Multiple Scattering Casimir Force Calculations: Layered and Corrugated Materials, Wedges, and CP Forces

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IJMPA **24**, 1757 ’09; JPCS **161**, 012001, 012008, 012022 ’09;

PRL **101**, 160402; PRD **78**, 065018, 065019; **77**, 045005 ’08

supported by NSF and DOE

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I. Multiple Scattering Technique

The multiple scattering approach starts from the well-known formula for the vacuum energy or Casimir energy (for simplicity here we first restrict attention to a massless scalar field)(τ is the “infinite” time that the configuration exists) [Schwinger, 1975]

$$E = \frac{i}{2\tau}\mathrm{Tr}\ln G \to \frac{i}{2\tau}\mathrm{Tr}\ln GG_0^{-1},$$

where G (G_0) is the Green’s function,

$$(-\partial^2 + V)G = 1, \quad +\mathrm{BC}, \quad -\partial^2 G_0 = 1.$$

T-matrix

Now we define the T-matrix,

$$T = S - 1 = V(1 + G_0 V)^{-1}.$$

If the potential has two disjoint parts, $V = V_1 + V_2$ it is easy to derive the interaction between the two bodies (potentials):

$$E_{12} = -\frac{i}{2\tau}\text{Tr}\ln(1 - G_0 T_1 G_0 T_2)$$
$$= -\frac{i}{2\tau}\text{Tr}\ln(1 - V_1 G_1 V_2 G_2),$$

where $G_i = (1 + G_0 V_i)^{-1} G_0, \quad i = 1, 2.$

Exact Results–Weak Coupling

In weak coupling it is possible to derive the exact (scalar) interaction between two potentials

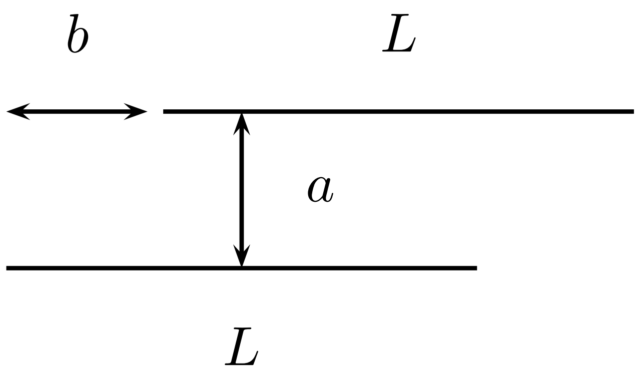
$$2D: \quad \frac{E}{L_z} = -\frac{1}{32\pi^3}\int (d\mathbf{r}_\perp)(d\mathbf{r}'_\perp)\frac{V_1(\mathbf{r}_\perp)V_2(\mathbf{r}'_\perp)}{|\mathbf{r}-\mathbf{r}'|^2},$$

$$3D: \quad E = -\frac{1}{64\pi^3}\int (d\mathbf{r})(d\mathbf{r}')\frac{V_1(\mathbf{r})V_2(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}.$$

Exact Results for Finite Plates

Consider two plates of finite length L, offset by an amount b, separated by a distance a:

$$V_1(\mathbf{r}_\perp) = \lambda_1 \delta(y)\theta(x)\theta(L-x),$$
$$V_2(\mathbf{r}'_\perp) = \lambda_2 \delta(y'-a)\theta(x'-b)\theta(L+b-x'),$$



Exact Results for Finite Plates (cont

This gives an explicit result for the energy between the plate

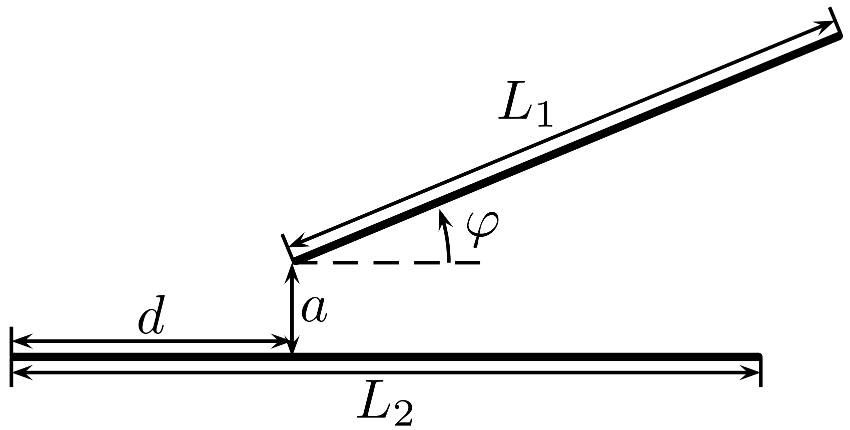
$$\frac{E}{L_z} = -\frac{\lambda_1\lambda_2}{32\pi^3}\left[-2g(b/a) + g((L-b)/a) + g((L+b)/a)\right]$$

where

$$g(x) = x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) = -\text{Re}(1+ix)\ln(1+ix).$$

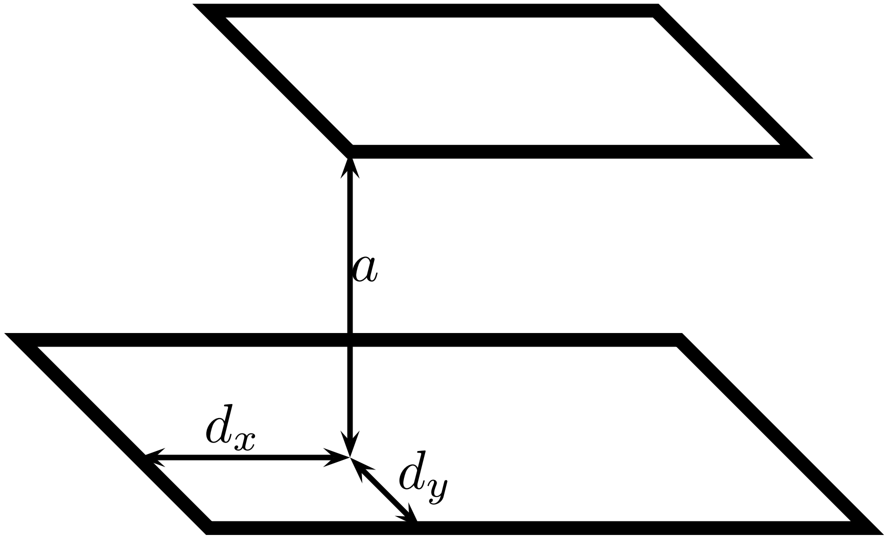
We can consider arbitrary lengths and orientations, in 3 dimensions, for the plates. [J. Wagner et al.]

Tilted plates



Explicit interaction energies can be given in terms of Ti_2, inverse tangent integral. For fixed CM distance from the lower plate D, for $L_1 \to L$, $L_2 \to \infty$, $d \to -\infty$, and $D > \frac{L}{2}$, the equilibrium position of the upper plate is at $\phi = \pi/2$.

Rectangular Parallel Plates



As $a \longrightarrow 0$,

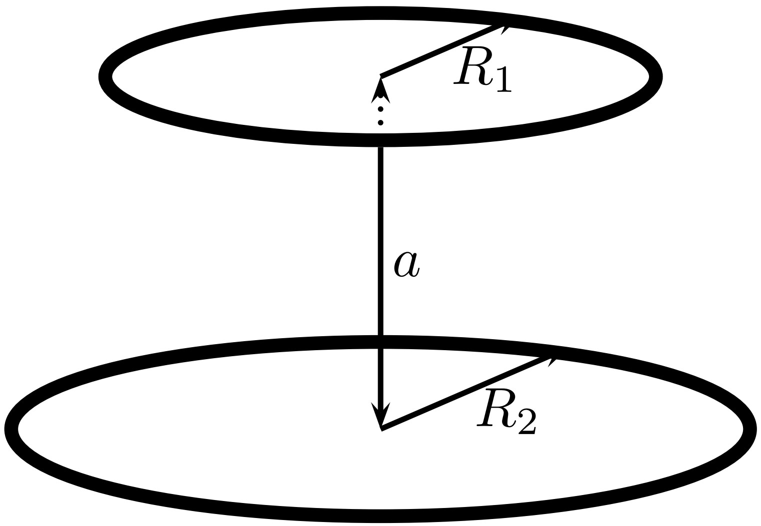
$$\frac{F}{A} = -\frac{\lambda_1 \lambda_2}{32\pi^2 a^2}\left(1 + c_1 a + c_2 a^2 + \dots\right)$$

