient Stability

- Minimize the distance to the steady-state dispatch while constraining the rotor angle

TABLE I
GENERATOR DISPATCH RESULTS TO MAINTAIN STANDARD 90 DEGREE ROTOR SWINGS

	6 Bus				14 Bus				30 Bus			
Gen. Reactance	Max.%	Max. MW	Avg. %	Avg. MW	Max.%	Max. MW	Avg. %	Avg. MW	Max.%	Max. MW	Avg. %	Avg. MW
0.02	0.00058	0.00066	0.00013	0.00012	0.00000	0.00000	0.00000	0.00000	0.00335	0.00067	0.00031	0.00009
0.06	0.00058	0.00066	0.00013	0.00012	0.00001	0.00000	0.00000	0.00000	0.00287	0.00060	0.00026	0.00008
0.10	0.00046	0.00051	0.00011	0.00010	0.00000	0.00000	0.00000	0.00000	0.00286	0.00060	0.00026	0.00008
0.14	0.00047	0.00051	0.00011	0.00010	0.00000	0.00000	0.00000	0.00000	0.00284	0.00060	0.00026	0.00008
0.20	0.00049	0.00054	0.00011	0.00010	0.00000	0.00000	0.00000	0.00000	0.00279	0.00060	0.00025	0.00008
	39 Bus				57 Bus							
Gen. Reactance	Max.%	Max. MW	Avg. %	Avg. MW	Max.%	Max. MW	Avg. %	Avg. MW				
0.02	0.00011	0.00069	0.00001	0.00009	0.00000	0.00000	0.00000	0.00000				
0.06	0.00011	0.00068	0.00001	0.00009	0.00000	0.00000	0.00000	0.00000				
0.10	0.00010	0.00061	0.00001	0.00008	0.00000	0.00000	0.00000	0.00000				
0.14	0.00009	0.00057	0.00001	0.00007	0.00000	0.00000	0.00000	0.00000				
0.20	0.00013	0.00081	0.00002	0.00010	9.49169	5.17440	0.25634	0.25016				

Power Restoration with Transient Stability

- Minimizing the rotor angle while maintaining a certain dispatching distance

TABLE II
MAXIMUM ROTOR SWINGS (IN DEGREE) ON VARIOUS GENERATOR DISPATCH LIMITS

	6 Bus				14 Bus				30 Bus			
	Maximum dispatch limit (L2)				Maximum dispatch limit (L2)				Maximum dispatch limit (L2)			
Gen. Reactance	10%	5%	1%	0.1%	10%	5%	1%	0.1%	10%	5%	1%	0.1%
0.02	0.57272	1.02414	1.54931	1.81033	3.15922	3.93163	4.82311	5.26211	1.93677	2.43144	3.09765	3.46969
0.06	1.81033	1.81033	1.81033	1.81033	5.26211	5.26211	5.26211	5.28046	3.46969	3.46969	3.46969	3.46969
0.10	1.81033	1.81033	1.81033	1.81033	5.28046	5.28046	5.28046	5.29695	3.46969	3.46969	3.46969	3.46969
0.14	1.81033	1.81033	1.81033	1.81033	5.29695	5.29695	5.29695	5.30077	3.46969	3.46969	3.46969	3.46969
0.20	1.81033	1.95477	2.52171	2.92749	5.30077	5.30077	5.30077	5.30077	3.46969	3.46969	3.65917	3.76947

	39 Bus				57 Bus			
	Maximum dispatch limit (L2)				Maximum dispatch limit (L2)			
Gen. Reactance	10%	5%	1%	0.1%	10%	5%	1%	0.1%
0.02	7.36854	8.62295	12.5102	14.94628	3.81116	4.48884	6.59446	10.31860
0.06	15.57954	18.86922	22.98914	25.31593	10.31860	10.31860	10.31860	13.58787
0.10	25.31593	25.31593	26.14855	29.3314	13.58787	13.58787	13.58787	16.67782
0.14	29.3314	29.3314	29.3314	29.3314	16.67782	16.67782	16.67782	19.50131
0.20	29.3314	29.3314	29.3314	29.3314	19.50131	19.50131	21.65008	24.25307

Beyond Steady States

- Using the dynamics to correct the steady state
  - hierarchical approach
  - feedback loop
- The power flow equations
  - need to be solved exactly
  - the relaxations are “cheating”
- Scalability is the next frontier
  - more realistic dynamic models
  - thermal limits
  - scaling performance

Open Issues

- Power Flows
  - Can we get scalable approximations with performance guarantees on these problems?
  - Can we get scalable relaxations?
  - Can we define what normal operating conditions means?
- Dynamics
  - Can we get scalable optimization approaches?
- Stochastic and robust optimisation
  - Generalizing the deterministic case