Correction to Lifshitz formula

- If upper plate is completely above lower plate, $c_1 = 0$.
- If plates are of the same size and aligned,

$$c_1 = -\frac{1}{\pi}\frac{\text{Perimeter}}{\text{Area}}.$$

Coaxial disks

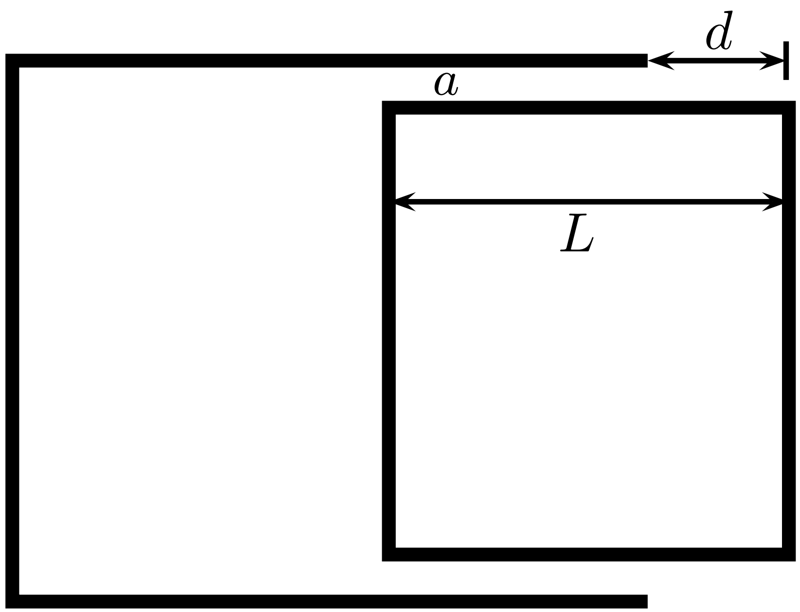


- If $R_1 < R_2$, $c_1 = 0$.
- If $R_1 = R_2$, $c_1 = -\frac{1}{\pi}\frac{\text{Perimeter}}{\text{Area}}$.

Salient Features–two thin plates

- Two plates of finite length experience a lateral force so that they wish to align in the position of maximum symmetry.
- In this symmetric configuration, there is a torque about the CM of a single plate so that it tends to seek perpendicular orientation with respect to the other plate.
- First correction to Lifshitz formula is geometrical.

Relevance to Casimir Pistol



S. A. Fulling, L. Kaplan, K. Kirsten, Z. H. Liu, and K. A. Milton, arXiv:0806.2468 [hep-th], J. Phys. A: Math. Theor. **42**, 155402 (2009).

Summing van der Waals forces

The (retarded dispersion) van der Waals potential between polarizable molecules is given by

$$V = -\frac{23}{4\pi}\frac{\alpha_1\alpha_2}{r^7}, \quad \alpha = \frac{\varepsilon - 1}{4\pi N}.$$

This allows us to consider in the same vein (electromagnetic) interaction between distinct dilute dielectric bodies of arbitrary shape.

Derivation of vdW interaction

This vdW potential may be directly derived from

$$W = -\frac{i}{2}\text{Tr}\,\ln \boldsymbol{\Gamma}\boldsymbol{\Gamma}_0^{-1} \approx -\frac{i}{2}\text{Tr}\,V_1\boldsymbol{\Gamma}_0V_2\boldsymbol{\Gamma}_0,$$

where $V = \epsilon - 1$ and

$$\boldsymbol{\Gamma}_0 = \boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{1}\frac{e^{-|\zeta||\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} - \mathbf{1}$$

$$= (\boldsymbol{\nabla}\boldsymbol{\nabla} - \mathbf{1}\zeta^2)G_0(\mathbf{r}-\mathbf{r}').$$

II. Interaction between ε, μ bodies

Consider material bodies characterized by a permittivity $\varepsilon(\mathbf{r})$ and a permeability $\mu(\mathbf{r})$, so we have corresponding electric and magnetic potentials

$$V_e(\mathbf{r}) = \varepsilon(\mathbf{r}) - 1, \quad V_m(\mathbf{r}) = \mu(\mathbf{r}) - 1.$$

Then the trace-log is ($\boldsymbol{\Phi}_0 = -\frac{1}{\zeta}\boldsymbol{\nabla} \times \boldsymbol{\Gamma}_0$)

$$\begin{aligned} \text{Tr} \ln \boldsymbol{\Gamma}\boldsymbol{\Gamma}_0^{-1} &= -\text{Tr} \ln(\mathbf{1} - \boldsymbol{\Gamma}_0 V_e) - \text{Tr} \ln(\mathbf{1} - \boldsymbol{\Gamma}_0 V_m) \\ &\quad -\text{Tr} \ln(\mathbf{1} + \boldsymbol{\Phi}_0 \mathbf{T}_e \boldsymbol{\Phi}_0 \mathbf{T}_m), \\ \mathbf{T}_{e,m} &= V_{e,m}(\mathbf{1} - \boldsymbol{\Gamma}_0 V_{e,m})^{-1}. \end{aligned}$$

Factorization

If we have *disjoint* electric bodies, the interaction term separates out:

$$\mathrm{Tr}\,\ln\left(\mathbf{1}-\mathbf{\Gamma}_0(V_1+V_2)\right) = -\mathrm{Tr}\,\ln(\mathbf{1}-\mathbf{\Gamma}_0\mathbf{T}_1) \\ -\mathrm{Tr}\,\ln(\mathbf{1}-\mathbf{\Gamma}_0\mathbf{T}_2) - \mathrm{Tr}\,\ln(\mathbf{1}-\mathbf{\Gamma}_0\mathbf{T}_1\mathbf{\Gamma}_0\mathbf{T}_2),$$

so only the latter term contributes to the interaction energy,

$$E_{\mathrm{int}} = \frac{i}{2}\mathrm{Tr}\,\ln(\mathbf{1}-\mathbf{\Gamma}_0\mathbf{T}_1\mathbf{\Gamma}_0\mathbf{T}_2).$$

ε-μ Lifshitz force

The same is true if one body is electric and the other magnetic,

$$E_{\rm int} = -\frac{i}{2}\mathrm{Tr}\ln(1 + \mathbf{\Phi}_0\mathbf{T}_1^e\mathbf{\Phi}_0\mathbf{T}_2^m).$$

Using this, it is easy to show that the Lifshitz energy between a parallel dielectric and diamagnetic slabs is

$$E_{\varepsilon\mu} = \frac{1}{16\pi^3}\int d\zeta \int d^2k \left[\ln\left(1 - r_1 r_2' e^{-2\kappa a}\right) + \ln\left(1 - r_1 r_2' e^{-2\kappa a}\right)\right]$$

Repulsive Casimir force

where

$$r_i = \frac{\kappa - \kappa_i}{\kappa + \kappa_i}, \quad r_i' = \frac{\kappa - \kappa_i'}{\kappa + \kappa_i'},$$

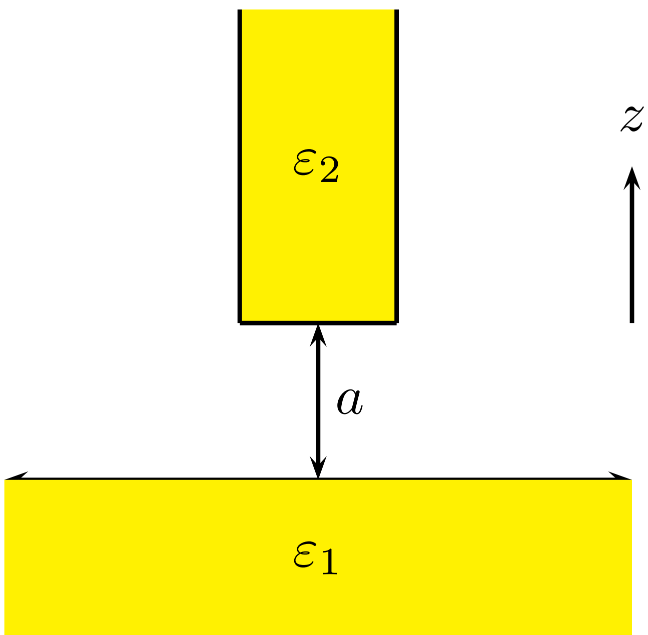
with $\kappa^2 = k^2 + \zeta^2$, $\kappa_i = k^2 + \varepsilon\zeta^2$, $\kappa_i' = \kappa_i/\varepsilon_i$. This means in the perfect reflecting limit, $\varepsilon \to \infty$, $\mu \to \infty$,

$$E_{\text{Boyer}} = +\frac{7}{8}\frac{\pi^2}{720a^3},$$

we get Boyer's repulsive result.

III. Dilute dielectrics

We now give some exact results for dilute dielectrics, $|\varepsilon - 1| \ll 1$. For example, consider



Force between slab/infinite plate

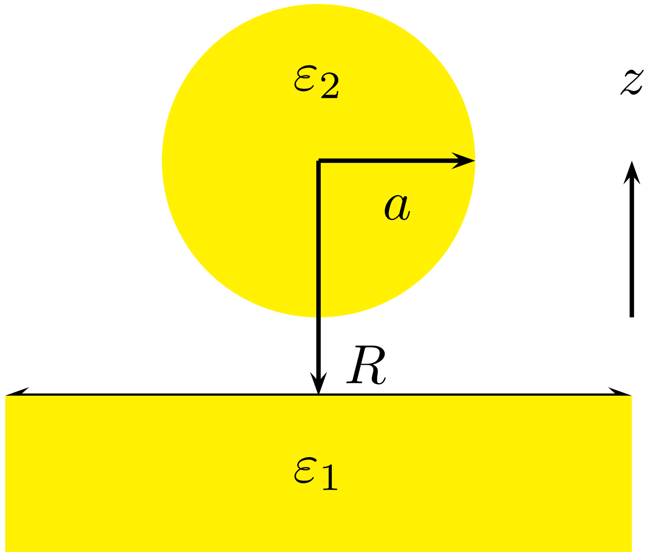
If the cross sectional area of the finite slab is A, the force between the slabs is

$$\frac{F}{A} = -\frac{23}{640\pi^2}\frac{1}{a^4}(\varepsilon_1 - 1)(\varepsilon_2 - 1),$$

the Lifshitz formula for infinite (dilute) slabs.

Note that there is no correction due to the finite area of the slab.

Force between sphere and plate



$$E = -\frac{23}{640\pi^2}(\varepsilon_1 - 1)(\varepsilon_2 - 1)\frac{4\pi a^3/3}{R^4}\frac{1}{(1 - a^2/R^2)^2},$$

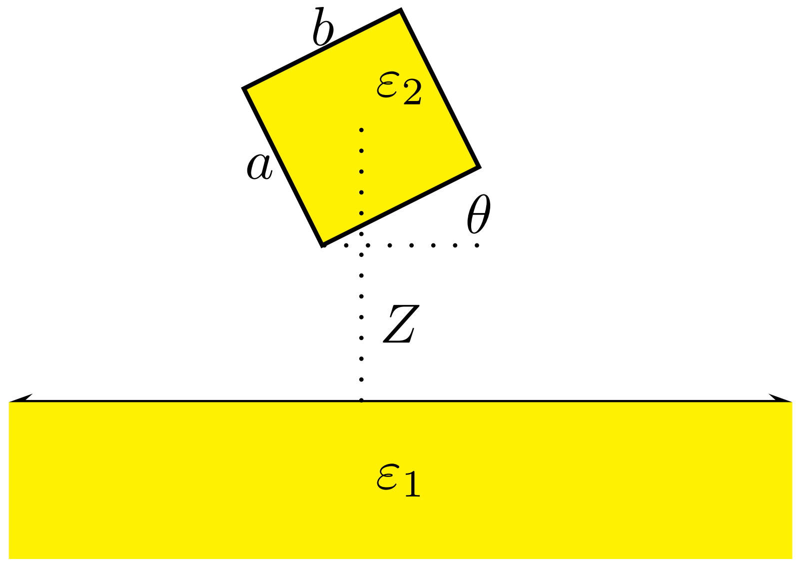
— comparison with PFA

which agrees with the PFA in the short separation limit, $R - a = \delta \ll a$:

$$F_{\rm PFA} = 2\pi a \mathcal{E}_{\parallel}(\delta) = -\frac{23}{640\pi^2}(\varepsilon_1 - 1)(\varepsilon_2 - 1)\frac{2\pi a}{3\delta^3},$$

with an exact correction, intermediate between that for scalar 1/2(Dirichlet+Neumann) and electromagnetic perfectly-conducting boundaries.

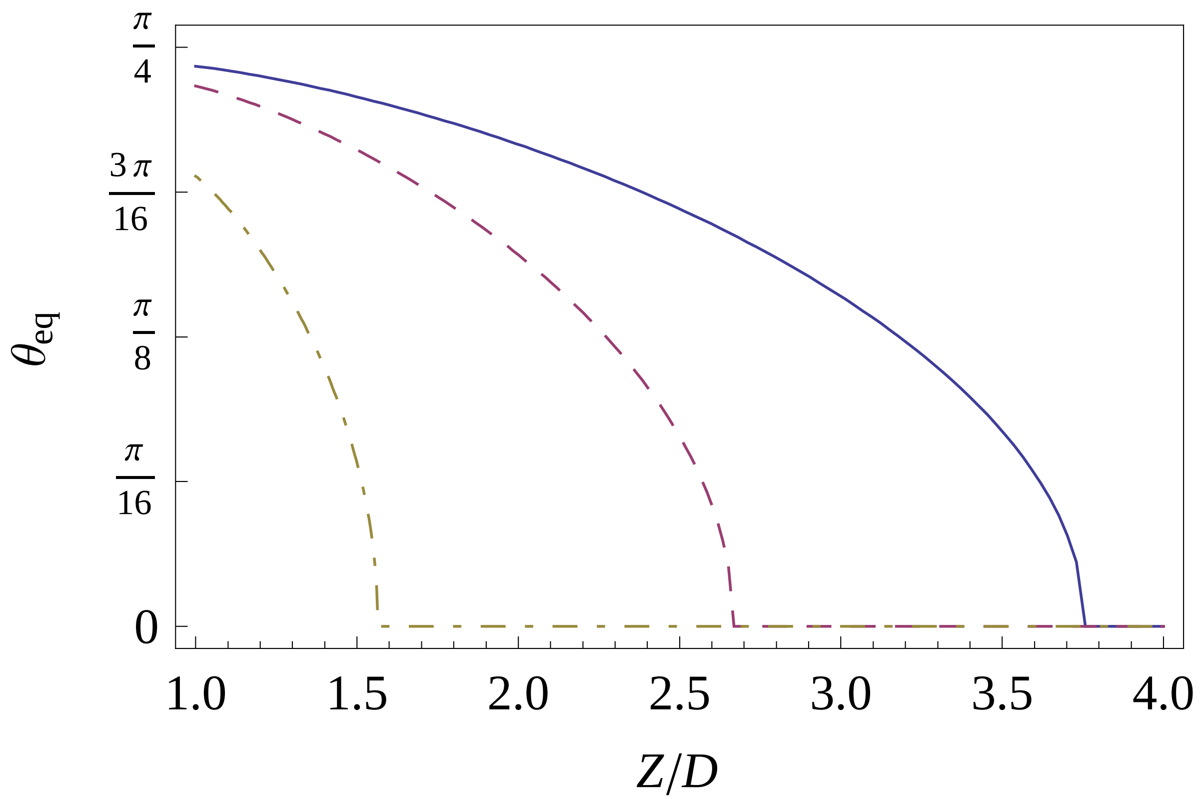
Energy between slab and plate



Torque between slab and plate

Generically, the shorter side wants to align with the plate, which is obvious geometrically, since that (for fixed center of mass position) minimizes the energy. However, if the slab has square cross section, the equilibrium position occurs when a corner is closest to the plate, also obvious geometrically. But if the two sides are close enough in length, a nontrivial equilibrium position between these extremes can occur.

Nontrivial equilibria



Stable equilibria

The stable equilibrium angle of a slab above an infinite plate for given b/a ratios 0.95, 0.9, and 0.7, respectively given by solid, dashed, and dot-dashed lines. For large enough separation, the shorter side wants to face the plate, but for

$$Z < Z_0 = \frac{a}{2}\sqrt{\frac{2a^2 + 5b^2 + \sqrt{9a^4 + 20a^2b^2 + 20b^4}}{5\left(a^2 - b^2\right)}}$$

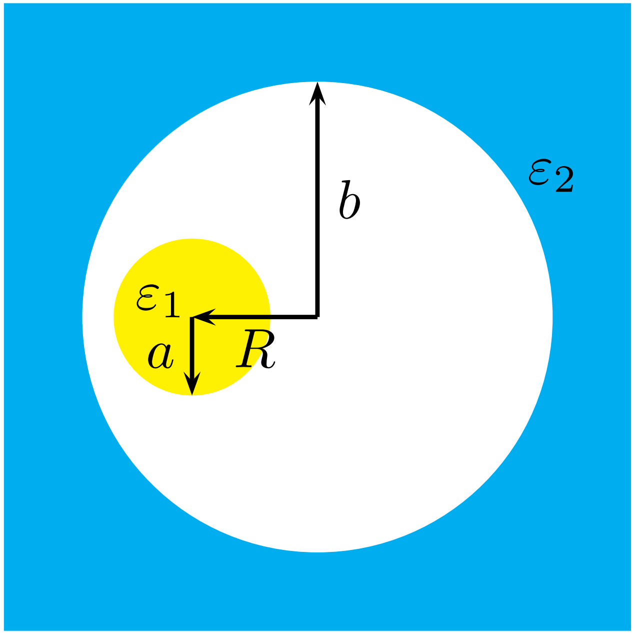
the equilibrium angle increases, until finally at $Z = D = \sqrt{a^2 + b^2}/2$ the slab touches the plate at an angle $\theta = \arctan b/a$.

Interaction between ∥ cylinders

$$\frac{E}{L} = -\frac{23}{60\pi}(\varepsilon_1 - 1)(\varepsilon_2 - 1)\frac{a^2b^2}{R^6}$$
$$\times \frac{1 - \frac{1}{2}\left(\frac{a^2+b^2}{R^2}\right) - \frac{1}{2}\left(\frac{a^2-b^2}{R^2}\right)^2}{\left[\left(1 - \left(\frac{a+b}{R}\right)^2\right)\left(1 - \left(\frac{a-b}{R}\right)^2\right)\right]^{5/2}}.$$

Eccentric interior cylinder

This result can be analytically continued to the case when one dielectric cylinder is entirely inside a hollowed-out cylinder within an infinite dielectric medium.



Interaction between spheres

$$E = -\frac{23}{1920\pi}\frac{(\varepsilon_1 - 1)(\varepsilon_2 - 1)}{R}\left\{ \ln\left(\frac{1-\left(\frac{a-b}{R}\right)^2}{1-\left(\frac{a+b}{R}\right)^2}\right) + \frac{4ab}{R^2}\frac{\frac{a^6-a^4b^2-a^2b^4+b^6}{R^6} - \frac{3a^4-14a^2b^2+3b^4}{R^4} + 3\frac{a^2+b^2}{R^2} - 1}{\left[\left(1-\left(\frac{a-b}{R}\right)^2\right)\left(1-\left(\frac{a+b}{R}\right)^2\right)\right]^2}\right\}$$

PFA and sphere-plate

This expression, which is rather ugly, may be verified to yield the proximity force theorem:

$$E \to U = -\frac{23}{640\pi}\frac{a(R-a)}{R\delta^2}, \quad \delta = R - a - b \ll a, b.$$

It also, in the limit $b \to \infty$, $R \to \infty$ with $R - b = Z$ held fixed, reduces to the result for the interaction of a sphere with an infinite plate.

IV. Exact temperature results

The scalar Casimir energy between two weak nonoverlapping potentials $V_1(\mathbf{r})$ and $V_2(\mathbf{r})$ at temperature T is

$$E_T = -\frac{T}{32\pi^2}\int (d\mathbf{r})(d\mathbf{r}')V_1(\mathbf{r})V_2(\mathbf{r}')\frac{\coth 2\pi T|\mathbf{r}-\mathbf{r}'|}{|\mathbf{r}-\mathbf{r}'|^2}.$$

Exact PFA

The energy between a semitransparent plane and an arbitrarily curved nonintersecting semitransparent surface:

$$E_T = -\frac{\lambda_1\lambda_2 T}{16\pi}\int dS \int_{2\pi T z(S)} dx \frac{\coth x}{x},$$

where the area integral is over the curved surface. This is precisely what one means by the PFA:

$$E_{\text{PFA}} = \int dS \mathcal{E}_\parallel(z(S)),$$

as proved by Decca et al.

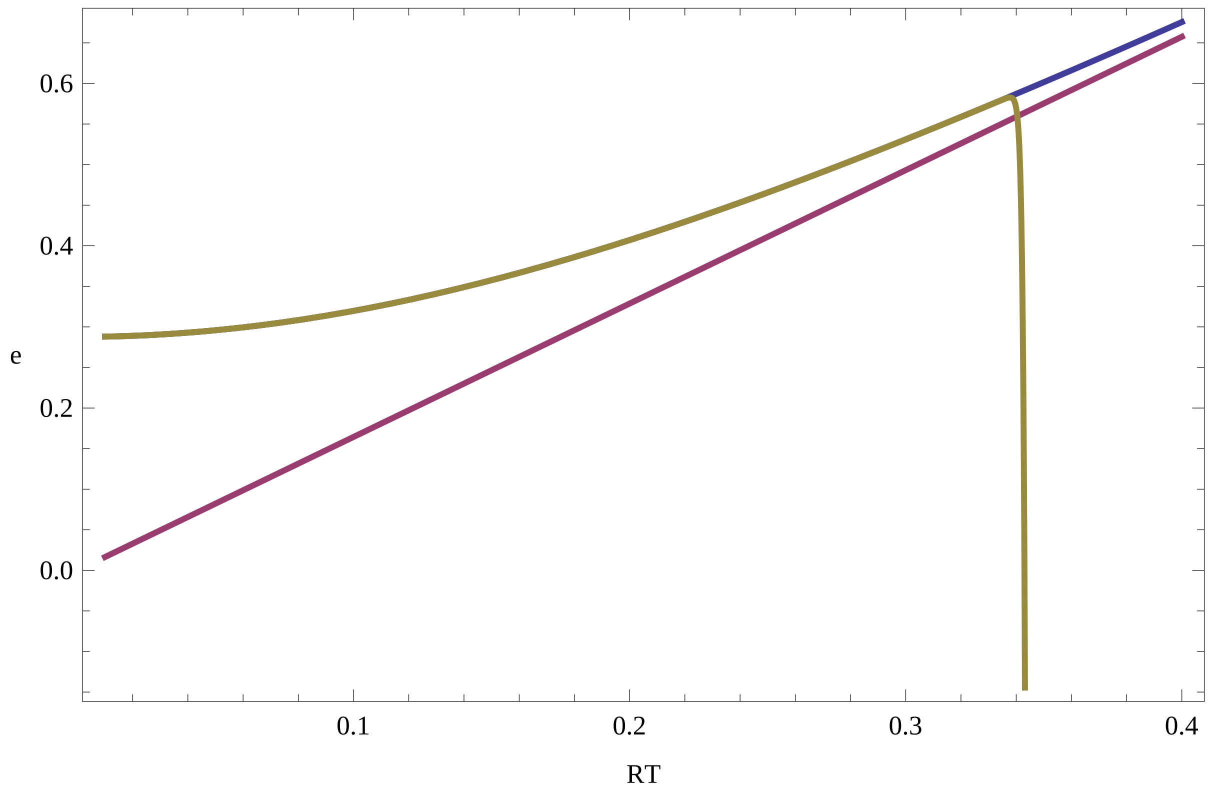
Interaction between ST spheres

$$E_T = -\frac{\lambda_1\lambda_2}{16\pi}\frac{ab}{R}\Bigg\{\ln\frac{1-(a-b)^2/R^2}{1-(a+b)^2/R^2} + f(2\pi T(R+a+b)) + f(2\pi T(R-a-b)) - f(2\pi T(R-a+b)) - f(2\pi T(R+a-b))\Bigg\},$$

where f is obtained from

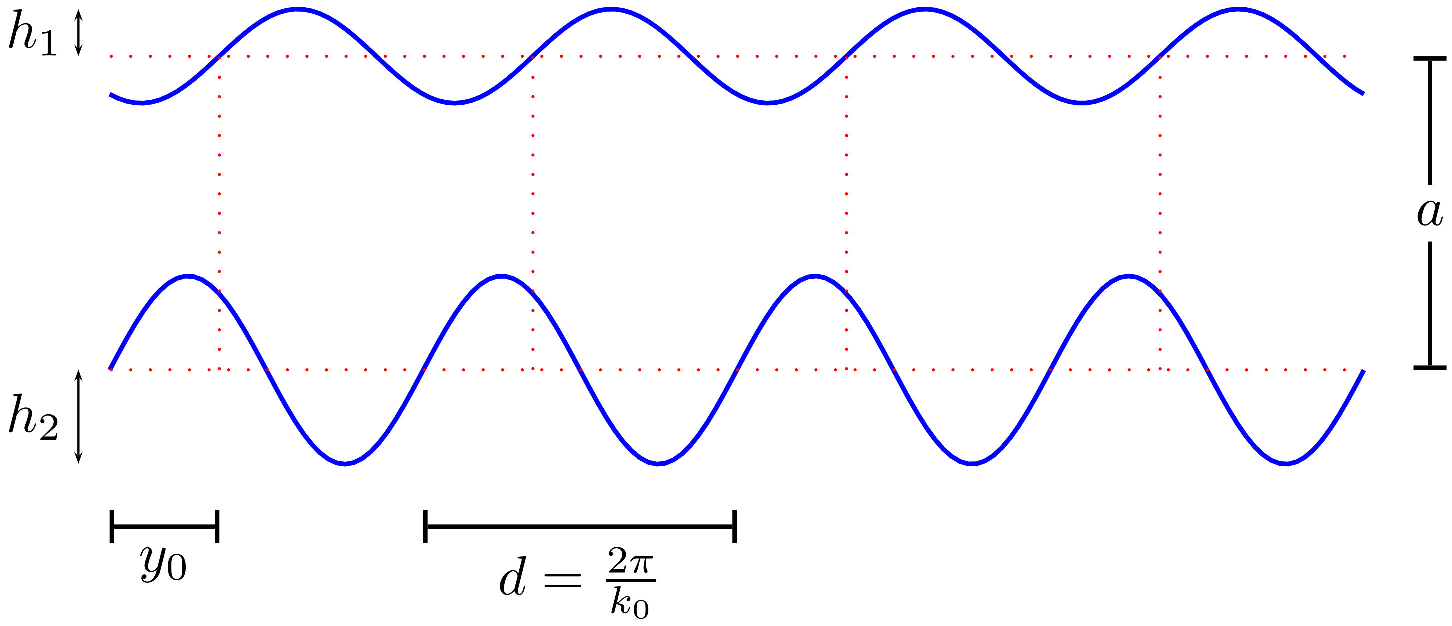
$$y\frac{d^2}{dy^2}f(y) = \coth y - \frac{1}{y}, \qquad f(0) = f'(0) = 0.$$

Numerical results



$a = b = R/4$. Exact, high T, and truncated series expansion.

V. Noncontact gears

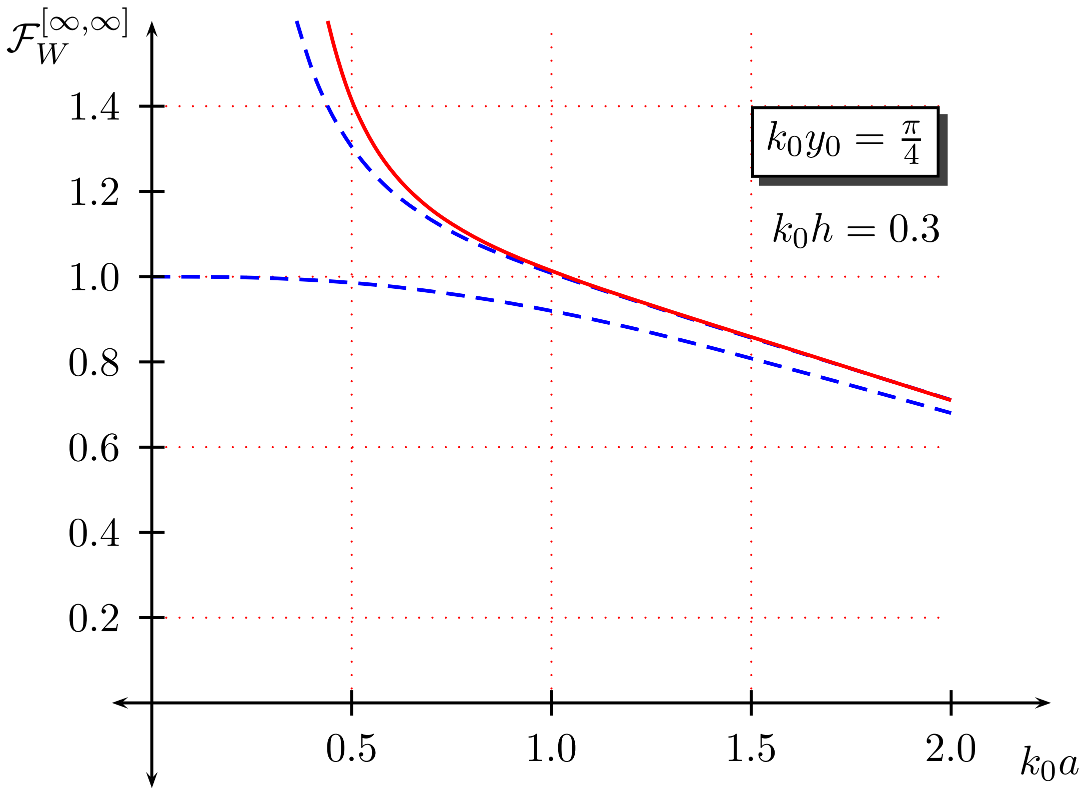


Perturbation theory

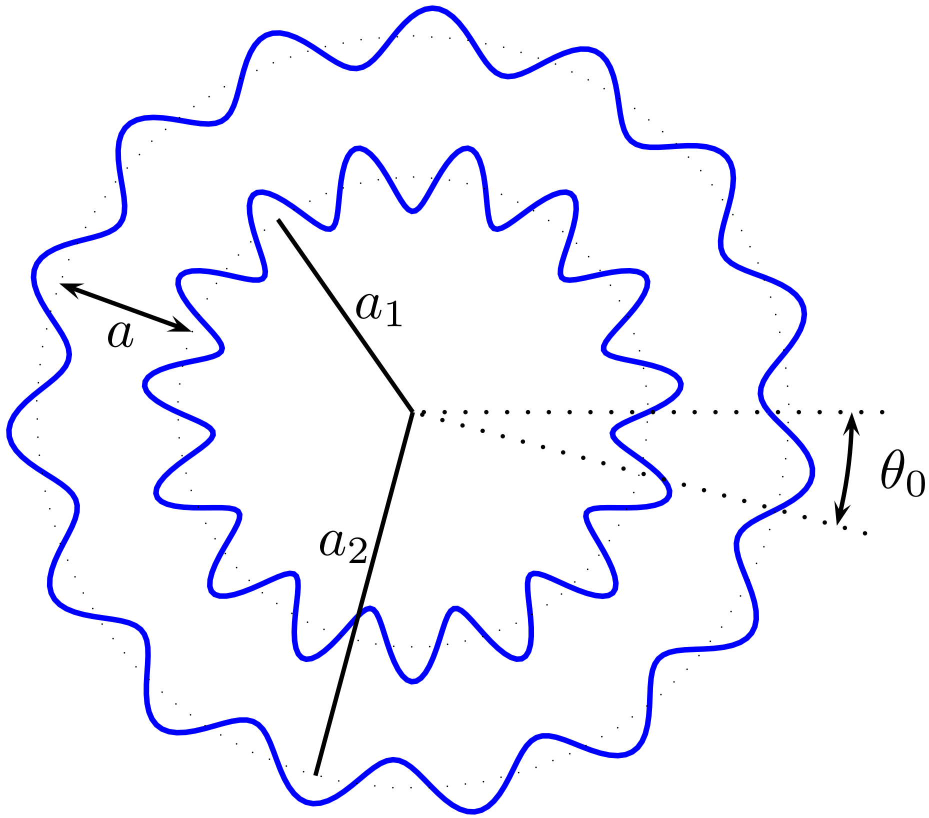
Here we compute the *lateral* force between the offset corrugated plates. The Dirichlet and electromagnetic cases were previously considered by Kardar and Emig, to second order in corrugation amplitudes. We have carried out the calculations to fourth order. In weak coupling we can calculate to all orders, and verify that fourth order is very accurate, provided $k_0 h \ll 1$.

$$\mathcal{F} = \frac{F_{\text{Lat}}}{|F^{(0)}_{\text{Cas}}|(h_1 h_2/a^2) k_0 a \sin(k_0 y)}$$

Weak coupling limit



Concentric corrugated cylinders



Casimir torque per unit area

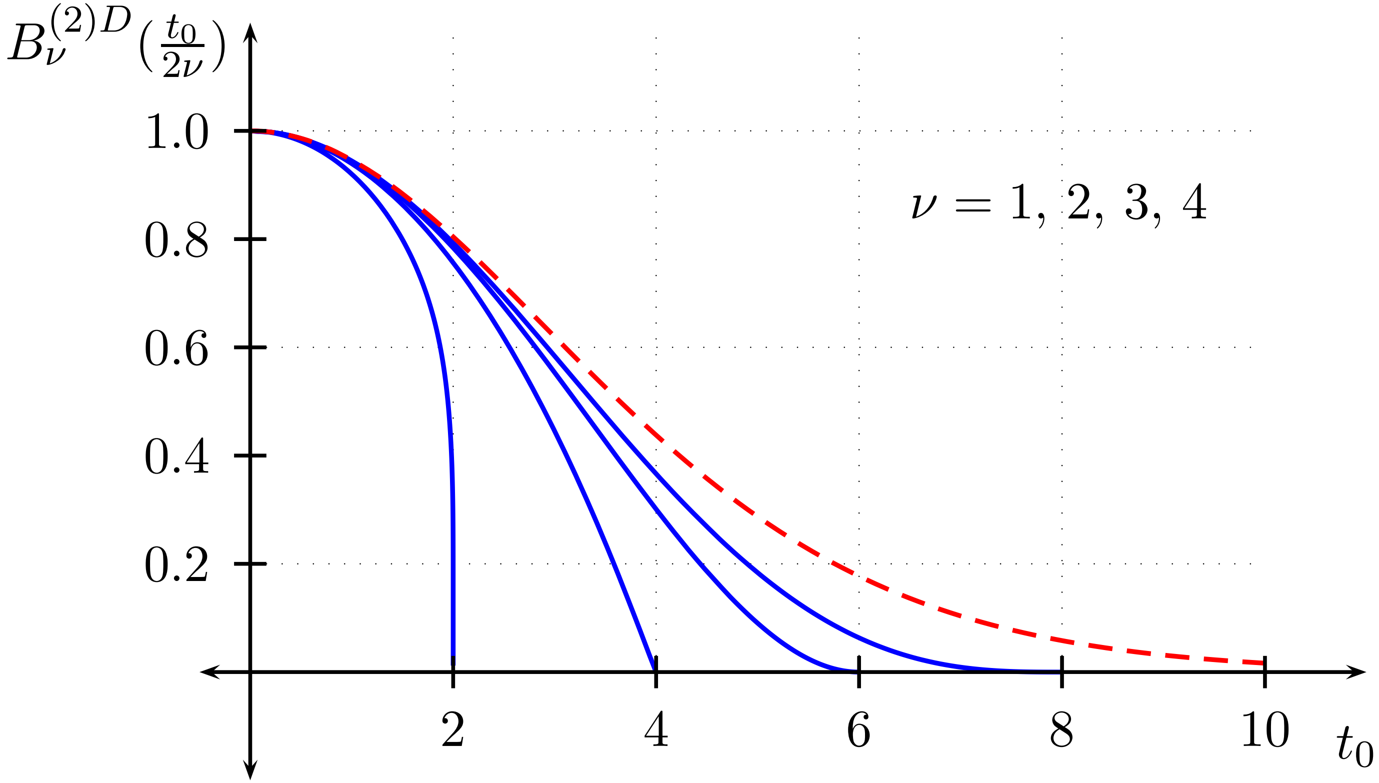
For corrugations given by δ-function potentials with sinusoidal amplitudes:

$$h_1(\theta) = h_1 \sin \nu(\theta + \theta_0),$$
$$h_2(\theta) = h_2 \sin \nu\theta$$

the torque to lowest order in the corrugations in strong coupling (Dirichlet limit) $(\alpha = (a_2 - a_1)/(a_2 + a_1))$

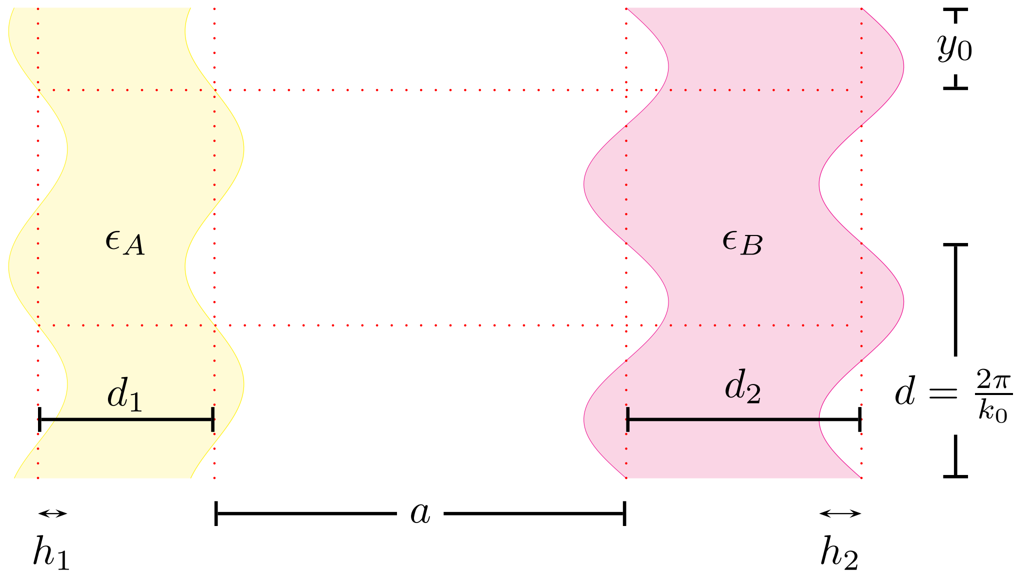
$$\frac{\tau^{(2)D}}{2\pi R L_z} = \nu \sin \nu\theta_0 \frac{\pi^2}{240 a^3} \frac{h_1}{a} \frac{h_2}{a} B_\nu^{(2)D}(\alpha).$$

Dirichlet limit of cylindrical gears



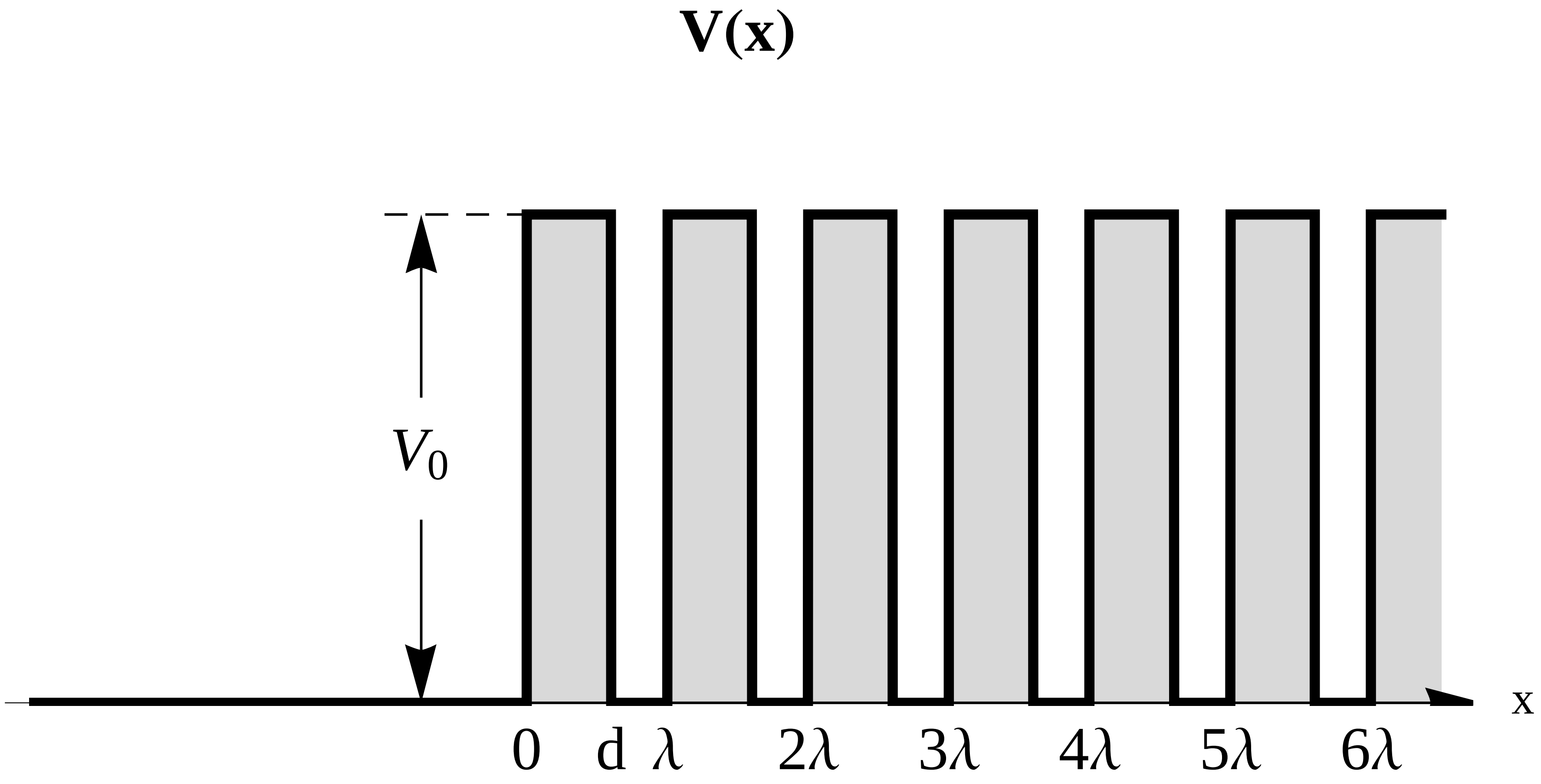
A similar result can be found for weak coupling, which, again, has a closed form.

Corrugated Dielectric Slabs



2nd and 4th order results should appear soon.

VI. Multilayered surfaces



Green's function

To the right of this array of potentials, the reduced Green's function has the form, in terms the reflection coefficient $\mathcal{R}$ for the array:

$$g(x,x') = \frac{1}{2\kappa}\left(e^{-\kappa|x-x'|} + \mathcal{R}e^{\kappa(x+x')}\right).$$

(We can actually find the Green's function everywhere, for any piecewise continuous potential. This will be described in detail in forthcoming papers by Jef Wagner.)

Reflection coefficient

The array reflection coeffient may be readily expressed in terms of the reflection and transmission coefficients for a single potential:

$$\mathcal{R} = R + Te^{-\kappa d}\mathcal{R}e^{-\kappa d}(1 - Re^{-2\kappa d}\mathcal{R})^{-1},$$

where d is the distance between the potentials, the result of summing multiple reflections, $\longrightarrow$

$$\mathcal{R} = \frac{1}{2R}\left[e^{2\kappa d} + R^2 - T^2 - \sqrt{(e^{2\kappa d} - R^2 - T^2)^2 - 4R^2T^2}\right].$$

Dielectric slabs

If the potentials consist of dielectric slabs, with dielectric constant ε and thickness a, the TE reflection and transmission coefficients for a single slab are ($\kappa' = \sqrt{\varepsilon\zeta^2 + k^2}$)

$$R^{\mathrm{TE}} = \frac{e^{2\kappa' a} - 1}{\left(\frac{1+\kappa'/\kappa}{1-\kappa'/\kappa}\right) e^{2\kappa' a} - \left(\frac{1-\kappa'/\kappa}{1+\kappa'/\kappa}\right)},$$

$$T^{\mathrm{TE}} = \frac{4(\kappa'/\kappa)e^{\kappa' a}}{(1+\kappa'/\kappa)^2 e^{2\kappa' a} - (1-\kappa'/\kappa)^2}.$$

TM: except in the exponents, $\kappa' \to \kappa'/\varepsilon$.

CP Force

Consider an atom, of polarizability $\alpha(\omega)$, a distance Z to the left of the array. The Casimir-Polder energy is

$$E = -\int d\zeta \int \frac{d^2k}{(2\pi)^2}\alpha(i\zeta)g_{kk}(Z,Z),$$

where apart from an irrelevant constant the trace is

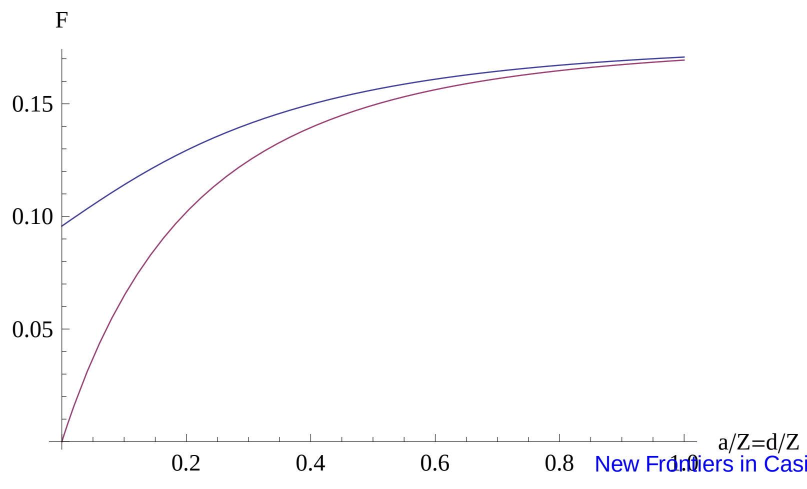
$$g_{kk}(Z,Z) \to \frac{1}{2\kappa}\left[-\zeta^2\mathcal{R}^{\mathrm{TE}} + (\zeta^2+2k^2)\mathcal{R}^{\mathrm{TM}}\right] e^{-2\kappa|Z|}.$$

Numerical Results ($\varepsilon = 2$)

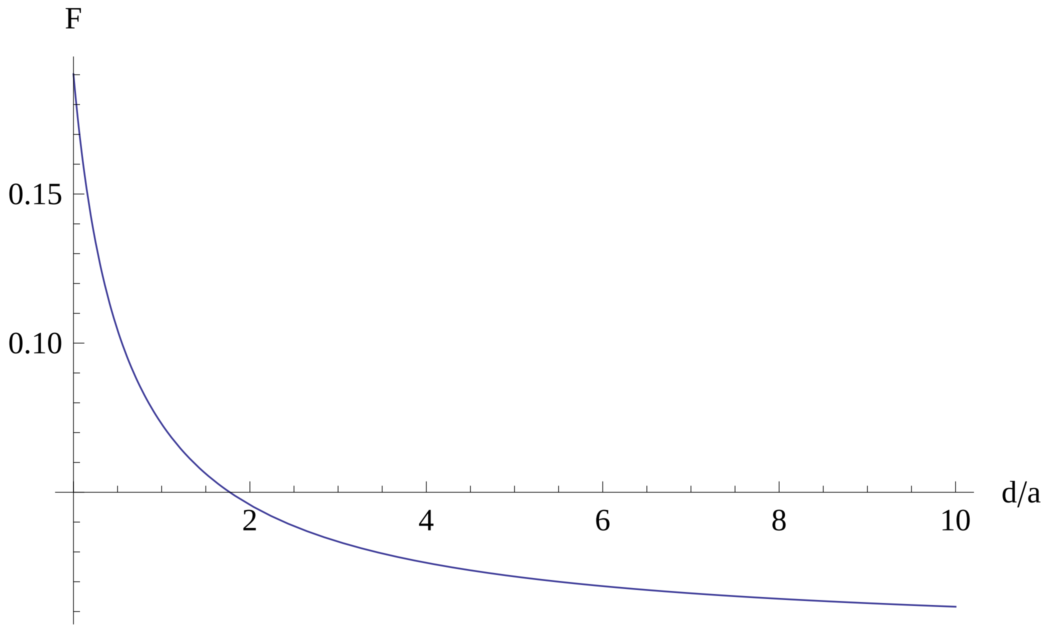
For example, in the static limit, where we disregard the frequency dependence of the polarizability,

$$E = -\frac{\alpha(0)}{2\pi}\frac{1}{Z^4}F(a/Z, d/Z).$$

This is compared with the single slab result:



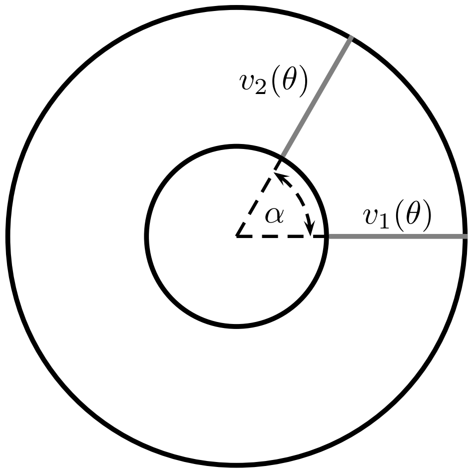
$Z \to \infty$ limit ($\varepsilon = 2$)



- When $d/a \to 0$ we recover the bulk limit.

VII. Annular pistons

The multiple scattering approach allows us to calculate the torque between annular pistons:



We use multiple scattering in the angular coordinates, and an eigenvalue condition in the radial coordinates—equally well solvable with radial Green's functions, but generalizable.

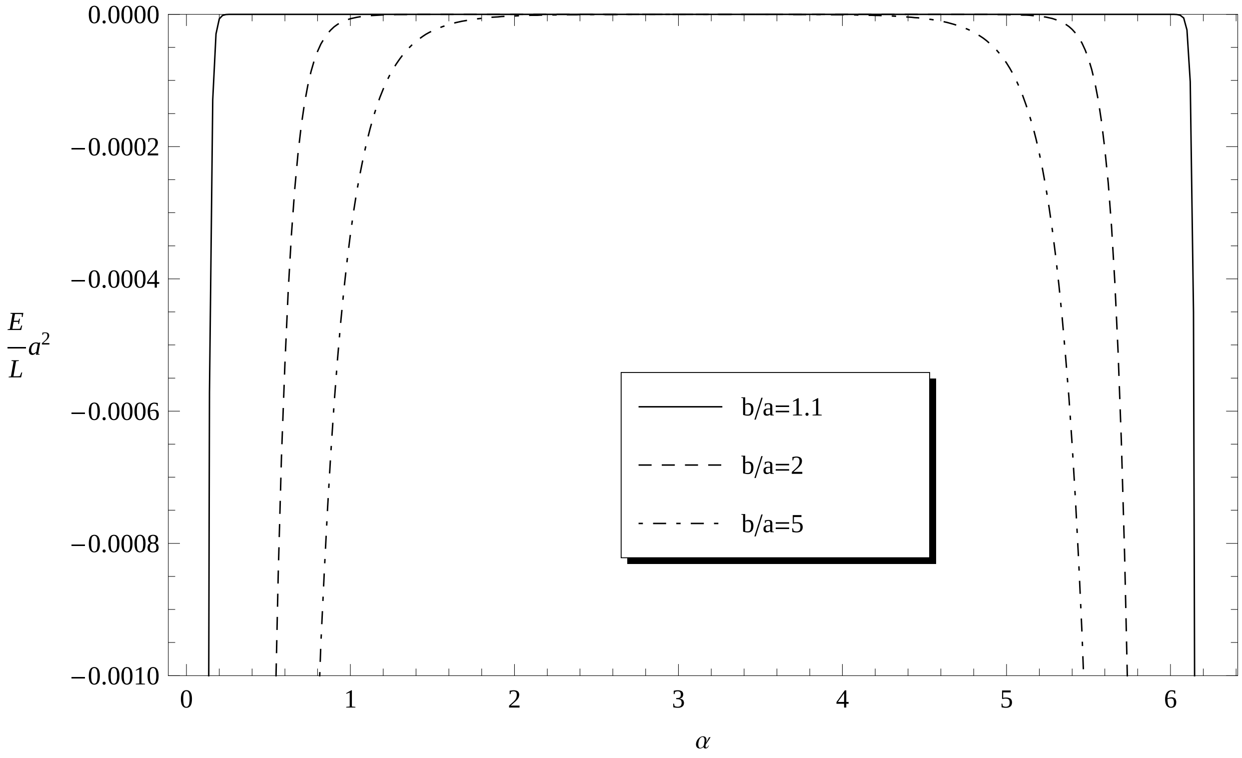
Energy for annular Casimir piston

Using the argument principle to determine the angular eigenvalues, we get the following expression for the energy for an annular Casimir piston: $\mathcal{E} =$

$$\int_0^\infty \frac{\kappa d\kappa}{8\pi^2 i} \int_\gamma d\eta \frac{\partial}{\partial\eta} \ln\left[K_{i\eta}(\kappa a)L_{i\eta}(\kappa b) - L_{i\eta}(\kappa a)K_{i\eta}(\kappa b)\right]$$

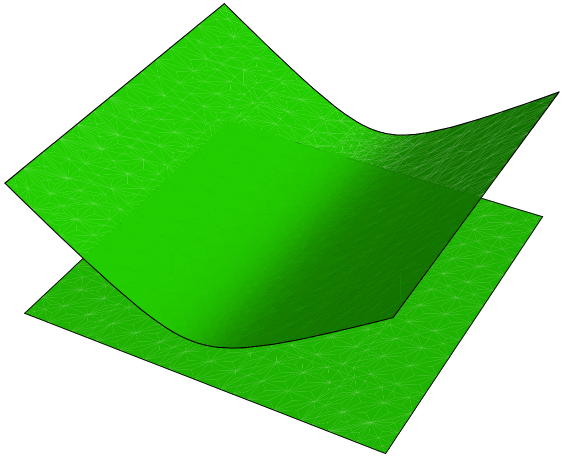
$$\times \ln\left(1 - \frac{\lambda_1\lambda_2 \cosh^2\eta(\pi-\alpha)/\cosh^2\eta\pi}{(2\eta\tanh\eta\pi+\lambda_1)(2\eta\tanh\eta\pi+\lambda_2)}\right).$$

Energy of Annular Piston

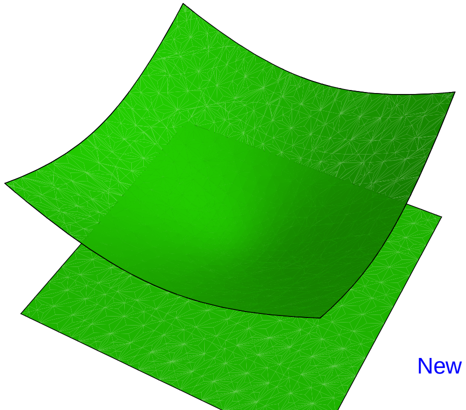


Hyperbolæ

Hyperbolic cylinder above plane:



Hyperbola of Revolution:



Conclusions

- Weak coupling results are laboratory for testing PFA.
- We have general results for the Green's functions for arbitrary piecewise continuous potentials in separable coordinates.
- From these we can calculate not only CP forces, but Casimir energies and torques for many geometries, including annular pistons, and forces between hyperbolic surfaces.
- New results for the electromagnetic non-contact gears, both for conductors and dielectrics, are in progress.